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# NONOSCILLATION THEOREMS FOR SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS OF EULER TYPE

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## 1. INTRODUCTION

The purpose of this paper is to improve nonoscillation criteria for the nonlinear differential equation

$$t^2x'' + g(x) = 0, \qquad t > 0, \tag{1.1}$$

where $g(x)$ satisfies a suitable smoothness condition for the uniqueness of solutions of the initial value problem and the signum condition

$$xg(x) > 0 \quad \text{if} \ x \neq 0. \tag{1.2}$$

As already has been shown in [3], under the assumption (1.2), every solution of (1.1) is continuable in the future. Thus, we may investigate the oscillatory behavoir of solutions of (1.1). A nontrivial solution $x(t)$ of (1.1) is said to be *oscillatory* if there exists a sequence $\{t_n\}$ tending to $\infty$ such that $x(t_n) = 0$. Otherwise, it is said to be *nonoscillatory*. Equation (1.1) is said to be oscillatory (resp., nonoscillatory) in case all nontrivial solutions are oscillatory (resp., nonoscillatory).

When $g(x) = \lambda x$, equation (1.1) becomes the famous Euler differential equation and it is well known that (1.1) is oscillatory if $\lambda > 1/4$ and is nonoscillatory if $\lambda \leq 1/4$. In this case, equation (1.1) does not allow the coexistence of oscillatory solutions and nonoscillatory solutions.

On the contrary, in the case that $g(x)$ is nonlinear, it is possible that equation (1.1) has both oscillatory solutions and nonoscillatory solutions at the same time because of lack of Sturm's separation theorem. However, Sugie and Hara [3] showed that there is no possibility of the coexistence, that is, if $g(x)/x \geq \lambda$ with $\lambda > 1/4$, then equation (1.1) is oscillatory; and if $g(x)/x \leq 1/4$, then (1.1) is nonoscillatory (see also [5]). They also pointed out that all nontrivial solutions of (1.1) have a tendency to be oscillatory as $g(x)/x$ grows larger in some sense and the most delicate case in the oscillation problem for equation (1.1) is

$$\frac{g(x)}{x} \searrow \frac{1}{4} \quad \text{as} \ |x| \to \infty. \tag{1.3}$$

Recently, transforming equation (1.1) into a system of Liénard type and using phase plane analysis of the Liénard system, Sugie and Kita [4] discussed the delicate problem and extended the results above as follows:

THEOREM A. *Assume* (1.2) *and suppose that there exists a* $\lambda$ *with* $\lambda > 1/4$ *such that*

$$\frac{g(x)}{x} \geq \frac{1}{4} + \frac{\lambda}{(2\log|x|)^2}$$

*for* $|x|$ *sufficiently large. Then equation* (1.1) *is oscillatory.*

THEOREM B. *Assume* (1.2) *and suppose that*

$$\frac{g(x)}{x} \le \frac{1}{4} + \frac{1}{16(\log|x|)^2}$$

*for* $x > 0$ *or* $x < 0$, $|x|$ *sufficiently large. Then equation* (1.1) *is nonoscillatory.*

Theorems A and B can be applied to the most part of (1.3). Unfortunately, however, they are inapplicable to the case

$$(2\log|x|)^2 \left\{\frac{g(x)}{x} - \frac{1}{4}\right\} \searrow \frac{1}{4} \quad \text{as } |x| \to \infty. \tag{1.4}$$

Note that (1.4) implies (1.3). Thus, the subcase (1.4) of (1.3) remains unsetted. Our problem has become more and more delicate.

In this paper, we give an infinite sequence of nonoscillation theorems which is applied even to the case (1.4). To this end, we introduce some condensed notation. Write

$$L_1(x) = 1, \qquad L_{n+1}(x) = L_n(x)\, l_n(x), \qquad n = 1, 2, \cdots,$$

where

$$l_1(x) = 2\log x, \qquad l_{n+1}(x) = \log\{l_n(x)\},$$

and set

$$S_n(x) = \sum_{k=1}^{n} \frac{1}{\{L_k(x)\}^2}.$$

Define $e_0 = 1$ and $e_n = \exp(e_{n-1})$. Then we have

$$l_{n+1}(x) = \log\{l_n(x)\} > 0 \quad \text{for } x > \sqrt{e_n},$$

and therefore, the function sequences $\{L_n(x)\}$, $\{l_n(x)\}$ and $\{S_n(x)\}$ are well-defined for a sufficiently large $x$. Our main result is stated in the following:

THEOREM 1.1. *Assume* (1.2) *and suppose that there exists a positive integer* $n$ *such that*

$$\frac{g(x)}{x} \le \frac{1}{4} S_n(|x|) \tag{1.5}$$

*for* $x > 0$ *or* $x < 0$, $|x|$ *sufficiently large. Then equation* (1.1) *is nonoscillatory.*

*Remark* 1.2. If $n = 1$, then condition (1.5) becomes $g(x)/x \le 1/4$ for $|x|$ sufficiently large. Also, Theorem 1.1 coincides with Theorem B when $n = 2$.

## 2. GENERAL SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS OF EULER TYPE

Consider the Riemann-Weber version of Euler differential equation

$$y'' + \frac{1}{t^2}\left\{\frac{1}{4} + \frac{\delta}{(\log t)^2}\right\} y = 0 \tag{$(E)_2$}$$

(refer to [1]). Then we see that equation $(E)_2$ has the general solution

$$y(t) = \begin{cases} \sqrt{t}\{K_1(\log t)^z + K_2(\log t)^{1-z}\} & \text{if } \delta \ne 1/4, \\ \sqrt{t\log t}\{K_3 + K_4 \log(\log t)\} & \text{if } \delta = 1/4, \end{cases} \tag{2.1}$$

where $K_i$ $(i = 1, 2, 3, 4)$ are arbitrary constants and $z$ is the root of

$$z(1 - z) = \delta. \tag{2.2}$$

From (2.1) we see that all nontrivial solutions of $(E)_2$ are nonoscillatory when $\delta \le 1/4$. In case $\delta > 1/4$, the characteristic equation (2.2) has conjugate roots $z = 1/2 \pm i\alpha/2$, where $\alpha = \sqrt{4\delta - 1}$. Hence, by Euler's formula, the real solution of $(E)_2$ can be written as

$$y(t) = \sqrt{t \log t}\left\{k_1 \cos\left(\frac{\alpha}{2} \log(\log t)\right) + k_2 \sin\left(\frac{\alpha}{2}\log(\log t)\right)\right\}.$$

If $(k_1, k_2) = (0,0)$, then $y(t)$ is the trivial solution. On the other hand, if $(k_1, k_2) \ne (0,0)$, then

$$y(t) = k_3 \sqrt{t \log t} \sin\left(\frac{\alpha}{2}\log(\log t) + \beta\right),$$

where $k_3 = \sqrt{k_1^2 + k_2^2} \ne 0$, $\sin\beta = k_1/k_3$ and $\cos\beta = k_2/k_3$. Thus, equation $(E)_2$ is classified into two types as follows:

PROPOSITION 2.1. *If $\delta > 1/4$, then equation $(E)_2$ is oscillatory, and otherwise it is nonoscillatory.*

Let us regard the most simple Euler differential equation

$$y'' + \frac{\delta}{t^2} y = 0 \qquad (E)_1$$

as the first stage. Then equation $(E)_2$ corresponds to the second stage. We go on to the $n$th stage of linear differential equations of Euler type. For this purpose, let

$$\log_0 t = t, \qquad \log_n t = \log(\log_{n-1} t), \qquad n = 1, 2, \cdots,$$

and consider

$$y'' + \left\{\frac{1}{4}\sum_{k=0}^{n-2}\left(\prod_{i=0}^{k} \log_i t\right)^{-2} + \delta\left(\prod_{i=0}^{n-1}\log_i t\right)^{-2}\right\} y = 0. \qquad (E)_n$$

Then we have the following formula.

PROPOSITION 2.2. *Equation $(E)_n$ with $n \ge 2$ has the general solution*

$$y(t) = \begin{cases} \left(\prod_{i=0}^{n-2} \log_i t\right)^{1/2}\left\{K_1(\log_{n-1} t)^z + K_2(\log_{n-1} t)^{1-z}\right\} & \text{if } \delta \ne 1/4, \\ \left(\prod_{i=0}^{n-1} \log_i t\right)^{1/2}\left\{K_3 + K_4 \log_n t\right\} & \text{if } \delta = 1/4, \end{cases}$$

*where $K_i$ $(i = 1,2,3,4)$ are arbitrary constants and $z$ is the root of the characteristic equation* (2.2).

*Proof*. We use mathematical induction on $n$. Let $n = 2$. Since $\log_0 t = t$, $\log_1 t = \log t$ and $\log_2 t = \log(\log t)$, equation $(E)_n$ becomes $(E)_2$ and the function $y(t)$ satisfies (2.1). Hence, the assertion is true for $n = 2$.

Assume the assertion is true for $n = p \ge 2$ and consider equation $(E)_n$ with $n = p + 1$. Changing variable $t = e^s$, we can rewrite equation $(E)_{p+1}$ as

$$\ddot{u}(s) - \dot{u}(s) + t^2\left\{\frac{1}{4}\sum_{k=0}^{p-1}\left(\prod_{i=0}^{k}\log_i t\right)^{-2} + \delta\left(\prod_{i=0}^{p}\log_i t\right)^{-2}\right\} u(s) = 0,$$

where $\dot{} = d/ds$ and $u(s) = y(e^s) = y(t)$. Arranging the left-hand of the above equality, we have

$$\begin{aligned}
&\ddot{u}(s) - \dot{u}(s) + t^2\left[\frac{1}{4}\left\{\left(\prod_{i=0}^{0}\log_i t\right)^{-2} + \sum_{k=1}^{p-1}\left(\prod_{i=0}^{k}\log_i t\right)^{-2}\right\} + \delta\left(\prod_{i=0}^{p}\log_i t\right)^{-2}\right]u(s)\\
&= \ddot{u}(s) - \dot{u}(s) + \left\{\frac{1}{4} + \frac{1}{4}\sum_{k=1}^{p-1}\left(\prod_{i=1}^{k}\log_i t\right)^{-2} + \delta\left(\prod_{i=1}^{p}\log_i t\right)^{-2}\right\}u(s)\\
&= \ddot{u}(s) - \dot{u}(s) + \frac{1}{4}u(s) + \left\{\frac{1}{4}\sum_{k=1}^{p-1}\left(\prod_{i=1}^{k}\log_{i-1} s\right)^{-2} + \delta\left(\prod_{i=1}^{p}\log_{i-1} s\right)^{-2}\right\}u(s)\\
&= \ddot{u}(s) - \dot{u}(s) + \frac{1}{4}u(s) + \left\{\frac{1}{4}\sum_{k=1}^{p-1}\left(\prod_{i=0}^{k-1}\log_i s\right)^{-2} + \delta\left(\prod_{i=0}^{p-1}\log_i s\right)^{-2}\right\}u(s)\\
&= \ddot{u}(s) - \dot{u}(s) + \frac{1}{4}u(s) + \left\{\frac{1}{4}\sum_{k=0}^{p-2}\left(\prod_{i=0}^{k}\log_i s\right)^{-2} + \delta\left(\prod_{i=0}^{p-1}\log_i s\right)^{-2}\right\}u(s).
\end{aligned}$$

Hence, equation $(E)_{p+1}$ is transformed into the equation

$$\ddot{u}(s) - \dot{u}(s) + \frac{1}{4}u(s) + \left\{\frac{1}{4}\sum_{k=0}^{p-2}\left(\prod_{i=0}^{k}\log_i s\right)^{-2} + \delta\left(\prod_{i=0}^{p-1}\log_i s\right)^{-2}\right\}u(s) = 0.$$

By setting $w(s) = u(s)\exp(-s/2)$, this equation becomes

$$\ddot{w}(s) + \left\{\frac{1}{4}\sum_{k=0}^{p-2}\left(\prod_{i=0}^{k}\log_i s\right)^{-2} + \delta\left(\prod_{i=0}^{p-1}\log_i s\right)^{-2}\right\}w(s) = 0$$

because

$$\ddot{w}(s) = \left\{\ddot{u}(s) - \dot{u}(s) + \frac{1}{4}u(s)\right\}\exp(-s/2),$$

and therefore, $w(s)$ satisfies equation $(E)_p$. Hence, by the inductive assumption, we see that

$$w(s) = \begin{cases} \left(\prod_{i=0}^{p-2}\log_i s\right)^{1/2}\left\{K_1(\log_{p-1} s)^z + K_2(\log_{p-1} s)^{1-z}\right\} & \text{if } \delta \neq 1/4,\\ \left(\prod_{i=0}^{p-1}\log_i s\right)^{1/2}\left\{K_3 + K_4\log_p s\right\} & \text{if } \delta = 1/4. \end{cases}$$

Since $y(t) = w(s)\exp(s/2) = w(\log t)\sqrt{t}$, we have

$$\begin{aligned}
y(t) &= \left(\prod_{i=0}^{p-2}\log_i(\log t)\right)^{1/2}\left\{K_1(\log_{p-1}(\log t))^z + K_2(\log_{p-1}(\log t))^{1-z}\right\}t^{1/2}\\
&= \left(\prod_{i=0}^{p-2}\log_{i+1} t\right)^{1/2}\left\{K_1(\log_p t)^z + K_2(\log_p t)^{1-z}\right\}(\log_0 t)^{1/2}\\
&= \left(\prod_{i=0}^{p-1}\log_i t\right)^{1/2}\left\{K_1(\log_p t)^z + K_2(\log_p t)^{1-z}\right\}
\end{aligned}$$

if $\delta \neq 1/4$ and

$$\begin{aligned} y(t) &= \Big(\prod_{i=0}^{p-1} \log_i(\log t)\Big)^{1/2}\Big\{K_3 + K_4 \log_p(\log t)\Big\}\, t^{1/2} \\ &= \Big(\prod_{i=0}^{p-1} \log_{i+1} t\Big)^{1/2}\Big\{K_3 + K_4 \log_{p+1} t\Big\}\, (\log_0 t)^{1/2} \\ &= \Big(\prod_{i=0}^{p} \log_i t\Big)^{1/2}\Big\{K_3 + K_4 \log_{p+1} t\Big\} \end{aligned}$$

if $\delta = 1/4$. Thus, the assertion is also true for $n = p+1$. This completes the proof. □

By Proposition 2.2, we can classify equation $(E)_n$ into two types as follows:

PROPOSITION 2.3. *If* $\delta > 1/4$, *then equation* $(E)_n$ *is oscillatory, and otherwise it is nonoscillatory.*

## 3. Positive orbits of a Liénard system

To prove our main result, Theorem 1.1, we will prepare an important lemma in this section. Changing variable $t = e^s$, we can transform equation (1.1) into the equation

$$\ddot{u} - \dot{u} + g(u) = 0, \qquad s \in \mathbf{R},$$

which is equivalent to the planar system

$$\begin{aligned} \dot{u} &= v + u, \\ \dot{v} &= -\,g(u). \end{aligned} \tag{3.1}$$

System (3.1) is of Liénard type. Sugie and Hara [3, Lemma 4.1] proved that all nontrivial solutions of (3.1) are unbounded.

We call the projection of a positive semitrajectory of (3.1) onto the phase plane a *positive orbit*. Under the assumption (1.2), the unique equilibrium of (3.1) is the origin, in other words, every solution is nontrivial except the zero solution. Taking the vector field of (3.1) into account, we see that if equation (1.1) has a nontrivial oscillatory solution $x(t)$, then the positive orbit of (3.1) corresponding to $x(t)$ rotates around the origin clockwise.

Numerous studies have been made on positive orbits of more general Liénard systems. There is a possibility that systems of Liénard type have both positive orbits rotating around the origin clockwise and positive orbits running to infinity, to put it another way, such systems have both oscillatory solutions and nonoscillatory solutions at the same time (for example, see [2]). As shown below, however, it is impossible that both oscillatory solutions and nonoscillatory solutions coexist in system (3.1). From this point of view, the following lemma plays the same role of Sturm's separation theorem in linear differential equations.

LEMMA 3.1. *Under the assumption* (1.2), *if equation* (1.1) *has a nontrivial oscillatory solution, then all nontrivial positive orbits of* (3.1) *keep on rotating around the origin clockwise.*

*Proof.* Let $(u(s), v(s))$ be a nontrivial oscillatory solution of (3.1) and let $A = (u(s_0), v(s_0))$ Then it follows from Lemma 4.1 in [3] that $(u(s), v(s))$ is unbounded. Define a Liapunov function

$$V(u,v) = \frac{1}{2}v^2 + \int_0^u g(\sigma)d\sigma$$

and consider the level curve $V(u,v) = H$ for any $H > 0$. Then there exist two points of intersection of the curve with the straight line $v = -u$. In fact, the function $V(u,-u)$ is increasing for $u > 0$ and decreasing for $u < 0$, and

$$V(0,0) = 0, \qquad V(u,-u) \to \infty \quad \text{as } |u| \to \infty.$$

Let $(-a,a)$ and $(b,-b)$ be the points of intersection, where $a > 0$ and $b > 0$. It is clear that numbers $a$ and $b$ are dependent of $H$ and increasing with respect to $H$, and satisfy

$$a(H) \to \infty, \qquad b(H) \to \infty$$

as $H \to \infty$. Define a domain $D_H$ by

$$D_H = \Big\{(u,v)\colon -a < u < b \ \text{ and } \ V(u,v) < H\Big\}.$$

Then the domain $D_H$ becomes larger as $H$ increases and covers the whole $(u,v)$-plane, that is,

$$D_{H_1} \subset D_{H_2} \ \text{ for } \ H_2 > H_1, \qquad \bigcup_{H>0} D_H = \mathbf{R}^2.$$

Since $(u(s),v(s))$ is unbounded, the positive orbit $\gamma^+_{(3.1)}(A)$ which corresponds to $(u(s),v(s))$ cannot stay in $D_H$. Hence, we choose a $\tau > 0$ such that $(u(\tau),v(\tau)) \in D^c_H$, where $D^c_H$ is the complement of $D_H$. From the vector field of (3.1), we see that $\gamma^+_{(3.1)}(A)$ does not cross the lines $u = b$ and $u = -a$ again. We also see that $\gamma^+_{(3.1)}(A)$ cannot cross the level curve $V(u,v) = H$ twice, because

$$\frac{d}{ds}V(u(s),v(s)) = u(s)g(u(s)) > 0$$

by (1.2). Hence, $\gamma^+_{(3.1)}(A)$ cannot return to $D_H$ for $s \geq \tau$. Since $H$ is arbitrary and $(u(s),v(s))$ is oscillatory, $\gamma^+_{(3.1)}(A)$ keeps on rotating around the origin clockwise and tending toward infinity. From the uniqueness of solutions for the initial value problem it follows that all nontrivial positive orbits of (3.1) must have the same property. □

## 4. Proof of Theorem 1.1

As mentioned in Section 1, Sugie and Hara [3] proved that Theorem 1.1 is true for $n = 1$, and thus, let $n \geq 2$. We prove only the case that condition (1.5) is satisfied for $x > 0$ sufficiently large, because the other case is carried out by the same manner.

We first prove the special case

$$\frac{g(x)}{x} = \frac{1}{4}S_n(x) \tag{4.1}$$

for $x > 0$ sufficiently large. The proof of this case is by contradiction. Assume that equation (1.1) with (4.1) has a nontrivial oscillatory solution. Consider system (3.1), which is equivalent to equation (1.1). For convenience' sake, we call it *system* (4.2) if $g(x)$ satisfies (4.1). System (4.2) coincides with the system

$$\dot{u} = v + u,$$
$$\dot{v} = -\frac{1}{4}S_n(u)u$$

for $u > 0$ sufficiently large. By the assumption of contradiction and Lemma 3.1, all nontrivial positive orbits of (4.2) keep on rotating around the origin in clockwise direction.

We now consider the linear differential equation

$$y'' + \left\{\frac{1}{4}\sum_{k=0}^{n-1}\left(\prod_{i=0}^{k}\log_i t\right)^{-2}\right\}y = 0, \tag{4.3}$$

which is equation $(E)_n$ with $\delta = 1/4$. Let $t_0$ be an arbitrary number with $t_0 > e_{n-1}$ and define

$$\mu^2 = 1 + \frac{1}{4}\left\{-1 + \sum_{k=1}^{n-1}\left(\prod_{i=1}^{k}\log_i t_0\right)^{-1}\right\}^2, \qquad y_0 = \frac{\sqrt{5t_0}}{2\mu} > 0. \tag{4.4}$$

Putting $K_3 = y_0\left(\prod_{i=0}^{n-1}\log_i t_0\right)^{-1/2}$ and $K_4 = 0$ in Proposition 2.2, we see that the function

$$y(t) = y_0\left(\prod_{i=0}^{n-1}\log_i t_0\right)^{-1/2}\left(\prod_{i=0}^{n-1}\log_i t\right)^{1/2} \tag{4.5}$$

is a nonoscillatory solution of (4.3).

*Claim* 1. $\left(\prod_{i=0}^{n-1}\log_i t\right)' = \sum_{k=1}^{n-1}\left(\prod_{i=k}^{n-1}\log_i t\right) + 1.$

We prove the claim by mathematical induction. Since

$$\left(\prod_{i=0}^{1}\log_i t\right)' = (t\log t)' = \log t + 1 = \sum_{k=1}^{1}\left(\prod_{i=k}^{1}\log_i t\right) + 1,$$

the claim holds when $n = 2$. Suppose that the claim is satisfied with $n = p$. Then we have

$$\begin{aligned}\left(\prod_{i=0}^{p}\log_i t\right)' &= \left\{\left(\prod_{i=0}^{p-1}\log_i t\right)\log_p t\right\}' = \left(\prod_{i=0}^{p-1}\log_i t\right)'\log_p t + \left(\prod_{i=0}^{p-1}\log_i t\right)\left(\prod_{i=0}^{p-1}\log_i t\right)^{-1}\\ &= \left\{\sum_{k=1}^{p-1}\left(\prod_{i=k}^{p-1}\log_i t\right) + 1\right\}\log_p t + 1 = \sum_{k=1}^{p}\left(\prod_{i=k}^{p}\log_i t\right) + 1.\end{aligned}$$

Hence, the claim is also satisfied with $n = p + 1$.

From Claim 1, we see that the solution $y(t)$ satisfies the initial conditions

$$y(t_0) = y_0, \qquad y'(t_0) = \frac{y_0}{2}\sum_{k=0}^{n-1}\left(\prod_{i=0}^{k}\log_i t_0\right)^{-1}.$$

In fact, we have

$$\begin{aligned}y'(t) &= \frac{y_0}{2}\left(\prod_{i=0}^{n-1}\log_i t_0\right)^{-1/2}\left(\prod_{i=0}^{n-1}\log_i t\right)^{-1/2}\left(\prod_{i=0}^{n-1}\log_i t\right)'\\ &= \frac{y_0}{2}\left(\prod_{i=0}^{n-1}\log_i t_0\right)^{-1/2}\left(\prod_{i=0}^{n-1}\log_i t\right)^{-1/2}\left\{\sum_{k=1}^{n-1}\left(\prod_{i=k}^{n-1}\log_i t\right) + 1\right\},\end{aligned} \tag{4.6}$$

and therefore,

$$\begin{aligned} y'(t_0) &= \frac{y_0}{2}\Big(\prod_{i=0}^{n-1}\log_i t_0\Big)^{-1}\Big\{\sum_{k=1}^{n-1}\Big(\prod_{i=k}^{n-1}\log_i t_0\Big)+1\Big\} \\ &= \frac{y_0}{2}\Big\{\sum_{k=1}^{n-1}\Big(\prod_{i=0}^{k-1}\log_i t_0\Big)^{-1}+\Big(\prod_{i=0}^{n-1}\log_i t_0\Big)^{-1}\Big\} \\ &= \frac{y_0}{2}\Big\{\sum_{k=0}^{n-2}\Big(\prod_{i=0}^{k}\log_i t_0\Big)^{-1}+\Big(\prod_{i=0}^{n-1}\log_i t_0\Big)^{-1}\Big\} \\ &= \frac{y_0}{2}\sum_{k=0}^{n-1}\Big(\prod_{i=0}^{k}\log_i t_0\Big)^{-1}. \end{aligned}$$

Let $s = \log t$. Then equation (4.3) is transferred into the system

$$\begin{aligned} \dot{u} &= v + u, \\ \dot{v} &= -\Big\{\frac{1}{4}+\frac{1}{4}\sum_{k=0}^{n-2}\Big(\prod_{i=0}^{k}\log_i s\Big)^{-2}\Big\}u. \end{aligned} \tag{4.7}$$

The change of variable also transfers the solution (4.5) to $(u(s), v(s))$ which is represented as

$$(u(s), v(s)) = (y(e^s), y'(e^s)e^s - y(e^s)).$$

Using (4.5), (4.6) and the fact that $\log_i e^s = \log_{i-1}(\log e^s) = \log_{i-1} s$, we obtain

$$\begin{aligned} \frac{v(s)}{u(s)} &= \frac{y'(t)\,t}{y(t)} - 1 = \frac{1}{2}\Big(\prod_{i=0}^{n-1}\log_i t\Big)^{-1}\Big\{\sum_{k=1}^{n-1}\Big(\prod_{i=k}^{n-1}\log_i t\Big)+1\Big\}t - 1 \\ &= \frac{1}{2}\Big\{\sum_{k=1}^{n-1}\Big(\prod_{i=0}^{k-1}\log_i t\Big)^{-1}+\Big(\prod_{i=0}^{n-1}\log_i t\Big)^{-1}\Big\}t - 1 = \frac{1}{2}\Big\{\sum_{k=1}^{n}\Big(\prod_{i=0}^{k-1}\log_i t\Big)^{-1}\Big\}t - 1 \\ &= \frac{1}{2}\Big\{1+\sum_{k=2}^{n}\Big(\prod_{i=1}^{k-1}\log_i t\Big)^{-1} - 2\Big\} = -\frac{1}{2}+\frac{1}{2}\sum_{k=2}^{n}\Big(\prod_{i=1}^{k-1}\log_i e^s\Big)^{-1} \\ &= -\frac{1}{2}+\frac{1}{2}\sum_{k=2}^{n}\Big(\prod_{i=0}^{k-2}\log_i s\Big)^{-1} = -\frac{1}{2}+\frac{1}{2}\sum_{k=0}^{n-2}\Big(\prod_{i=0}^{k}\log_i s\Big)^{-1}. \end{aligned}$$

Let $s_0 = \log t_0 > e_{n-2}$. Then we get

$$u(s_0) = y(t_0) = y_0, \qquad v(s_0) = \frac{y_0}{2}\Big\{-1+\sum_{k=0}^{n-2}\Big(\prod_{i=0}^{k}\log_i s_0\Big)^{-1}\Big\}.$$

*Claim* 2. $\displaystyle\sum_{k=0}^{n-2}\Big(\prod_{i=0}^{k}\log_i s_0\Big)^{-1} < 1.$

Noticing $\log_i s_0 > e_{n-2-i}$ for $i = 0, 1, \cdots, n-2$, we have

$$\Big(\prod_{i=0}^{k}\log_i s_0\Big)^{-1} < \Big(\prod_{i=0}^{k} e_{n-2-i}\Big)^{-1} < \frac{1}{e_{n-2}} \le \Big(\frac{1}{e}\Big)^{n-2},$$

and therefore,

$$\sum_{k=0}^{n-2}\Big(\prod_{i=0}^{k}\log_i s_0\Big)^{-1} < \sum_{k=0}^{n-2}\Big(\frac{1}{e}\Big)^{n-2} = (n-1)\Big(\frac{1}{e}\Big)^{n-2} \le 1.$$

It turns out from Claim 2 that

$$(u(s_0), v(s_0)) \in R_1 \stackrel{\text{def}}{=} \Big\{(u, v) \colon u > 0 \text{ and } -\frac{1}{2}u < v < 0\Big\}.$$

Since

$$\frac{v(s)}{u(s)} \searrow -\frac{1}{2} \quad \text{as } s \to \infty,$$

we conclude that

$$(u(s), v(s)) \in R_1 \quad \text{for } s \geq s_0. \tag{4.8}$$

Letting $u = \rho\cos\varphi$ and $v = \rho\sin\varphi$, we can transform system (4.7) into the system

$$\begin{aligned} \dot{\rho} &= \rho\left\{f_1(\varphi) - \frac{\sin\varphi\cos\varphi}{4}\sum_{k=0}^{n-2}\left(\prod_{i=0}^{k}\log_i s\right)^{-2}\right\}, \\ \dot{\varphi} &= f_2(\varphi) - \frac{\cos^2\varphi}{4}\sum_{k=0}^{n-2}\left(\prod_{i=0}^{k}\log_i s\right)^{-2}, \end{aligned} \tag{4.9}$$

where

$$f_1(\varphi) = (\sin\varphi + \cos\varphi)\cos\varphi - \frac{1}{4}\sin\varphi\cos\varphi,$$

$$f_2(\varphi) = -(\sin\varphi + \cos\varphi)\sin\varphi - \frac{1}{4}\cos^2\varphi.$$

Let $(\rho(s), \varphi(s))$ be the solution of (4.9) which corresponds to $(u(s), v(s))$. From (4.8) we see that

$$-\theta^* < \varphi(s) < 0 \quad \text{for } s \geq s_0, \tag{4.10}$$

where $\theta^*$ is the number satisfying $0 < \theta^* < \pi/2$ and $\tan\theta^* = 1/2$.

Returning now to the nonlinear system (4.2), we consider the positive orbit $\gamma^+_{(4.2)}(A)$ starting at the point $A(u(s_0), v(s_0))$ at $s = s_0$. Recall that all nontrivial orbits of (4.2) keep on rotating around the origin clockwise, and so does $\gamma^+_{(4.2)}(A)$. Hence, it meets the line $v = -u/2$ infinitely many times. Let $s_1 > s_0$ be the first intersecting time of $\gamma^+_{(4.2)}(A)$ with the line.

Consider the system

$$\begin{aligned} \dot{r} &= r\left[f_1(\theta) - \frac{\sin\theta\cos\theta}{4}\sum_{k=2}^{n}\frac{1}{\{L_k(r\cos\theta)\}^2}\right], \\ \dot{\theta} &= f_2(\theta) - \frac{\cos^2\theta}{4}\sum_{k=2}^{n}\frac{1}{\{L_k(r\cos\theta)\}^2}. \end{aligned} \tag{4.11}$$

Let $(r(s), \theta(s))$ be the solution of (4.11) corresponding to $\gamma^+_{(4.2)}(A)$. Note that the starting point $A$ is in the region $R_1$. Then we see that

$$\theta(s_1) = -\theta^*, \qquad -\theta^* < \theta(s) < 0 \quad \text{for } s_0 \leq s < s_1. \tag{4.12}$$

Since the function $f_1(\theta)$ is increasing for $-\theta^* \leq \theta < 0$, we have

$$f_1(\theta(s)) \geq f_1(-\theta^*) = \frac{1}{2} \quad \text{for } s_0 \leq s < s_1,$$

and therefore,

$$\dot{r}(s) = r(s)\left[f_1(\theta(s)) - \frac{\sin\theta(s)\cos\theta(s)}{4}\sum_{k=2}^{n}\frac{1}{\{L_k(r(s)\cos\theta(s))\}^2}\right] \geq \frac{1}{2}r(s)$$

for $s_0 \leq s \leq s_1$. Integrating this inequality from $s_0$ to $s \leq s_1$, we get

$$r(s) \geq r(s_0)\exp\left\{\frac{1}{2}(s - s_0)\right\} \quad \text{for } s_0 \leq s \leq s_1,$$

$$r(s_0) = \sqrt{\{u(s_0)\}^2 + \{v(s_0)\}^2} = y_0\sqrt{1 + \frac{1}{4}\left\{ -1 + \sum_{k=0}^{n-2}\left(\prod_{i=0}^{k} \log_i s_0\right)^{-1}\right\}^2}$$
$$= y_0\sqrt{1 + \frac{1}{4}\left\{ -1 + \sum_{k=1}^{n-1}\left(\prod_{i=1}^{k} \log_i t_0\right)^{-1}\right\}^2} = \mu y_0 = \frac{\sqrt{5t_0}}{2}.$$

Hence, together with (4.12), we obtain

$$\log\left(r(s)\cos\theta(s)\right) \geq \frac{1}{2}(s - s_0) + \log\frac{\sqrt{5t_0}}{2} + \log\left(\cos\theta^*\right)$$
$$= \frac{1}{2}(s - s_0) + \log\frac{\sqrt{5t_0}}{2} + \log\frac{2}{\sqrt{5}} = \frac{1}{2}s \tag{4.13}$$

for $s_0 \leq s \leq s_1$.

*Claim* 3. $l_i(r(s)\cos\theta(s)) \geq \log_{i-1} s$ for $s_0 \leq s \leq s_1$ and $i = 1, 2, \cdots, n-1$.

The proof is by mathematical induction. The claim is true for $i = 1$ because

$$l_1(r(s)\cos\theta(s)) = 2\log(r(s)\cos\theta(s)) \geq s = \log_0 s \quad \text{for } s_0 \leq s \leq s_1$$

by (4.13). Suppose that the claim is satisfied with $i = p$. Then we have

$$l_{p+1}(r(s)\cos\theta(s)) = \log\{l_p(r(s)\cos\theta(s))\} \geq \log\left(\log_{p-1} s\right) = \log_p s$$

for $s_0 \leq s \leq s_1$, namely, the claim is also satisfied with $i = p + 1$.

From (4.11) and Claim 3, we conclude that

$$\dot{\theta}(s) = f_2(\theta(s)) - \frac{\cos^2\theta(s)}{4}\sum_{k=2}^{n}\frac{1}{\{L_k(r(s)\cos\theta(s))\}^2}$$
$$= f_2(\theta(s)) - \frac{\cos^2\theta(s)}{4}\sum_{k=2}^{n}\left(\prod_{i=1}^{k-1} l_i(r(s)\cos\theta(s))\right)^{-2}$$
$$\geq f_2(\theta(s)) - \frac{\cos^2\theta(s)}{4}\sum_{k=2}^{n}\left(\prod_{i=1}^{k-1}\log_{i-1} s\right)^{-2}$$
$$= f_2(\theta(s)) - \frac{\cos^2\theta(s)}{4}\sum_{k=0}^{n-2}\left(\prod_{i=0}^{k}\log_i s\right)^{-2}$$

for $s_0 \leq s \leq s_1$. Comparing this differential inequality and the second equation in system (4.9), we see that

$$\varphi(s) \leq \theta(s) \quad \text{for } s_0 \leq s \leq s_1.$$

Hence, by (4.10) we obtain

$$\theta(s) > -\theta^* \quad \text{for } s_0 \leq s \leq s_1,$$

which is a contradiction to (4.12) at $s = s_1$. Thus, equation (1.1) is nonoscillatory in the special case (4.1).

Next, we consider the case that (4.1) does not hold. Then there exists a sequence $\{x_k\}$ tending to $\infty$ such that

$$\frac{g(x_k)}{x_k} < \frac{1}{4}S_n(x_k), \qquad k = 1, 2, \cdots. \tag{4.14}$$

Of course, condition (1.5) is satisfied for $x > 0$ sufficiently large. We prove the remaining case (4.14) by contradiction. Suppose that equation (1.1) has a nontrivial oscillatory solution. Then, from Lemma 3.1 it turns out that all nontrivial positive orbits of

$$\begin{aligned} \dot{u} &= v + u, \\ \dot{v} &= -\,g(u) \end{aligned} \tag{4.15}$$

rotate around the origin clockwise.

As proved above, all nontrivial solutions of (1.1) with (4.1) are nonoscillatory. Hence, without loss of generality, we can choose a solution $\zeta(t)$ which is positive for $t \geq T$, $T$ sufficiently large. Since $\zeta(t)$ is a solution of (1.1) with (4.1), we have

$$t^2\zeta''(t) = -\frac{1}{4}S_n(\zeta(t))\zeta(t) < 0 \quad \text{for } t \geq T,$$

that is, $\zeta'(t)$ is strictly decreasing for $t \geq T$. If there exists a $t_1 > T$ such that $\zeta'(t_1) \leq 0$, then we can choose a $t_2 > t_1$ such that

$$\zeta'(t) \leq \zeta'(t_2) < 0 \quad \text{for } t \geq t_2,$$

and therefore,

$$\zeta(t) \leq \zeta(t_2) + \zeta'(t_2)(t - t_2) \to -\infty$$

as $t \to \infty$. This contradicts the assumption that $\zeta(t) > 0$ for $t \geq T$. We then conclude that $\zeta'(t) > 0$ for $t \geq T$.

Consider again system (4.2) which is equivalent to (1.1) with (4.1). Let $(\xi(s), \eta(s))$ be the solution of (4.2) corresponding to $\zeta(t)$. Then

$$(\xi(s), \eta(s)) = (\zeta(e^s), \zeta'(e^s)e^s - \zeta(e^s)).$$

Since $\zeta(t) > 0$ and $\zeta'(t) > 0$ for $t \geq T$, we see that

$$(\xi(s), \eta(s)) \in R_2 \stackrel{\text{def}}{=} \big\{(u, v) \colon u > 0 \text{ and } v > -u\big\} \tag{4.16}$$

and $\dot{\xi}(s) = \zeta'(e^s)e^s > 0$ for $s \geq \log T$. Taking notice that system (4.2) has no equilibria in the region $R_2$, we also see that

$$\xi(s) \to \infty \quad \text{as } s \to \infty.$$

Hence, there exist an $s_2 > 0$ and a positive integer $m$ such that

$$\xi(s_2) = x_m. \tag{4.17}$$

For simplicity, let

$$u_1 = \xi(s_2), \qquad v_1 = \eta(s_2), \qquad B = (u_1, v_1) \in R_2$$

and consider the positive orbit $\gamma^+_{(4.2)}(B)$, which corresponds to $(\xi(s), \eta(s))$. Then, from (4.16) it follows that $\gamma^+_{(4.2)}(B)$ remains in the region $R_2$.

To compare with the positive orbit $\gamma^+_{(4.2)}(B)$, we consider the positive orbit $\gamma^+_{(4.15)}(B)$. The slopes of $\gamma^+_{(4.15)}(B)$ and $\gamma^+_{(4.2)}(B)$ at the point $B$ are

$$-\frac{g(u_1)}{v_1 + u_1}, \qquad -\frac{S_n(u_1)u_1/4}{v_1 + u_1},$$

respectively. Hence, by (4.14) and (4.17) we see that both are negative and the former is gentle than the latter. Since all nontrivial positive orbits of (4.15) go around the origin, we also see that $\gamma^+_{(4.15)}(B)$ crosses the boundary line $v = -u$ of $R_2$. Consequently, $\gamma^+_{(4.15)}(B)$ and $\gamma^+_{(4.2)}(B)$ have a point of intersection in the region $R_2$. Let $C(u_2, v_2)$ be the first intersecting point.

Positive orbits $\gamma^+_{(4.15)}(B)$ and $\gamma^+_{(4.2)}(B)$ can be regarded as the graphs of $v = \psi(u)$ and $v = \omega(u)$ which are solutions of the equations

$$\frac{dv}{du} = -\frac{g(u)}{v+u}, \qquad \frac{dv}{du} = -\frac{S_n(u)u/4}{v+u}$$

satisfying $\psi(u_1) = \omega(u_1) = v_1$, respectively. Since $\psi(u_2) = \omega(u_2) = v_2$ and $\omega(u) < \psi(u)$ for $u_1 < u < u_2$, we have

$$\begin{aligned} v_1 - v_2 &= \int_{u_1}^{u_2} \frac{g(u)}{\psi(u)+u} du \le \int_{u_1}^{u_2} \frac{S_n(u)u/4}{\psi(u)+u} du \\ &< \int_{u_1}^{u_2} \frac{S_n(u)u/4}{\omega(u)+u} du = v_1 - v_2 \end{aligned}$$

by (1.5). This is a contradiction. Thus, equation (1.1) is nonoscillatory even in the case (4.14). We have completed the proof of Theorem 1.1. □

Judging from results in Theorems A, B and our main result, it seems reasonable to infer as follows:

CONJECTURE 4.1. *Assume* (1.2) *and suppose that there exist a* $\lambda$ *with* $\lambda > 1/4$ *and a positive integer* $n$ *with* $n \geq 3$ *such that*

$$\frac{g(x)}{x} \geq \frac{1}{4} S_{n-1}(|x|) + \frac{\lambda}{\{L_n(|x|)\}^2}$$

*for* $|x|$ *sufficiently large. Then equation* (1.1) *is oscillatory.*

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