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The Necessary and Sufficient Condition for Global Stability of a Lotka-Volterra Cooperative or Competition System with Delays

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1. Introduction

Global stability of Lotka-Volterra delay systems has been studied by a lot of authors (see, [2-6, 9-11] and the reference cited therein). Most of the papers consider the situations at which undelayed intraspecific competitions are present (see, for example, [2, 3, 6, 9, 11]). In these cases, either a Liapunov functional is used ([3, 6, 9, 11]) or comparison theorems can be applied ([2]) to obtain the global asymptotic stability of a positive equilibrium point. Essentially, the point is globally asymptotically stable if there is the domination of the undelayed intraspecific competition over the delayed intra- (and inter-) specific competition. However, we find few papers referring to how the sharp domination is. In other words, there are few studies giving necessary and sufficient conditions for the global stability of Lotka-Volterra delay systems.

In this paper we consider the following symmetrical Lotka-Volterra system with delays including both cooperative and competition cases:

$$\begin{aligned} x'(t) &= x(t)[r_1 - ax(t) + \alpha x(t-\tau_{11}) + \beta y(t-\tau_{12})] \\ y'(t) &= y(t)[r_2 - ay(t) + \beta x(t-\tau_{21}) + \alpha y(t-\tau_{22})]. \end{aligned} \tag{1}$$

The initial condition of (1) is given as

$$\begin{aligned} x(s) &= \phi(s) \geq 0, -\Delta \leq s \leq 0\,; \phi(0) > 0 \\ y(s) &= \psi(s) \geq 0, -\Delta \leq s \leq 0\,; \psi(0) > 0. \end{aligned} \tag{2}$$

Here r_1, r_2, a, α, β, and τ_{ij} $(i,j=1,2)$ are constants with $r_1>0$, $r_2>0$, $a>0$, and $\tau_{ij}\geq 0$. ϕ and ψ are continuous functions and $\Delta = \max\{\tau_{ij} | i,j=1,2\}$. (1) is called a *cooperative system* if $\beta > 0$ and is called a *competition system* if $\beta < 0$.

We assume that (1) has a unique positive equilibrium (x^*, y^*), that is

$$x^* = \frac{(a-\alpha)r_1 + \beta r_2}{(a-\alpha)^2 - \beta^2} > 0, \qquad y^* = \frac{\beta r_1 + (a-\alpha)r_2}{(a-\alpha)^2 - \beta^2} > 0. \tag{3}$$

The positive equilibrium (x^*, y^*) is said to be globally asymptotically stable if (x^*, y^*) is stable and attracts any solution of (1) with (2). The purpose of this paper is to seek the necessary and sufficient condition for the global asymptotic stability of (x^*, y^*) of (1) for all delays τ_{ij} $(i, j = 1, 2)$, making the best use of the symmetry of the system. The result is the following:

Theorem 1. *The positive equilibrium (x^*, y^*) of (1) is globally asymptotically stable for all $\tau_{ij} \geq 0$ $(i, j = 1, 2)$ if and only if*

$$|\beta| < a - \alpha \qquad \textit{and} \qquad |\beta| \leq a + \alpha$$

hold.

In the case when there are no delays in system (1), that is $\tau_{ij} = 0$ $(i, j = 1, 2)$, (x^*, y^*) is globally asymptotically stable if and only if $|\beta| < a - \alpha$ holds (the proof is omitted for the sake of page restrictions). So we can see that the condition $|\beta| < a - \alpha$ and $|\beta| \leq a + \alpha$ in Theorem 1 reflects the delay effects.

When $\alpha > 0$, we notice that the positive delayed feedback terms $\alpha x(t - \tau_{11})$ and $\alpha y(t - \tau_{22})$ in the right-hand side of (1) play a role of *destabilizer* of the system. Biologically, $\alpha x(t - \tau_{11})$ and $\alpha y(t - \tau_{22})$ with $\alpha > 0$ may be viewed as *recycling* of population.

Gopalsamy [3] and Weng, Ma and Freedman [11] showed that if $|\alpha| + |\beta| < a$ holds, then the positive equilibrium (x^*, y^*) is globally asymptotically stable for all $\tau_{ij} \geq 0$ $(i, j = 1, 2)$. It is clear that Theorem 1 has the slight improvement of their results for (1). Recently, Lu and Wang [7] also considered the global asymptotic stability of (x^*, y^*) for (1) with $\alpha = 0$.

The proof of the sufficiency of Theorem 1 is done with a different-type Liapunov functional from ones used in the related papers mentioned above, and with an extended LaSalle's invariance principle (cf. [8; Lemma 3.1]). The necessity of Theorem 1 is proved by analyzing the roots of characteristic equations for linearized systems corresponding to the system (1).

2. The proof of Theorem 1

In this section, we will prove Theorem 1.

Sufficiency. When $\beta = 0$, the system (1) becomes the two scalar delay differential equations

$$\begin{aligned} x'(t) &= x(t)[r_1 - ax(t) + \alpha x(t - \tau_{11})] \\ y'(t) &= y(t)[r_2 - ay(t) + \alpha y(t - \tau_{22})]. \end{aligned} \tag{4}$$

By [5, pp.34-37], we see that $0 < a - \alpha$ and $0 \leq a + \alpha$ imply the global asymptotic stability of the positive equilibrium (x^*, y^*) of (4) for all nonnegative τ_{11} and τ_{22}.

When $\alpha = 0$, the system (1) becomes

$$\begin{aligned} x'(t) &= x(t)[r_1 - ax(t) + \beta y(t - \tau_{12})] \\ y'(t) &= y(t)[r_2 - ay(t) + \beta x(t - \tau_{21})] \end{aligned} \tag{5}$$

and, by (3), the positive equilibrium is give as

$$\left(\frac{ar_1 + \beta r_2}{a^2 - \beta^2}, \frac{\beta r_1 + ar_2}{a^2 - \beta^2} \right).$$

It follows from [11; Theorem 2.1] that (5) is globally asymptotically stable for all nonnegative τ_{12} and τ_{21} if $|\beta| < a$ holds. Therefore, we have only to consider the case $|\alpha| > 0$ and $|\beta| > 0$.

By the transformation

$$\bar{x} = x - x^*, \quad \bar{y} = y - y^*,$$

the system (1) is reduced to

$$\begin{aligned} x'(t) &= (x^* + x(t))[-ax(t) + \alpha x(t - \tau_{11}) + \beta y(t - \tau_{12})] \\ y'(t) &= (y^* + y(t))[-ay(t) + \beta x(t - \tau_{21}) + \alpha y(t - \tau_{22})] \end{aligned} \tag{6}$$

where we used $x(t)$ and $y(t)$ again instead of $\bar{x}(t)$ and $\bar{y}(t)$, respectively. Using [8; Lemma 3.1] we will prove that the trivial solution of (6) is globally asymptotically stable for all $\tau_{ij} \geq 0$ $(i, j = 1, 2)$. Define $C = C([-\Delta, 0], R^2)$ and

$$G = \left\{ (\phi, \psi) \in C \;\middle|\; \begin{array}{ll} \phi(s) + x^* \geq 0, & \phi(0) + x^* > 0 \\ \psi(s) + y^* \geq 0, & \psi(0) + y^* > 0 \end{array} \right\}.$$

Clearly, $\bar{G}$ (the closure of G) is positively invariant for (6). Construct the Liapunov functional V defined on G as

$$\begin{aligned} V(\phi, \psi) = {} & 2aX \left[\phi(0) - x^* \log \frac{\phi(0) + x^*}{x^*} \right] + 2aY \left[\psi(0) - y^* \log \frac{\psi(0) + y^*}{y^*} \right] \\ & + \alpha^2(X + 1) \int_{-\tau_{11}}^{0} \phi^2(\theta)d\theta + (\alpha^2 + \beta^2 Y) \int_{-\tau_{21}}^{0} \phi^2(\theta)d\theta \\ & + \beta^2 X(X + 1) \int_{-\tau_{12}}^{0} \psi^2(\theta)d\theta + Y(\alpha^2 + \beta^2 Y) \int_{-\tau_{22}}^{0} \psi^2(\theta)d\theta \end{aligned} \tag{7}$$

where X and Y are positive constants determined later. Then, it is clear that V is continuous on G and that for any $(\phi, \psi) \in \partial G$ (the boundary of G), the limit $l(\phi, \psi)$

$$l(\phi, \psi) = \lim_{\substack{(\Phi, \Psi) \to (\phi, \psi) \in \partial G \\ (\Phi, \Psi) \in G}} V(\Phi, \Psi)$$

exists or is $+\infty$. Furthermore,

$$\begin{aligned}
\dot{V}_{(6)}(\phi,\psi) =& 2aX\left[-a\phi(0)+\alpha\phi(-\tau_{11})+\beta\psi(-\tau_{12})\right]\phi(0)\\
&+2aY\left[-a\psi(0)+\beta\phi(-\tau_{21})+\alpha\psi(-\tau_{22})\right]\psi(0)\\
&+\alpha^2(X+1)\left[\phi^2(0)-\phi^2(-\tau_{11})\right]+(\alpha^2+\beta^2Y)\left[\phi^2(0)-\phi^2(-\tau_{21})\right]\\
&+\beta^2X(X+1)\left[\psi^2(0)-\psi^2(-\tau_{12})\right]+Y(\alpha^2+\beta^2Y)\left[\psi^2(0)-\psi^2(-\tau_{22})\right]\\
=&-X\left[-a\phi(0)+\alpha\phi(-\tau_{11})+\beta\psi(-\tau_{12})\right]^2\\
&-Y\left[-a\psi(0)+\beta\phi(-\tau_{21})+\alpha\psi(-\tau_{22})\right]^2\\
&-\left[\alpha\phi(-\tau_{11})-\beta X\psi(-\tau_{12})\right]^2-\left[\alpha\phi(-\tau_{21})-\beta Y\psi(-\tau_{22})\right]^2\\
&-\left[(a^2-\alpha^2)X-\beta^2Y-2\alpha^2\right]\phi^2(0)\\
&-\left[-\beta^2X^2-\beta^2Y^2-\beta^2X+(a^2-\alpha^2)Y\right]\psi^2(0).
\end{aligned} \tag{8}$$

Let

$$\begin{aligned}
f(X,Y) &= (a^2-\alpha^2)X-\beta^2Y-2\alpha^2,\\
g(X,Y) &= -\beta^2X^2-\beta^2Y^2-\beta^2X+(a^2-\alpha^2)Y,
\end{aligned}$$

which are the coefficients of $\phi^2(0)$ and $\psi^2(0)$ in the last two expressions of (8), respectively. Then, the global asymptotic stability of the trivial solution of (6) will be proven for all $\tau_{ij}\geq 0$ $(i,j=1,2)$ if there exist $X>0$ and $Y>0$ such that

$$f(X,Y)\geq 0 \qquad \text{and} \qquad g(X,Y)\geq 0. \tag{9}$$

In fact, if there exist $X>0$ and $Y>0$ such that (9) holds, then we have

$$\dot{V}_{(2.3)}(\phi,\psi)\leq 0 \quad \text{on } G. \tag{10}$$

From (7) and (10), we see that the trivial solution of (6) is stable and that every solution is bounded.

Further, let

$$E=\{(\phi,\psi)\in\bar{G} \mid l(\phi,\psi)<\infty \text{ and } \dot{V}_{(6)}(\phi,\psi)=0\},$$

M : the largest subset in E that is invariant with respect to (6).

Then, for $(\phi,\psi)\in M$, the solution $z_t(\phi,\psi)=(x(t+\theta),y(t+\theta))$ $(-\Delta\leq\theta\leq 0)$ of (6) through $(0,\phi,\psi)$ remains in M for $t\geq 0$ and satisfies for $t\geq 0$,

$$\dot{V}_{(6)}(z_t(\phi,\psi))=0.$$

Hence, for $t\geq 0$,

$$\begin{aligned}
-ax(t)+\alpha x(t-\tau_{11})+\beta y(t-\tau_{12})&=0\\
-ay(t)+\beta x(t-\tau_{21})+\alpha y(t-\tau_{22})&=0,
\end{aligned} \tag{11}$$

which implies that for $t \geq 0$,

$$x'(t) = y'(t) = 0.$$

Thus, for $t \geq 0$,

$$x(t) = c_1, \quad y(t) = c_2 \tag{12}$$

for some constants c_1 and c_2. From (11) and (12), we obtain

$$\begin{bmatrix} -a+\alpha & \beta \\ \beta & -a+\alpha \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This implies that $c_1 = c_2 = 0$ by the assumption (3) and thus we have

$$x(t) = y(t) = 0 \quad \text{for } t \geq 0.$$

Therefore, for any $(\phi, \psi) \in M$, we have

$$(\phi(0), \psi(0)) = (x(0), y(0)) = (0, 0).$$

By [8; Lemma 3.1], any solution $z_t = (x(t+\theta), y(t+\theta))$ tends to M. Thus

$$\lim_{t \to +\infty} x(t) = \lim_{t \to +\infty} y(t) = 0.$$

Hence, the trivial solution of (6) is globally asymptotically stable for all $\tau_{ij} \geq 0$ $(i, j = 1, 2)$.

That is why we have only to show that there exist $X > 0$ and $Y > 0$ such that (9) holds. (9) can be equivalently written as

$$\begin{aligned} &Y \leq \frac{a^2 - \alpha^2}{\beta^2} X - \frac{2\alpha^2}{\beta^2} \\ &\left(X + \frac{1}{2}\right)^2 + \left(Y - \frac{a^2 - \alpha^2}{2\beta^2}\right)^2 \leq \frac{(a^2 - \alpha^2)^2 + \beta^4}{4\beta^4}. \end{aligned} \tag{13}$$

Now let us define

$$L: \ Y = \frac{a^2 - \alpha^2}{\beta^2} X - \frac{2\alpha^2}{\beta^2},$$

$$\Gamma: \ \left(X + \frac{1}{2}\right)^2 + \left(Y - \frac{a^2 - \alpha^2}{2\beta^2}\right)^2 = \frac{(a^2 - \alpha^2)^2 + \beta^4}{4\beta^4}.$$

Then we see that there exist $X > 0$ and $Y > 0$ such that (9) holds if and only if the line L intersects with the circle Γ in the first quadrant, except X and Y axes, of XY-plane (Fig. 1). Investigating the radius of Γ and the distance between the center of Γ and the line L, we have that there exists a pair of the real roots (X, Y) of (13) if and only if

$$|a^2 - \alpha^2 - \beta^2| \geq 2|\alpha\beta| \tag{14}$$

holds. (14) means either $a^2 - \alpha^2 - \beta^2 \geq 2|\alpha\beta|$ or $a^2 - \alpha^2 - \beta^2 \leq -2|\alpha\beta|$. We will now prove that the former just shows the line L intersects with the circle Γ in the first quadrant exept X and Y axes. In fact, the former implies $a^2 - \alpha^2 > \beta^2$ by $|\alpha| > 0$ and $|\beta| > 0$, and the gradient of the line L is greater than 1. On the other hand, the gradient of the tangent line of the circle Γ at the origin is $\frac{\beta^2}{a^2-\alpha^2}$ which, in this case, becomes less than 1. Thus, $a^2 - \alpha^2 - \beta^2 \geq 2|\alpha\beta|$ shows that the line L intersects with the circle Γ in the first open quadrant.

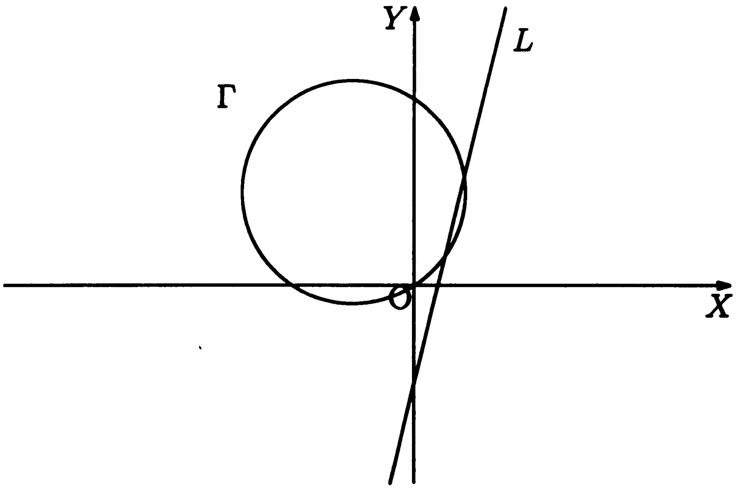

Fig. 1

It is easy to see that $|\beta| < a-\alpha$ and $|\beta| \leq a+\alpha$ imply $a^2-\alpha^2-\beta^2 \geq 2|\alpha\beta|$. Therefore, it is proved that there exist $X > 0$ and $Y > 0$ such that (9) holds. Hence, (x^*, y^*) is globally asymptotically stable for all $\tau_{ij} \geq 0$ $(i, j = 1, 2)$.

Necessity. Assume the assertion is false, that is, let (x^*, y^*) be globally asymptotically stable for all $\tau_{ij} \geq 0$ $(i, j = 1, 2)$ but $|\beta| \geq a - \alpha$ or $|\beta| > a + \alpha$.

Linearizing (6), we have

$$\begin{aligned} x'(t) &= x^*[-ax(t) + \alpha x(t - \tau_{11}) + \beta y(t - \tau_{12})] \\ y'(t) &= y^*[-ay(t) + \beta x(t - \tau_{21}) + \alpha y(t - \tau_{22})]. \end{aligned} \tag{15}$$

Now, we will show that there exists a characteristic root λ_0 of (15) such that

$$Re(\lambda_0) > 0 \tag{16}$$

for some τ_{ij} $(i, j = 1, 2)$, which implies that the trivial solution of (6) is not stable (see [1, pp.160, 161]).

We note that the case $|\beta| = a - \alpha$ is excluded from consideration because of the assumption (3). In the case $|\beta| > a-\alpha$, we see that (x^*, y^*) is not globally asymptotically stable when $\tau_{ij} = 0$ $(i, j = 1, 2)$ (the proof is omitted for the sake of page restrictions). Therefore, we have only to consider the case $|\beta| < a - \alpha$ and $|\beta| > a + \alpha$. The proof is divided by three cases

(I) The case $-a \le \alpha < 0$. Let $\tau_{11} = \tau_{12} = \tau_{21} = \tau_{22} = \tau$; then the characteristic equation of (15) takes the form

$$\lambda^2 + p\lambda + q + (r + s\lambda)e^{-\lambda\tau} + ve^{-2\lambda\tau} = 0 \tag{17}$$

where $p = a(x^* + y^*)$, $q = a^2x^*y^*$, $r = -2a\alpha x^*y^*$, $s = -\alpha(x^* + y^*)$ and $v = (\alpha^2 - \beta^2)x^*y^*$.

Substituting $\lambda = iy$ into (17), we have

$$(-y^2 + piy + q)e^{iy\tau} + r + siy + ve^{-iy\tau} = 0. \tag{18}$$

By separating the real and imaginary parts of (18), we obtain

$$\begin{aligned} &[(-y^2+q)^2 - v^2 + p^2y^2]\cos(y\tau) = (r - sp)y^2 - r(q - v) \\ &[(-y^2+q)^2 - v^2 + p^2y^2]\sin(y\tau) = sy^3 + [rp - s(q+v)]y. \end{aligned} \tag{19}$$

From (19) we have

$$\left[(-y^2+q)^2 - v^2 + p^2y^2\right]^2 = \left[(r - sp)y^2 - r(q-v)\right]^2 + \left[sy^3 + [rp - s(q+v)]y\right]^2.$$

To solve y in (19), define the following function

$$\begin{aligned} f_1(Y) = {}&[(-Y+q)^2 - v^2 + p^2Y]^2 - [(r-sp)Y - r(q-v)]^2 \\ &- Y[sY + rp - s(q+v)]^2 \end{aligned} \tag{20}$$

where $Y = y^2$. Then f_1 is a quartic fuction such that $f_1 \to +\infty$ as $|Y| \to +\infty$ and

$$f_1(0) = [a^2 - \alpha^2 + \beta^2]^2[(a+\alpha)^2 - \beta^2][(a-\alpha)^2 - \beta^2](x^*y^*)^4 < 0.$$

Thus, there can exist some positive zeros of (20).

Let Y_0 be such a positive zero. Substituting y_0, which satisfies $Y_0 = y_0^2$, into (19), we can get some τ_0 such that (17) has a characteristic root iy_0 at τ_0.

Let

$$P_1(\lambda, \tau) = \lambda^2 + p\lambda + q + (r + s\lambda)e^{-\lambda\tau} + ve^{-2\lambda\tau}.$$

Clearly, $P_1(iy_0, \tau_0) = 0$. From (17), we have

$$\begin{aligned} \frac{\partial P_1(iy_0,\tau_0)}{\partial \tau} &= 2iy_0(-y_0^2 + piy_0 + q) + iy_0(r + siy_0)e^{-iy_0\tau_0}, \\ \frac{\partial P_1(iy_0,\tau_0)}{\partial \lambda} &= 2iy_0 + p + 2\tau_0(-y_0^2 + piy_0 + q) + [s + \tau_0(r + siy_0)]e^{-iy_0\tau_0}. \end{aligned}$$

Now, we will consider the following value:

$$K_1 = 1 + \frac{(a^2 - a\alpha\cos(y_0\tau_0))(x^* - y^*)^2}{(p + s\cos(y_0\tau_0))^2 + (2y_0 - s\sin(y_0\tau_0))^2}.$$

We obtain

$$\begin{aligned} &(a^2 - a\alpha\cos(y_0\tau_0))(x^* - y^*)^2 \ge 0, \\ &(p + s\cos(y_0\tau_0))^2 + (2y_0 - s\sin(y_0\tau_0))^2 \ne 0 \end{aligned}$$

since $|\alpha| \le a$, and we get $K_1 > 0$. Then, $\frac{\partial P_1(iy_0,\tau_0)}{\partial \tau} \neq 0$ holds because

$$0 < K_1 = Re\left[\frac{-2iy_0(-y_0^2 + piy_0 + q) - iy_0(r + siy_0)e^{-iy_0\tau_0}}{p + s\cos(y_0\tau_0) + i(2y_0 - s\sin(y_0\tau_0))}\right]$$
$$= Re\left[\frac{-\frac{\partial P_1(iy_0,\tau_0)}{\partial \tau}}{p + s\cos(y_0\tau_0) + i(2y_0 - s\sin(y_0\tau_0))}\right].$$

Furthermore, $Re\left(-\frac{\frac{\partial P_1(iy_0,\tau_0)}{\partial \lambda}}{\frac{\partial P_1(iy_0,\tau_0)}{\partial \tau}}\right) > 0$ holds because

$$\text{sign}K_1 = \text{sign}\left[Re\left\{\left(\frac{-\frac{\partial P_1(iy_0,\tau_0)}{\partial \tau}}{p + s\cos(y_0\tau_0) + i(2y_0 - s\sin(y_0\tau_0))}\right)^{-1}\right\}\right]$$
$$= \text{sign}\left[Re\left(\frac{p + s\cos(y_0\tau_0) + i(2y_0 - s\sin(y_0\tau_0))}{-\frac{\partial P_1(iy_0,\tau_0)}{\partial \tau}} - \frac{\tau_0}{iy_0}\right)\right]$$
$$= \text{sign}\left[Re\left(-\frac{\frac{\partial P_1(iy_0,\tau_0)}{\partial \lambda}}{\frac{\partial P_1(iy_0,\tau_0)}{\partial \tau}}\right)\right].$$

Hence, we have $\frac{\partial P_1(iy_0,\tau_0)}{\partial \lambda} \neq 0$. Thus, by the well-known implicit function theorem, we have

$$\text{sign}\left[Re\left(\left.\frac{d\lambda}{d\tau}\right|_{\lambda=iy_0,\tau=\tau_0}\right)\right] = \text{sign}\left[Re\left(-\frac{\frac{\partial P_1(iy_0,\tau_0)}{\partial \tau}}{\frac{\partial P_1(iy_0,\tau_0)}{\partial \lambda}}\right)\right]$$
$$= \text{sign}\left[Re\left\{\left(-\frac{\frac{\partial P_1(iy_0,\tau_0)}{\partial \tau}}{\frac{\partial P_1(iy_0,\tau_0)}{\partial \lambda}}\right)^{-1}\right\}\right] = \text{sign}K_1 > 0.$$

This implies that (16) holds. Therefore, the trivial solution of (6) is not stable, that is, (x^*, y^*) is not stable near τ_0, which is a contradiction.

(II) The case $\alpha < -a$. Here, we can take $r_1 \le r_2$ without loss of generality. From (3), it is easy to see that $r_1 \le r_2$ if and only if $x^* \le y^*$. Let $\tau_{11} = \tau$ and $\tau_{12} = \tau_{21} = \tau_{22} = 0$; then the characteristic equation of (15) takes the form

$$\lambda^2 + \tilde{p}\lambda + \tilde{q} + (\tilde{r} + \tilde{s}\lambda)e^{-\lambda\tau} = 0 \tag{21}$$

where $\tilde{p} = ax^* + (a-\alpha)y^*$, $\tilde{q} = [a(a-\alpha) - \beta^2]x^*y^*$, $\tilde{r} = -\alpha(a-\alpha)x^*y^*$, and $\tilde{s} = -\alpha x^*$. Let us use p, q, r and s again instead of $\tilde{p}$, $\tilde{q}$, $\tilde{r}$ and $\tilde{s}$, respectively. Substituting $\lambda = iy$ into (21), we obtain

$$-y^2 + piy + q + (r + siy)e^{-iy\tau} = 0. \tag{22}$$

By separating the real and imaginary parts of (22), we have

$$\begin{aligned}(r^2 + s^2y^2)\cos(y\tau) &= r(y^2 - q) - spy^2\\ (r^2 + s^2y^2)\sin(y\tau) &= sy(y^2 - q) + pry\end{aligned} \tag{23}$$

$$\left[r^2+s^2y^2\right]^2=\left[r(y^2-q)-spy^2\right]^2+\left[sy(y^2-q)+pry\right]^2.$$

Define the following function

$$f_2(Y)=Y\left[s(Y-q)+pr\right]^2+\left[r(Y-q)-spY\right]^2-\left[r^2+s^2Y\right]^2$$

where $Y=y^2$, then f_2 is an upwards cubic function to the right and

$$f_2(0)=[\alpha(a-\alpha)]^2[(a-\alpha)^2-\beta^2][a^2-\alpha^2-\beta^2](x^*y^*)^4<0.$$

Thus, there can exist some positive roots of $f_2(Y)=0$.

Let Y_0 be such a positive root. Substituting y_0, which satisfies $Y_0=y_0^2$, into (23), we can get some τ_0 such that (21) has a characteristic root iy_0 at τ_0.

Let

$$P_2(\lambda,\tau)=\lambda^2+p\lambda+q+(r+s\lambda)e^{-\lambda\tau}.$$

Then, $P_2(iy_0,\tau_0)=0$ and we obtain from (21) that

$$\begin{aligned}\frac{\partial P_2(iy_0,\tau_0)}{\partial\tau}&=-iy_0(-y_0^2+piy_0+q),\\ \frac{\partial P_2(iy_0,\tau_0)}{\partial\lambda}&=2iy_0+p+[s-\tau_0(r+siy_0)]e^{-iy_0\tau_0}.\end{aligned}$$

Clearly, $\frac{\partial P_2(iy_0,\tau_0)}{\partial\tau}\neq 0$. We will now consider the following value:

$$K_2=\frac{s^2y_0^4+2r^2y_0^2-s^2q^2-2r^2q+p^2r^2}{[(py_0)^2+(y_0^2-q)^2][r^2+(sy_0)^2]}.$$

We get $K_2>0$ since we have

$$\begin{aligned}&-s^2q^2-2r^2q+p^2r^2\\ &\quad=[a^2x^{*2}+(a-\alpha)^2y^{*2}+2\beta^2x^*y^*][\alpha(a-\alpha)]^2x^{*2}y^{*2}-\alpha^2[a(a-\alpha)-\beta^2]^2x^{*4}y^{*2}\\ &\quad=\alpha^2\beta^2[2a(a-\alpha)-\beta^2]x^{*4}y^{*2}+[(a-\alpha)^2y^{*2}+2\beta^2x^*y^*]\alpha^2(a-\alpha)^2x^{*2}y^{*2}\\ &\quad\geq\alpha^2\beta^2[2a(a-\alpha)-\beta^2]x^{*4}y^{*2}+[(a-\alpha)^2x^{*2}+2\beta^2x^{*2}]\alpha^2(a-\alpha)^2x^{*2}y^{*2}\\ &\quad=\alpha^2\left[(a-\alpha)^4-\beta^4+2(a-\alpha)(2a-\alpha)\beta^2\right]x^{*4}y^{*2}>0.\end{aligned}$$

Furthermore,

$$\begin{aligned}\text{sign}K_2&=\text{sign}\left[Re\left(\frac{2iy_0+p}{-iy_0(-y_0^2+piy_0+q)}+\frac{s}{iy_0(r+siy_0)}-\frac{\tau_0}{iy_0}\right)\right]\\ &=\text{sign}\left[Re\left(-\frac{\frac{\partial P_2(iy_0,\tau_0)}{\partial\lambda}}{\frac{\partial P_2(iy_0,\tau_0)}{\partial\tau}}\right)\right].\end{aligned}$$

Hence, we can obtain $\frac{\partial P_2(iy_0,\tau_0)}{\partial \lambda} \neq 0$ and, by the same reason as above,

$$\text{sign}\left[Re\left(\left.\frac{d\lambda}{d\tau}\right|_{\lambda=iy_0,\tau=\tau_0}\right)\right] = \text{sign}\left[Re\left(-\frac{\frac{\partial P_2(iy_0,\tau_0)}{\partial\tau}}{\frac{\partial P_2(iy_0,\tau_0)}{\partial\lambda}}\right)\right]$$
$$= \text{sign}\left[Re\left\{\left(-\frac{\frac{\partial P_2(iy_0,\tau_0)}{\partial\tau}}{\frac{\partial P_2(iy_0,\tau_0)}{\partial\lambda}}\right)^{-1}\right\}\right] = \text{sign}K_2 > 0.$$

This implies that (16) holds, which is a contradiction. The proof of Theorem 1 is thus completed.

Remark 1. We are interested in giving necessary and sufficient conditions for the global stability of several systems which have more generality than the system (1). However, it becomes much more complicated and has not been solved yet. This problem is left for a future work.

Here, we give the following three portraits of the trajectory of (1) with (2), drawn by a computer using the Runge-Kutta method, to illustrate Theorem 1 ($r_1 = 10$, $r_2 = 10$, $\tau_{11} = 45$, $\tau_{12} = 46$, $\tau_{21} = 47$, $\tau_{22} = 48$, and $(\phi, \psi) = (3.7 + 0.05t,\ 2.9 + 0.8\sin(0.7t))$).

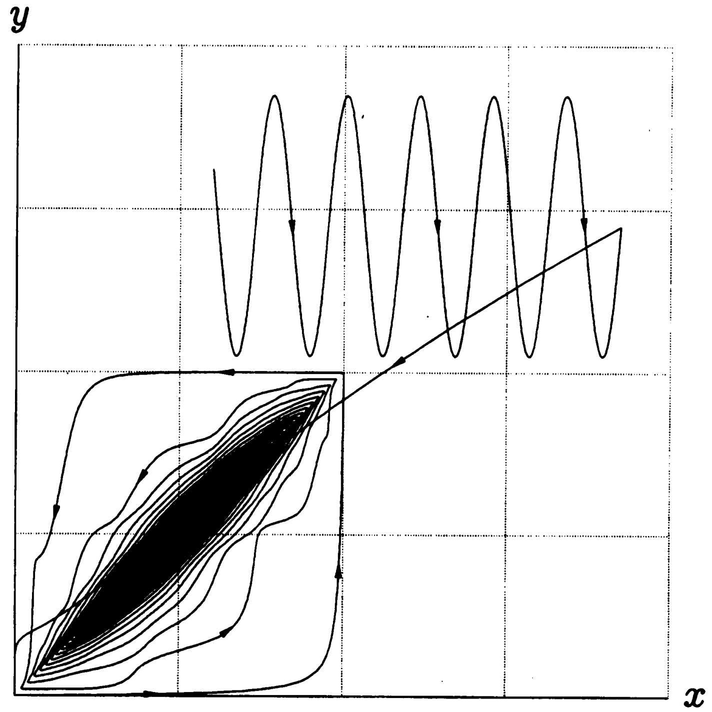

Fig.2 $a = 5$, $\alpha = -2$, $\beta = -2.9$ $(|\beta| < a + \alpha)$
(x^*, y^*) nearly equals $(1.01, 1.01)$.

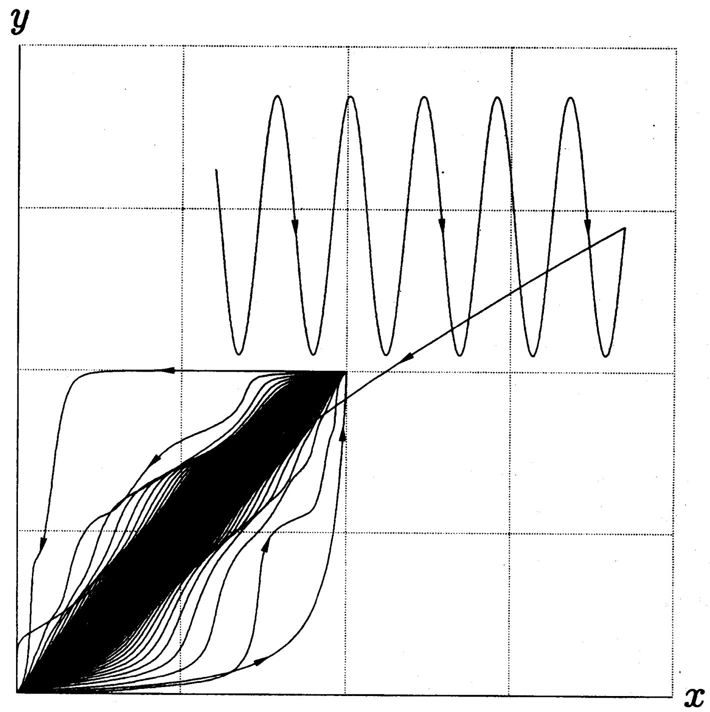

Fig.3 $a = 5,\ \alpha = -2,\ \beta = -3\quad (|\beta| = a + \alpha)$
$(x^*, y^*) = (1, 1).$

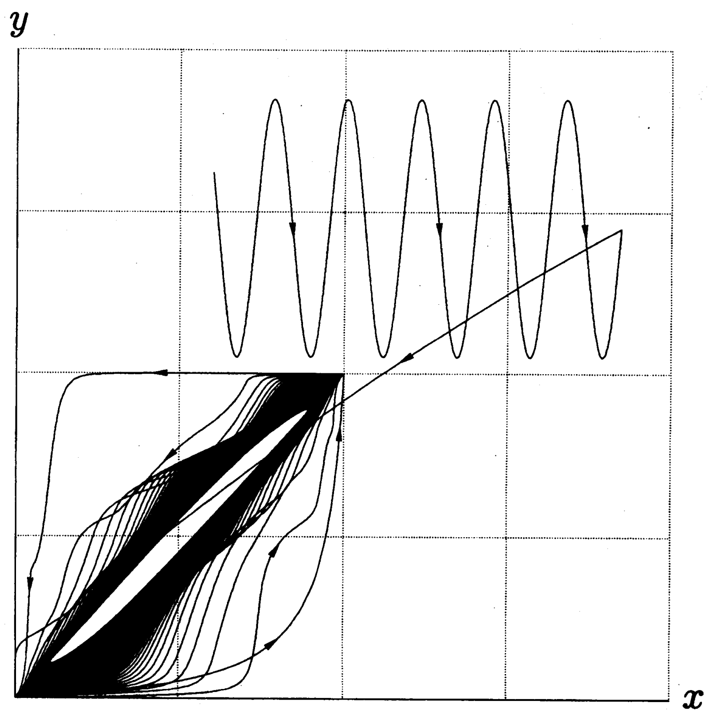

Fig.4 $a = 5,\ \alpha = -2,\ \beta = -3.02\quad (|\beta| > a + \alpha)$
(x^*, y^*) nearly equals $(0.99, 0.99)$.

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