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Multiple interior layers of solutions to elliptic Sine-Gordon type ODE

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1 Introduction

We consider the perturbed elliptic Sine-Gordon equation on an interval

$$\begin{aligned} -u''(t)+\lambda\sin u(t) &= \mu f(u(t)),\ u(t)>0 \quad t\in I:=(-T,T), \\ u(\pm T) &= 0, \end{aligned} \tag{1.1}$$

where $\lambda, \mu > 0$ are parameters and $T > 0$ is a constant. Throughout this paper, we assume:
(A.1) f is locally Lipschitz continuous, odd in u. Furthermore, $f(u) > 0$ for $u > 0$.
(A.2) There exist constants $C > 0$ and $p > 1$ such that $|f(u)| \leq C(1 + |u|^p)$ for $u \in \mathbf{R}$.
(A.3) $f(u) \leq Cu$ for $0 < u \ll 1$, where $C > 0$ is a constant.
(A.4) There exists a constant $m > 1$ such that for $u \in \mathbf{R}$

$$f(u)u \geq mF(u) := m\int_0^u f(s)ds.$$

The typical examples of $f(u)$ are:

$$f(u) = |u|^{p-1}u, \quad (p>1), \quad f(u) = |u|^{p-1}u + |u|^{q-1}u, \quad (p,q>1).$$

The aim here is to investigate the layer structure of the solutions to (1.1) for $\lambda \gg 1$ by using variational approach. To be more precise, we show the existence of the solutions u_λ which have $2n$ multiple interior layers in I for $\lambda \gg 1$. The location of multiple interior layers of u_λ as $\lambda \to \infty$ are also determined. Further, we show the existence of solutions u_λ with boundary layers.

We explain the variational framework. We consider the variational problem (M) subject to the constraint depending on λ:
(M) Minimize

$$L_\lambda(u) := \frac{1}{2}\int_I |u'(t)|^2 dt + \lambda\int_I (1-\cos u(t))dt \tag{1.2}$$

under the constraint

$$u \in M_\alpha := \left\{ u \in H_0^1(I) : K(u) := \int_I F(u(t))dt = 2TF(\alpha) \right\}, \tag{1.3}$$

where $\alpha > 0$ is a *fixed constant*, $H_0^1(I)$ is the usual real Sobolev space. Then by the Lagrange multiplier theorem, we obtain solution triple $(\lambda, \mu(\lambda), u_\lambda) \in \mathbf{R}_+^2 \times M_\alpha$ of (1.1) (and consequently $u_\lambda \in C^2(\bar{I})$ by a standard regularity theorem) corresponding to the problem (**M**).

Theorem 0 [5]. *Assume (A.1)-(A.4). Let $0 < \alpha < 2\pi$ satisfy $F(\alpha) < F(2\pi)/2$. Then:*
(i) $u_\lambda \to 2\pi$ locally uniformly on $(-T_{\alpha,0}, T_{\alpha,0})$ as $\lambda \to \infty$, where $T_{\alpha,0} := F(\alpha)T/F(2\pi)$.
(ii) $u_\lambda \to 0$ locally uniformly on $I \setminus [-T_{\alpha,0}, T_{\alpha,0}]$ as $\lambda \to \infty$.
(iii) $\mu(\lambda) \to 0$ as $\lambda \to \infty$.

We next remove the restriction $F(\alpha) < F(2\pi)/2$ in Theorem 0. To do this, we introduce the condition (A.5.n) for *a given* $n \in \mathbf{N}$:

(A.5.n) $H(n) := F(2(n+1)\pi) - 2nF(2n\pi) + 2\sum_{k=0}^{n-1} F(2k\pi) > 0.$

Note that *"Assume (A.5.n)"* implies that the assumption (A.5.n) holds only for *a given* n. The example of f which satisfies (A.1)–(A.5.n) for a fixed $n \in \mathbf{N}$ is $f(u) = |u|^{p-1}u$ for $p > p_n$, where $p_n > 1$ is a constant depending on a given n.

Theorem 1 [6]. *Assume (A.1)-(A.4) and (A.5.1). Let $0 < \alpha < 2\pi$ satisfy $F(\alpha) \geq F(2\pi)/2$. Then the assertions (i)-(iii) in Theorem 0 hold.*

We next show the existence of the solutions u_λ which have $2(n+1)$ multiple interior transition layers at $t = \pm T_{\alpha,n}, \pm(T - T_{\alpha,n}), \pm(T - 3T_{\alpha,n}), \cdots, \pm(T - (2n-1)T_{\alpha,n})$ as $\lambda \to \infty$, where

$$T_{\alpha,n} := (F(\alpha) - F(2n\pi))T/H(n).$$

For $D \subset \mathbf{R}$, let $-D := \{-t : t \in D\} \subset \mathbf{R}$ and $|D|$ be the Lebesgue measure of D.

Theorem 2 [6]. *Let $n \in \mathbf{N}$ be given. Assume (A.1)-(A.4) and (A.5.n). If α satisfies $2n\pi < \alpha < 2(n+1)\pi$ and*

$$F(2n\pi) < F(\alpha) < \frac{1}{2(n+1)}F(2(n+1)\pi) + \frac{1}{(n+1)}\sum_{k=0}^{n} F(2k\pi), \tag{1.4}$$

then as $\lambda \to \infty$:
(i) $\|u_\lambda\|_\infty < 2(n+1)\pi$.
(ii) $u_\lambda \to 2(n+1)\pi$ locally uniformly on $(-T_{\alpha,n}, T_{\alpha,n})$.
(iii) $u_\lambda \to 2n\pi$ locally uniformly on $\pm(T_{\alpha,n}, T - (2n-1)T_{\alpha,n})$.
(iv) $u_\lambda \to 2k\pi$ locally uniformly on $\pm(T-(2k+1)T_{\alpha,n}, T-(2k-1)T_{\alpha,n})$ for $k = 1, \cdots, n-1$.
(v) $u_\lambda \to 0$ locally uniformly on $\pm(T - T_{\alpha,n}, T]$.
(vi) There exist constants $C_1, C_2 > 0$ such that

$$\mu(\lambda) \leq C_1 \lambda e^{-C_2\sqrt{\lambda}}. \tag{1.5}$$

Note that if (A.5.n) is satisfied, then there exists $\alpha > 0$ which satisfies $2n\pi < \alpha < 2(n+1)\pi$ and (1.4) for n.

We now consider the case where the condition (1.4) does not hold. Namely, we consider $\alpha > 0$ which satisfies $2n\pi < \alpha < 2(n+1)\pi$ and

$$\frac{1}{2(n+1)}F(2(n+1)\pi) + \frac{1}{(n+1)}\sum_{k=0}^{n} F(2k\pi) \leq F(\alpha). \tag{1.6}$$

In this case, u_λ has multiple interior layers at $t = \pm(T - (2k-1)S_{\alpha,n})(k = 1, \cdots, n+1)$ as $\lambda \to \infty$, where

$$S_{\alpha,n} := \frac{(F(2(n+1)\pi) - F(\alpha))T}{(2n+1)F(2(n+1)\pi) - 2\sum_{k=0}^{n} F(2k\pi)}.$$

Theorem 3 [6]. *Let $n \in \mathbf{N}$ be given. Assume (A.1)-(A.4), (A.5.n) and (A.5.n+1). Let $2n\pi < \alpha < 2(n+1)\pi$ satisfy (1.6). Then as $\lambda \to \infty$:*
(i) $\|u_\lambda\|_\infty \to 2(n+1)\pi$.
(ii) $u_\lambda \to 2(n+1)\pi$ locally uniformly on $(-(T-(2n+1)S_{\alpha,n}), T-(2n+1)S_{\alpha,n})$.
(iii) $u_\lambda \to 2k\pi$ locally uniformly on $\pm(T-(2k+1)S_{\alpha,n}, T-(2k-1)S_{\alpha,n})$ for $k = 1, \cdots, n$.
(iv) $u_\lambda \to 0$ locally uniformly on $\pm(T - S_{\alpha,n}, T]$.
(v) The formula (1.5) holds.

Finally, we show the existence of solutions which have boundary layers.

Theorem 4 [6]. *Let $n \in \mathbf{N}$ be given. Assume (A.1)-(A.4) and (A.5.n). If $\alpha = 2n\pi$, then $\|u_\lambda\|_\infty < 2(n+1)\pi$ for $\lambda \gg 1$ and $u_\lambda \to 2n\pi$ locally uniformly on $(-T, 0) \bigcup (0, T)$ as $\lambda \to \infty$.*

The idea of the proof of Theorems 2 is as follows. By using the variational characterization of u_λ, we find that the shape of u_λ for $\lambda \gg 1$ is like step function, each height of the steps are 2π. We first establish an estimate $\|u_\lambda\|_\infty < 2(n+1)\pi$ for $\lambda \gg 1$ by using (A.5.n). Then u_λ must cross the line $u = 2\pi, 4\pi, \ldots, 2n\pi$. By using this fact, we secondly establish that $|I_{\lambda,k}| \sim 2|I_{\lambda,0}|$ for $\lambda \gg 1$, where $I_{\lambda,k} \subset (0,T)$ $(k = 1, \cdots n-1)$ are the intervals on which $u_\lambda \to 2k\pi$ locally uniformly as $\lambda \to \infty$. Finally, by using an estimate $\|u_\lambda\|_\infty < 2(n+1)\pi$, we prove that $|I_{\lambda,2(n+1)}| \sim |I_{\lambda,0}|$ for $\lambda \gg 1$. To prove Theorem 3, we show that $|I_{\lambda,k}| \sim 2|I_{\lambda,0}|$ for $k = 1, 2, \cdots, n$ and $\lambda \gg 1$.

The rest of this paper is organized as follows. We introduce some fundamental lemmas in Section 2. Based on these lemmas, we prove Theorem 2 (i) for $n = 1$ in Section 3.

2 Preliminaries

In this section, we introduce some fundamental lemmas. For the full proofs, we refer to [5]. We know by [2] that a solution u of (1.1) satisfies

$$u(t) = u(-t) \quad \text{for } t \in [0, T]. \tag{2.1}$$

$$u'(t) < 0 \quad \text{for } t \in (0, T], \tag{2.2}$$

$$u'(0) = 0, u(0) = \|u\|_\infty, \tag{2.3}$$

For $0 \le r \le \|u_\lambda\|_\infty$, let $t_{r,\lambda} \in [0, T]$ satisfy $u_\lambda(t_{r,\lambda}) = r$, which exists uniquely by (2.2). The following notation will be used repeatedly. For a fixed $0 < \epsilon \ll 1$, let

$$l_{\lambda,\epsilon} := t_{2\pi,\lambda} - t_{2\pi+\epsilon,\lambda}, \quad m_{\lambda,\epsilon} := t_{2\pi-\epsilon,\lambda} - t_{2\pi,\lambda}, \quad \delta_{\lambda,\epsilon} := T - t_{\epsilon,\lambda}.$$

In what follows, we always fix $0 < \epsilon \ll 1$ first. Then let $\lambda \to \infty$. Therefore, the standard notation $o(1)$ will be used for $\lambda \gg 1$. Furthermore, the notation $l_{\lambda,\epsilon} = \delta_{\lambda,\epsilon} + O(\epsilon) + o(1)$ (for instance) means that $|l_{\lambda,\epsilon} - \delta_{\lambda,\epsilon}| \le C\epsilon + o(1)$ for $0 < \epsilon \ll 1$ fixed and $\lambda \gg 1$.

Lemma 2.1 *Assume that $(\lambda, \mu, u) \in \mathbf{R}_+ \times \mathbf{R} \times C^2(\bar{I})$ satisfies (1.1). Then $\mu > 0$. Further, for $t \in \bar{I}$,*

$$\frac{1}{2}u'(t)^2 + \mu F(u(t)) + \lambda \cos u(t) = \frac{1}{2}u'(T)^2 + \lambda = \mu F(\|u\|_\infty) + \lambda \cos \|u\|_\infty. \tag{2.4}$$

Proof. Multiply the equation in (1.1) by $u'(t)$. Then we have

$$\frac{d}{dt}\left\{\frac{1}{2}u'(t)^2 + \mu F(u(t)) + \lambda \cos u(t)\right\} = 0, \qquad t \in \bar{I}.$$

Hence, for $t \in \bar{I}$,

$$\frac{1}{2}u'(t)^2 + \mu F(u(t)) + \lambda \cos u(t) \equiv \text{constant}. \tag{2.5}$$

By putting $t = 0, T$ in (2.5), we obtain (2.4) by (2.3). Then by (2.4), we obtain

$$\mu F(\|u\|_\infty) = \frac{1}{2}u'(T)^2 + \lambda(1 - \cos \|u\|_\infty) > 0. \tag{2.6}$$

Since $F(\|u\|_\infty) > 0$ by (A.1), $\mu > 0$ follows from (2.6). ∎

Lemma 2.2 *Let $\alpha > 0$ and $\lambda > 0$ be fixed. Then there exists $(\mu(\lambda), u_\lambda) \in \mathbf{R}_+ \times (M_\alpha \cap C^2(\bar{I}))$ which satisfies (1.1) and $L_\lambda(u_\lambda) = \beta(\lambda) := \inf_{u \in M_\alpha} L(u)$.*

Lemma 2.2 can be proved easily by choosing a minimizing sequence.

Lemma 2.3 *Let $\alpha > 0$ be fixed. Then $L_\lambda(u_\lambda) \le C\lambda^{\frac{m+2}{2(m+1)}}$ for $\lambda \gg 1$.*

Lemma 2.3 can be proved by finding an appropriate test function $\phi \in M_\alpha$

Lemma 2.4 *Let $\alpha > 0$ be fixed. Then $\mu(\lambda) = o(\lambda)$ for $\lambda \gg 1$.*

Lemma 2.4 is a consequence of Lemma 2.3. By Lemma 2.3, we obtain the following (2.7). Put $J_{\lambda,k,\delta} := \{t \in I : 2(k-1)\pi + \delta < u_\lambda(t) < 2k\pi - \delta\}$ for $0 < \delta \ll 1$ and $k \in \mathbf{N}$. By Lemma 2.3, as $\lambda \to \infty$,

$$\begin{aligned} |J_{\lambda,k,\delta}| &\le \frac{1}{1 - \cos\delta} \int_{J_{\lambda,k,\delta}} (1 - \cos u_\lambda(t))dt \\ &\le \frac{\lambda^{-1}}{1 - \cos\delta} L_\lambda(u_\lambda) \le C\lambda^{-m/(2(m+1))} \to 0. \end{aligned} \tag{2.7}$$

Lemma 2.5 *Let $\alpha > 0$ be fixed. Then $|u'_\lambda(T)|^2/\lambda \to 0$ as $\lambda \to \infty$.*

Lemma 2.5 follows from (2.4) and Lemma 2.4.

Lemma 2.6 *Let $\alpha > 0$ and $0 < \epsilon \ll 1$ be fixed. Then for $\lambda \gg 1$*

$$u'_\lambda(T)^2 \le C\lambda e^{-2\delta_{\lambda,\epsilon}\sqrt{(1-2\epsilon)\lambda}}. \tag{2.8}$$

Lemma 2.6 can be proved by (2.4) and Lemma 2.5 and the following Lemma 2.7 follows from Lemma 2.6.

Lemma 2.7 *Let $\alpha > 0$ and $0 < \epsilon \ll 1$ be fixed. Assume that there exists a subsequence $\{\lambda_j\}$ of $\{\lambda\}$ ($\lambda_j \to \infty$ as $j \to \infty$) such that $\|u_{\lambda_j}\|_\infty \ge 2\pi$. Then*

$$m_{\lambda_j,\epsilon} \ge \sqrt{1-2\epsilon}\,\delta_{\lambda_j,\epsilon} - o(1). \tag{2.9}$$

3 Proof of Theorem 2 (i) for $n = 1$

Lemma 3.1 *Assume (A.1)-(A.4). Let $\alpha > 0$ and $0 < \epsilon \ll 1$ be fixed. Then for $\lambda \gg 1$*

$$u'_\lambda(T)^2 \ge C_\epsilon \lambda e^{-2\delta_{\lambda,\epsilon}\sqrt{\lambda}}. \tag{3.1}$$

Proof. By (1.1),

$$u''_\lambda(t) + \mu(\lambda)f(u_\lambda(t)) = \lambda \sin u_\lambda(t) \le \lambda u_\lambda(t) \qquad \text{for } t \in [t_{\lambda,\epsilon}, T].$$

By this and (2.2), we obtain

$$\frac{dS_{\lambda,2}(t)}{dt} := \frac{d}{dt}\left\{\frac{1}{2}u'_\lambda(t)^2 + \mu(\lambda)F(u_\lambda(t)) - \frac{\lambda u_\lambda(t)^2}{2}\right\} \ge 0 \qquad \text{for } t \in [t_{\lambda,\epsilon}, T].$$

This implies that $S_{\lambda,2}(t)$ is increasing on $[t_{\lambda,\epsilon}, T]$. Then

$$\frac{1}{2}u'_\lambda(t)^2 + \mu(\lambda)F(u_\lambda(t)) - \frac{\lambda u_\lambda(t)^2}{2} \le \frac{1}{2}u'_\lambda(T)^2 \qquad \text{for } t \in [t_{\lambda,\epsilon}, T].$$

Then for $t \in [t_{\lambda,\epsilon}, T]$,

$$-u'_\lambda(t) \le \sqrt{u'_\lambda(T)^2 + \lambda u_\lambda(t)^2 - 2\mu(\lambda)F(u_\lambda(t))} \le \sqrt{u'_\lambda(T)^2 + \lambda u_\lambda(t)^2}. \tag{3.2}$$

Therefore, by (3.2), we obtain

$$\begin{aligned}
\delta_{\lambda,\epsilon} &= T - t_{\epsilon,\lambda} = \int_{t_{\epsilon,\lambda}}^T 1dt \ge \int_{t_{\epsilon,\lambda}}^T \frac{-u'_\lambda(t)}{\sqrt{u'_\lambda(T)^2 + \lambda u_\lambda(t)^2}}dt \\
&= \int_0^\epsilon \frac{ds}{\sqrt{u'_\lambda(T)^2 + \lambda s^2}} = \frac{1}{\sqrt{\lambda}}\log\left(\frac{\left|\epsilon + \sqrt{\epsilon^2 + X_{\lambda,2}^2}\right|}{X_{\lambda,2}}\right) \\
&\ge \frac{1}{\sqrt{\lambda}}\log\left(\frac{2\epsilon}{X_{\lambda,2}}\right),
\end{aligned}$$

where $X_{\lambda,2} := |u'_\lambda(T)|/\sqrt{\lambda}$. This yields (3.1). ∎

Lemma 3.2 *Assume (A.1)–(A.4). Let $\alpha > 0$ and $0 < \epsilon \ll$ be fixed. Suppose that there exists a subsequence $\{\lambda_j\}_{j=1}^{\infty}$ such that $\lambda_j \to \infty$ as $j \to \infty$ and $\|u_{\lambda_j}\|_\infty \ge 4\pi$. Then*

$$u'_{\lambda_j}(t_{2\pi,\lambda_j})^2 \le C\lambda_j e^{-2l_{\lambda_j,\epsilon}\sqrt{(1-\epsilon)\lambda_j}}, \tag{3.3}$$

$$t_{4\pi-\epsilon,\lambda_j} - t_{4\pi,\lambda_j} \ge \sqrt{(1-\epsilon)}l_{\lambda_j,\epsilon} - o(1). \tag{3.4}$$

Lemma 3.2 can be proved by the similar arguments as those used to prove Lemma 2.6.

Lemma 3.3 *Assume (A.1)–(A.4). Let $\alpha > 0$ and $0 < \epsilon \ll 1$ be fixed. Suppose that there exists a subsequence $\{\lambda_j\}$ such that $\lambda_j \to \infty$ as $j \to \infty$, and $\|u_{\lambda_j}\|_\infty \ge 2\pi$. Then*

$$u'_{\lambda_j}(t_{2\pi,\lambda})^2 \le C\lambda_j e^{-2m_{\lambda_j,\epsilon}\sqrt{(1-\epsilon)\lambda}}. \tag{3.5}$$

Proof. We write $\lambda = \lambda_j$, for short. For $t \in [t_{2\pi,\lambda}, t_{2\pi-\epsilon,\lambda}]$, by (1.1),

$$\begin{aligned} u''_\lambda(t) + \mu(\lambda)f(u_\lambda(t)) &= \lambda \sin u_\lambda(t) = -\lambda\sin(2\pi - u_\lambda(t)) \\ &\le -\lambda(1-\epsilon)(2\pi - u_\lambda(t)) = \lambda(1-\epsilon)(u_\lambda(t) - 2\pi). \end{aligned} \tag{3.6}$$

Then for $t \in [t_{2\pi,\lambda}, t_{2\pi-\epsilon,\lambda}]$, by (2.2) and (3.6),

$$\{u''_\lambda(t) + \mu(\lambda)f(u_\lambda(t)) - \lambda(1-\epsilon)(u_\lambda(t) - 2\pi)\}u'_\lambda(t) \ge 0.$$

This implies that for $t \in [t_{2\pi,\lambda}, t_{2\pi-\epsilon,\lambda}]$,

$$\frac{dS_{\lambda,4}(t)}{dt} := \frac{d}{dt}\left\{\frac{1}{2}u'_\lambda(t) + \mu(\lambda)F(u_\lambda(t)) - \frac{1-\epsilon}{2}(u_\lambda(t) - 2\pi)^2\right\} \ge 0.$$

So $S_{\lambda,4}(t)$ is non-decreasing in $[t_{2\pi,\lambda}, t_{2\pi-\epsilon,\lambda}]$. Then for $t \in [t_{2\pi,\lambda}, t_{2\pi-\epsilon,\lambda}]$, we obtain

$$\frac{1}{2}u'_\lambda(t) + \mu(\lambda)F(u_\lambda(t)) - \frac{1-\epsilon}{2}(u_\lambda(t) - 2\pi)^2 \ge \frac{1}{2}u'_\lambda(t_{2\pi,\lambda})^2 + \mu(\lambda)F(2\pi),$$

which implies

$$\frac{1}{2}u'_\lambda(t)^2 \ge \frac{1}{2}u_\lambda(t_{2\pi,\lambda})^2 + \frac{1-\epsilon}{2}(u_\lambda(t) - 2\pi)^2. \tag{3.7}$$

By (3.7) and the same calculation as those used to prove Lemma 2.6, we obtain (3.5). ∎

Lemma 3.4 *Assume (A.1)–(A.4). Let $\alpha > 0$ and $0 < \epsilon \ll 1$ be fixed. Suppose that there exists a subsequence $\{\lambda_j\}$ such that $\lambda_j \to \infty$ as $j \to \infty$, and $\|u_{\lambda_j}\|_\infty \ge 4\pi$. Then*

$$t_{4\pi-\epsilon,\lambda_j} - t_{4\pi,\lambda_j} \ge \sqrt{1-\epsilon}\,m_{\lambda_j,\epsilon} - o(1) \qquad \textit{for } \lambda_j \gg 1. \tag{3.8}$$

Lemma 3.5 *Assume (A.1)–(A.4). Let $\alpha > 0$ and $0 < \epsilon \ll 1$ be fixed. Assume that there exists a subsequence $\{\lambda_j\}$ such that $\lambda_j \to \infty$ as $j \to \infty$, and $\|u_{\lambda_j}\|_\infty \ge 2\pi + \epsilon$. Then*

$$l_{\lambda_j,\epsilon} = t_{2\pi,\lambda_j} - t_{2\pi+\epsilon,\lambda_j} \ge \sqrt{1-2\epsilon}\,\delta_{\lambda_j,\epsilon} - o(1) \qquad \textit{for } \lambda_j \gg 1. \tag{3.9}$$

Proof. We abrebiate λ_j as λ. For $t \in [t_{2\pi+\epsilon,\lambda}, t_{2\pi,\lambda}]$, by (2.4), we obtain

$$\begin{aligned}\frac{1}{2}u_\lambda'(t)^2 &\le \frac{1}{2}u_\lambda'(T)^2 + \lambda(1-\cos u_\lambda(t)) = \frac{1}{2}u_\lambda'(T)^2 + \lambda(1-\cos(u_\lambda(t)-2\pi)) \\ &\le \frac{1}{2}u_\lambda'(T)^2 + \frac{1}{2}\lambda(u_\lambda(t)-2\pi)^2.\end{aligned}$$

This implies

$$-u_\lambda'(t) \le \sqrt{\lambda(u_\lambda(t)-2\pi)^2 + u_\lambda'(T)^2}$$

for $t \in [t_{2\pi+\epsilon,\lambda}, t_{2\pi,\lambda}]$. Therefore,

$$l_{\lambda,\epsilon} = t_{2\pi,\lambda} - t_{2\pi+\epsilon,\lambda} \ge \int_{t_{2\pi+\epsilon,\lambda}}^{t_{2\pi,\lambda}} \frac{-u_\lambda'(t)}{\sqrt{\lambda(u_\lambda(t)-2\pi)^2 + u_\lambda'(T)^2}}dt = \int_0^\epsilon \frac{1}{\sqrt{\lambda s^2 + u_\lambda'(T)^2}}dt.$$

By this, we easily obtain (3.9). ∎

Lemma 3.6 *Assume (A.1)-(A.4). Let $\alpha > 0$ and $0 < \epsilon \ll 1$ be fixed. Assume that there exists a subsequence $\{\lambda_j\}$ such that $\lambda_j \to \infty$ as $j \to \infty$, and $\|u_{\lambda_j}\|_\infty \ge 4\pi$. Then*

$$t_{4\pi-\epsilon,\lambda_j} - t_{4\pi,\lambda_j} \ge \sqrt{1-2\epsilon}\,\delta_{\lambda_j,\epsilon} - o(1) \qquad \text{for } \lambda_j \gg 1. \tag{3.10}$$

Proof of Theorem 2.1 (i) for $n = 1$. We assume (A.1)-(A.4) and (A.5.1). Let $2\pi < \alpha < 4\pi$ which satisfies (1.4) for $n = 1$ be fixed. We assume that there exists a subsequence of $\{\lambda\}$, denoted by $\{\lambda\}$ again, such that $\lambda \to \infty$ and $\|u_\lambda\|_\infty \ge 4\pi$, and derive a contradiction. Let $0 < \epsilon \ll 1$ be fixed. By (2.7), we see that as $\lambda \to \infty$

$$|t_{\epsilon,\lambda} - t_{2\pi-\epsilon,\lambda}|, |t_{2\pi+\epsilon,\lambda} - t_{4\pi-\epsilon,\lambda}| \to 0. \tag{3.11}$$

Then by (3.11),

$$\begin{aligned} T &= T - t_{\epsilon,\lambda} + (t_{\epsilon,\lambda} - t_{2\pi-\epsilon,\lambda}) + (t_{2\pi-\epsilon,\lambda} - t_{2\pi,\lambda}) + (t_{2\pi,\lambda} - t_{2\pi+\epsilon,\lambda}) \qquad (3.12)\\ &+ (t_{2\pi+\epsilon,\lambda} - t_{4\pi-\epsilon,\lambda}) + t_{4\pi-\epsilon,\lambda} \\ &= \delta_{\lambda,\epsilon} + l_{\lambda,\epsilon} + m_{\lambda,\epsilon} + t_{4\pi-\epsilon,\lambda} + (t_{\epsilon,\lambda} - t_{2\pi-\epsilon,\lambda}) + (t_{2\pi+\epsilon,\lambda} - t_{4\pi-\epsilon,\lambda}) \\ &= \delta_{\lambda,\epsilon} + l_{\lambda,\epsilon} + m_{\lambda,\epsilon} + t_{4\pi-\epsilon,\lambda} + o(1). \end{aligned}$$

Therefore, by (3.4), (3.12), Lemmas 3.4 and 3.6,

$$T \le 3(t_{4\pi-\epsilon,\lambda} - t_{4\pi,\lambda}) + t_{4\pi-\epsilon,\lambda} + O(\epsilon) + o(1) \le 4t_{4\pi-\epsilon,\lambda} + O(\epsilon) + o(1).$$

This implies that for $\lambda \gg 1$

$$\frac{T}{4} \le t_{4\pi-\epsilon,\lambda} + O(\epsilon) + o(1). \tag{3.13}$$

On the other hand, by Lemmas 2.7, 3.5, (3.12) and (3.13),

$$\begin{aligned} 3\delta_{\lambda,\epsilon} &\le \delta_{\lambda,\epsilon} + m_{\lambda,\epsilon} + l_{\lambda,\epsilon} + O(\epsilon) + o(1) = T - t_{4\pi-\epsilon,\lambda} + O(\epsilon) + o(1) \\ &\le \frac{3}{4}T + O(\epsilon) + o(1). \end{aligned}$$

This implies that for $\lambda \gg 1$

$$\delta_{\lambda,\epsilon} \le \frac{1}{4}T + O(\epsilon) + o(1). \tag{3.14}$$

It is clear that

$$\begin{aligned} TF(\alpha) = \sum_{k=1}^{4} B_{k,\lambda,\epsilon} &:= \int_0^{T/4-C\epsilon} F(u_\lambda(t))dt + \int_{T/4-C\epsilon}^{t_{2\pi-\epsilon,\lambda}} F(u_\lambda(t))dt \\ &+ \int_{t_{2\pi-\epsilon,\lambda}}^{t_{\epsilon,\lambda}} F(u_\lambda(t))dt + \int_{t_{\epsilon,\lambda}}^{T} F(u_\lambda(t))dt \end{aligned} \tag{3.15}$$

By (3.11), we obtain that $B_{3,\lambda,\epsilon} \to 0$ as $\lambda \to \infty$. It is clear that $B_{4,\lambda,\epsilon} \le C\epsilon$. By (3.13), we see that $T/4 - C\epsilon \le t_{4\pi-\epsilon,\lambda}$ for $\lambda \gg 1$. Then by this,

$$B_{1,\lambda,\epsilon} \ge F(4\pi - \epsilon)\left(\frac{T}{4} - C\epsilon\right) \ge \frac{TF(4\pi)}{4} - C\epsilon.$$

By (3.11) and (3.14),

$$\begin{aligned} B_{2,\lambda,\epsilon} &\ge F(2\pi - \epsilon)(t_{2\pi-\epsilon,\lambda} - T/4 + C\epsilon) \\ &= F(2\pi - \epsilon)((t_{2\pi-\epsilon,\lambda} - t_{\epsilon,\lambda}) + T - \delta_{\lambda,\epsilon} - T/4 + C\epsilon) \\ &\ge \frac{TF(2\pi)}{2} - C\epsilon - o(1). \end{aligned}$$

By these inequalities and (3.15),

$$F(\alpha) \ge \frac{F(4\pi)}{4} + \frac{F(2\pi)}{2} - C\epsilon - o(1). \tag{3.16}$$

Choose ϵ sufficiently small. Then this contradicts (1.4) for $n = 1$. Thus the proof is complete. ∎

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