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ASYMPTOTIC PERIODIC SOLUTIONS FOR A TWO-DIMENSIONAL LINEAR DIFFERENCE SYSTEM WITH TWO DELAYS

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1. Introduction

Consider the linear delay difference system of dimension two

$$x_{n+1}-x_n+A(x_{n-\ell}+x_{n-k})=0, \quad n\in\mathbf{Z}_+=\{0,1,\cdots\}, \tag{1}$$

where A denotes a 2×2 constant real matrix and delays ℓ and k are positive integers. For convenience we assume the condition $\ell\leq k$, so that solutions of (1) are uniquely determined by $(k+1)$-initial values: $x_{-k}, x_{-k+1},\cdots, x_0\in\mathbf{R}^2$.

The system (1) is originated from the scalar difference equation

$$u_{n+1}-u_n+pu_{n-k}=0, \quad n\in\mathbf{Z}_+, \tag{2}$$

which often appears, related to some population dynamics, in mahtematical biology; a necessary and sufficient condition for the asymptotic stability of (2) was given by Levin and May [4] (see also [1; p.182], [2; p.12], [3], and [7]). Recently, the author [6] has obtained necessary and sufficient conditions for the asymptotic stability of (1), which improve the result ([4]) for (2) and also generalize those ([5]) for the system (1) with $\ell=k$. Under the assumption that the matrix A is either of the Jordan forms

$$\text{(i)}\quad p\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}, \qquad \text{(ii)}\quad \begin{pmatrix}p_1 & q\\ 0 & p_2\end{pmatrix},$$

we showed in [6] the following theorems, where p, θ, p_1, p_2 and q are all real constants and θ satisfies the condition $0<|\theta|\leq\pi/2$.

Thmorem 1.([6]) *Suppose that A is of the form* (i). *Then the system* (1) *is asymptotically stable if and only if*

$$0 < p < \frac{\sin\{(\pi/2 - |\theta|)/(\ell + k + 1)\}}{\cos\{(k - \ell)(\pi/2 - |\theta|)/(\ell + k + 1)\}}.$$

Thmorem 2.([6]) *Suppose that A is of the form* (ii). *Then the system* (1) *is asymptotically stable if and only if*

$$0 < p_1,\, p_2 < \frac{\sin\{(\pi/2(\ell + k + 1))\}}{\cos\{\pi(k - \ell)/2(\ell + k + 1)\}}.$$

Theorems 1 and 2 assert that in case (i) the stability region, with ℓ and k fixed, of the system (1) is given by the bounded set in the (θ, p)-plane:

$$S_1 = \{(\theta, p) \in \mathbf{R}^2 \,|\, 0 < p < p^*,\, 0 < |\theta| < \pi/2\},$$

and that in case (ii) it is given as the square in the (p_1, p_2)-plane:

$$S_2 = \{(p_1, p_2) \in \mathbf{R}^2 \,|\, 0 < p_1,\, p_2 < p_0^*\},$$

where p^* and p_0^* are the critical values in Theorems 1 and 2 respectively.

This papar investigates the behavior of solutions of (1) on the boundaries of the stability regions above. Even for the scalar equation (2), we can not find such kind of results. More specifically, we are concerned with solutions on

$$\Gamma_1 = \{(\theta, p) \in \partial S_1 \,|\, p = p^*\}$$

and

$$\Gamma_2 = \{(p_1, p_2) \in \partial S_2 \,|\, p_1 \text{ or } p_2 = p_0^*\},$$

corresponding to cases (i) and (ii) respectively.

We shall show that in case (i) every solution of (1) on Γ_1 is asymptotically periodic, i.e., asymptotically equivalent to some periodic solution and that this periodic solution admits an explicit expression (Theorem 3). On the other hand, in case (ii) the behavior of solutions depends on the form of the triangle matrix; if $p_1 = p_2$ and $q \neq 0$, on Γ_2 the system (1) possesses possibly unbounded solutions besides periodic solutions, and if $q = 0$, every solution of (1) on Γ_2 is asymptotically periodic. We also give explicit expressions of those periodic solutions in case (ii)

(Theorems 4 and 5). Our results particularly yield its asymptotic form for every solution of the scalar difference equation (2) in the critical case $p = 2\cos\{k\pi/(2k+1)\}$ (Corollary 1).

In the next section, we briefly discuss the system of first order, equivalent to (1), and then give the asymptotic form of each solution of the system (1). In section 3 we summarize distributions of the characteristic roots of (1) on the boundaries of stability regions. In section 4, we state our main results, giving explicit expressions of asymptotic periodic solutions of (1) for each coefficient matrix A.

2. Preliminaries

In this section we discuss the structure of solutions of the system (1). Let $\{z^n\} \subset \mathbf{R}^m$, m being $2(k+1)$, be the sequence defined by

$$z^n = {}^t({}^tz_0^n,\ {}^tz_1^n, \cdots,\ {}^tz_k^n) := {}^t({}^tx_{n-k},\ {}^tx_{n-k+1}, \cdots,\ {}^tx_n) \quad \text{for} \quad n \in \mathbf{Z}_+.$$

Then it follows from (1) that

$$z^{n+1} = {}^t({}^tx_{n+1-k}, \cdots,\ {}^tx_n,\ {}^tx_{n+1}) = {}^t({}^tz_1^n, \cdots,\ {}^tz_k^n,\ {}^t(z_k^n - A(z_0^n + z_{k-\ell}^n))),$$

so that the system (1) is equivalent to the m-dimensional system of first order:

$$z^{n+1} = \hat{A}z^n, \qquad n \in \mathbf{Z}_+, \tag{3}$$

where $\hat{A}$ is the $m \times m$ matrix of the form

$$\hat{A} = \begin{pmatrix} O & I_2 & O & \cdots & \cdots & \cdots & O \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & & & & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & O \\ O & \cdots & \cdots & \cdots & \cdots & O & I_2 \\ -A & O & \cdots & \underbrace{-A}_{k-\ell+1} & \cdots & O & I_2 \end{pmatrix}, \quad I_\ell \ : \text{ the } \ell\times\ell \text{ identity matrix.}$$

The structure of solutions of (3) is determined by the eigenvalues of the matrix $\hat{A}$. Let $\sigma(\hat{A})$ be the set of the eigenvalues of $\hat{A}$, and σ_-, σ_0 and σ_+ denote those of the eigenvalues which belong to the interior, the boundary and the exterior of the unit disk respectively. And let P :

$\mathbf{C}^m \to \bigoplus_{\lambda\in\sigma_0\cup\sigma_+} E(\lambda)$ be the projection, where $E(\lambda)$ is the generalized eigenspace of $\hat{A}$ associated with $\lambda \in \sigma(\hat{A})$. Then the solution $z^n = \hat{A}^n z^0$ of (3) with initial value z^0 is asymptotically equivalent to Pz^n in the sense that

$$\|z^n - Pz^n\| \leq C\varepsilon^n, \quad n \in \mathbf{Z}_+,$$

where ε is a constant such that $\max\{|\lambda| \mid \lambda \in \sigma_+\} < \varepsilon < 1$ and C is a positive constant depending on ε. Note that Pz^n is also a solution of (3) since P commutes with $\hat{A}$. The solution Pz^n is expressed explicitly in terms of a basis of $\bigoplus_{\lambda\in\sigma_0\cup\sigma_+} E(\lambda)$ and its dual. For this, we use the following lemma, well known in linear algebra.

Lemma 1. *Let V be an m-dimensional vector space over $\mathbf{C}$ and T a linear transformation on V. Then for eigenvalues λ, $\mu \in \sigma(T)$, the following hold:*

(i) $E^*(\lambda) \subset E(\mu)^\perp$, *for* $\lambda \neq \mu$;

(ii) $E^*(\lambda) \cap E(\lambda)^\perp = \{0\}$,

where $E^*(\lambda)$ is the generalized eigenspace of T^*, the adjoint of T, associated with $\lambda \in \sigma(T)$, i.e., $E^*(\lambda) = \bigcup_{\nu\geq 1} \operatorname{Coker}(\lambda I_m - T)^\nu$, and for a subspace $W \subset V$, $W^\perp \subset V^*$ is the subspace of covectors that vanish on W, i.e., $W^\perp = \{\psi \in V^* \mid \langle\phi, \psi\rangle = 0$ for all $\phi \in W\}$.

Now let $\{\psi_1^\lambda, \cdots, \psi_{n(\lambda)}^\lambda\}$ and $\{\phi_1^\lambda, \cdots, \phi_{n(\lambda)}^\lambda\}$ be bases of $E^*(\lambda)$ and $E(\lambda)$, the generalized eigenspaces of $\hat{A}^* = {}^t\hat{A}$ and $\hat{A}$ associated with $\lambda \in \sigma(\hat{A})$ respectively. Then the dual basis of $\{\phi_1^\lambda, \cdots, \phi_{n(\lambda)}^\lambda\}$, $\{\tilde{\psi}_1, \cdots, \tilde{\psi}_{n(\lambda)}\}$, is constructed in the following way. Assume that

$$c_1\psi_1^\lambda + \cdots + c_{n(\lambda)}\psi_{n(\lambda)}^\lambda \in E(\lambda)^\perp \tag{4}$$

with complex numbers $c_1, \cdots, c_{n(\lambda)}$. Then it immediately follows from Lemma 1 (ii) that $c_1 = \cdots = c_{n(\lambda)} = 0$. Since (4) is equivalent to

$$c_1\langle\phi_j^\lambda, \psi_1^\lambda\rangle + \cdots + c_{n(\lambda)}\langle\phi_j^\lambda, \psi_{n(\lambda)}^\lambda\rangle = 0, \quad \text{for} \quad j = 1, \cdots, n(\lambda),$$

we see that the matrix $\Psi^\lambda\Phi^\lambda = (\langle\phi_j^\lambda, \psi_i^\lambda\rangle)$ is nonsingular, where Ψ^λ and Φ^λ are $n(\lambda) \times n(\lambda)$ matrices given by

$$\Psi^\lambda = \begin{pmatrix} \psi_1^\lambda \\ \vdots \\ \psi_{n(\lambda)}^\lambda \end{pmatrix}, \qquad \Phi^\lambda = (\phi_1^\lambda, \cdots, \phi_{n(\lambda)}^\lambda).$$

The dual basis $\{\tilde{\psi}_1^\lambda, \cdots, \tilde{\psi}_{n(\lambda)}^\lambda\}$ is then obtained as

$$\tilde{\psi}_j^\lambda = (c_1^j, \cdots, c_{n(\lambda)}^j)\Psi^\lambda = c_1^j\psi_1^\lambda + \cdots + c_{n(\lambda)}^j\psi_{n(\lambda)}^\lambda, \tag{5}$$

where $(c_1^j, \cdots, c_{n(\lambda)}^j)$ is the solution of a linear equation

$$(c_1^j, \cdots, c_{n(\lambda)}^j)\Psi^\lambda\Phi^\lambda = (0, \cdots, 0, \underset{\hat{j}}{1}, 0, \cdots, 0) \in (\mathbf{C}^{n(\lambda)})^*$$

This, together with Lemma 1(i), shows that the projection P is represented, via $\{\tilde{\psi}_1^\lambda, \cdots, \tilde{\psi}_{n(\lambda)}^\lambda\}$ and $\{\phi_1^\lambda, \cdots, \phi_{n(\lambda)}^\lambda\}$, as follows

$$P = \sum_{\lambda\in\sigma_0\cup\sigma_+} \left(\phi_1^\lambda\tilde{\psi}_1^\lambda + \cdots + \phi_{n(\lambda)}^\lambda\tilde{\psi}_{n(\lambda)}^\lambda\right), \tag{6}$$

and therefore the solution Pz^n is given as

$$Pz^n = P\hat{A}^n z^0 = \sum_{\lambda\in\sigma_0\cup\sigma_+} \left(\phi_1^\lambda\tilde{\psi}_1^\lambda + \cdots + \phi_{n(\lambda)}^\lambda\tilde{\psi}_{n(\lambda)}^\lambda\right) \hat{A}^n z^0. \tag{7}$$

In particular, the solution x_n of the system (1) with initial values x_{-k}, x_{-k+1}, $\cdots, x_0$ is asymptotically equivalent to the solution $\mathrm{pr}(Pz^n)$, more precisely, $\|x_n - \mathrm{pr}(Pz^n)\|$ converges exponentially to 0 as n tends to infinity, where $\mathrm{pr} : \mathbf{C}^{m-2} \times \mathbf{C}^2 \to \mathbf{C}^2$ is the projection.

3. Characteristic roots in the critical cases.

In this section we consider the characteristic equation of (1) in the critical cases mentioned in section 1. Here the coefficient matrix A of the system (1) is assumed to be either of the forms below:

$$\text{(I)}\ p\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \text{(II)}\ \begin{pmatrix} p & 1 \\ 0 & p \end{pmatrix}, \qquad \text{(III)}\ \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}.$$

The characteristic equation of (3), or equivalently (1), is given by

$$F(\lambda) := \det(\lambda I_m - \hat{A}) = \det((\lambda^{k+1} - \lambda^k)I_2 + (\lambda^{\ell-k} + 1)A) = 0, \tag{8}$$

and its roots analysis has been done in section 2 ([6]). We summarize below the distribution of the roots of the characteristic equation (8) in the critical cases. We first consider case (I). In this case,

$$F(\lambda) = (\lambda^{k+1} - \lambda^k + pe^{i\theta}(\lambda^{\ell-k} + 1))(\lambda^{k+1} - \lambda^k + pe^{-i\theta}(\lambda^{\ell-k} + 1)).$$

When $(\theta, p) \in \Gamma_1$, we have the following lemma.

Lemma 2. *Let $p = p^*$ hold. Then the equation* (8) *has simple roots $e^{i\omega}$, $e^{-i\omega}$ on the unit circle, and the rest of the roots in the interior of the unit disk, where $\omega = (2\theta - \pi)/(\ell + k + 1)$.*

In case (II), the characteristic equation becomes

$$F(\lambda) = (\lambda^{k+1} - \lambda^k + p(\lambda^{\ell-k} + 1))^2 = 0.$$

So, when $p = p_0^*$, we have the following.

Lemma 3. *Let $p = p_0^*$ hold. Then the equation* (8) *has double roots $e^{i\omega_0}$, $e^{-i\omega_0}$ and the rest of the roots in the interior of the unit disk, where $\omega_0 = -\pi/(\ell + k + 1)$.*

And in case (III), the equation (8) is written as

$$F(\lambda) = (\lambda^{k+1} - \lambda^k + p_1(\lambda^{\ell-k} + 1))(\lambda^{k+1} - \lambda^k + p_2(\lambda^{\ell-k} + 1)) = 0.$$

Particularly when $(p_1, p_2) \in \Gamma_2$, we get:

Lemma 4. *Let $(p_1, p_2) \in \Gamma_2$. Then the following hold.*
(a) *If $p_1 = p_0^*$ and $0 < p_2 < p_0^*$, or $0 < p_1 < p_0^*$ and $p_2 = p_0^*$, the equation* (8) *has simple roots $e^{i\omega_0}$, $e^{-i\omega_0}$, and the rest of the roots in the interior of the unit disk.*
(b) *If $p_1 = p_2 = 2\cos\{k\pi/(2k+1)\}$, the equation* (8) *has double roots $e^{i\omega_0}$, $e^{-i\omega_0}$ with the rest of the roots in the interior of the unit disk.*

Thus we see that, on the boundaries of stability regions, $\sigma_0 = \{e^{i\omega}, e^{-i\omega}\}$ and $\sigma_+ = \emptyset$ for case (I), and that $\sigma_0 = \{e^{i\omega_0}, e^{-i\omega_0}\}$ and $\sigma_+ = \emptyset$ for cases (II) and (III). In the next section we shall give explicit expressions of asymptotic periodic solutions of (1) for each coefficient matrix A.

4. Explicit expressions of asymptotic periodic solutions.

Based on the results in sections 2 and 3, we can obtain explicit expessions of asymptotic periodic solutions of the system (1) in the critical cases. In case (I) we have the next theorem.

Theorem 3. *Suppose that $0 < |\theta| < \pi/2$ and $p = p^*$ hold. Then the solution x_n of (1) with initial values $x_{-k}, x_{-k+1}, \cdots, x_0$ satisfies*

$$x_n \to R(n\omega)\sum_{j=0}^{k} K_1(j)x_{-j} \quad \textit{exponentially as} \quad n \to \infty,$$

where

$$K_1(j) = \begin{cases} \left(I_2 + (\ell R(k\omega) + kR(\ell\omega))\,{}^tA\right)^{-1}, & j = 0; \\ \left(I_2 + (\ell R(k\omega) + kR(\ell\omega))\,{}^tA\right)^{-1} \cdot \ (R((\ell+j)\omega) + R((k+j)\omega)), & \\ & j = 1, \cdots, \ell; \\ \left(I_2 + (\ell R(k\omega) + kR(\ell\omega))\,{}^tA\right)^{-1} \cdot R((\ell+j)\omega), & j = \ell+1, \cdots, k. \end{cases}$$

and $R(\alpha)$ denotes the matrix $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$.

Proof. By Lemma 2, in this case, $\sigma_+ = \emptyset$ and the roots on the unit circle of the equation (8) are $\lambda := e^{i\omega}$ and its conjugate $\overline{\lambda}$, which are simple. Let ϕ_1 and ψ_1 be an eigenvector and an eigen-covector of $\hat{A}$ associated with λ. Direct calculations show that ϕ_1 and ψ_1 are given by

$$\phi_1 = \begin{pmatrix} \lambda^{-k}\begin{pmatrix} 1 \\ -i \end{pmatrix} \\ \vdots \\ \lambda^{-1}\begin{pmatrix} 1 \\ -i \end{pmatrix} \\ \begin{pmatrix} 1 \\ -i \end{pmatrix} \end{pmatrix}, \tag{9}$$

and

$$\psi_1 = \Big(\lambda^{\ell+k}p^*e^{-i\theta}(1,\,i),\ \cdots,\ \lambda^{2\ell+1}p^*e^{-i\theta}(1,\,i),\ (\lambda^\ell + \lambda^k)\lambda^\ell p^*e^{-i\theta}(1,\,i), \\ \cdots, (\lambda^\ell + \lambda^k)\lambda p^*e^{-i\theta}(1,\,i),\ (1,\,i)\Big) \tag{10}$$

Note that an eigenvector (an eigen-covector resp.) associated with $\overline{\lambda}$ is given by $\overline{\phi_1}$ ($\overline{\psi_1}$ resp.) since $\hat{A}$ is real. So $\{\phi_1, \overline{\phi_1}\}$ is a basis of $E(\lambda)\oplus E(\overline{\lambda})$

and, from (5), its dual is given by $\{\tilde{\psi}_1, \overline{\tilde{\psi}_1}\}$, where $\tilde{\psi}_1$ is defined by

$$\tilde{\psi}_1 = \frac{1}{\langle \phi_1, \psi_1 \rangle} \psi_1 = \frac{1}{2(1 + (\ell\lambda^k + k\lambda^\ell)p^* e^{-i\theta})} \psi_1.$$

It follows from (6) and (7) that the projection $P : \mathbf{C}^m \to E(\lambda) \oplus E(\overline{\lambda})$ is obtained as

$$P = \phi_1 \tilde{\psi}_1 + \overline{\phi_1}\, \overline{\tilde{\psi}_1},$$

and that

$$Pz^n = \lambda^n \phi_1 \tilde{\psi}_1 + \overline{\lambda}^n\, \overline{\phi_1}\, \overline{\tilde{\psi}_1} z^0 = 2\left\{ \mathrm{Re}(\lambda^n \phi_1 \tilde{\psi}_1) \right\} z^0.$$

So the solution x_n is asymptotically equivalent to x_n^* given by

$$\begin{aligned} x_n^* &:= \mathrm{pr}(Pz^n) \\ &= \mathrm{Re}\Bigg[\frac{\lambda^n}{1 + (\ell\lambda^k + k\lambda^\ell)p^* e^{-i\theta}} \Big\{ p^* e^{-i\theta} \Big(\lambda^{\ell+k} K z_0^0 + \cdots + \lambda^{2\ell+1} K z_{k-\ell-1}^0 \\ &\qquad + (\lambda^\ell + \lambda^k)\lambda^\ell K z_{k-\ell}^0 + \cdots + (\lambda^\ell + \lambda^k)\lambda K z_{k-1}^0 \Big) + K z_k^0 \Big\} \Bigg], \end{aligned}$$

where K is a 2×2 matrix defined by $\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$.

Now let $z_j^0 = {}^t(\xi_j, \eta_j)$ and $\zeta_j = \xi_j + i\eta_j$ for $j = 0, \cdots, k$. Note that

$$K z_j^0 = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} \xi_j \\ \eta_j \end{pmatrix} = \begin{pmatrix} \zeta_j \\ -i\zeta_j \end{pmatrix}.$$

It then follows that

$$\begin{aligned} x_n^* &= \mathrm{Re}\Bigg[\frac{\lambda^n}{1 + (\ell\lambda^k + k\lambda^\ell)p^* e^{-i\theta}} \Big\{ p^* e^{-i\theta} \Big(\sum_{j=0}^{k-1} \lambda^{\ell+k-j} \begin{pmatrix} \zeta_j \\ -i\zeta_j \end{pmatrix} \\ &\qquad + \sum_{j=0}^{\ell-1} \lambda^{\ell+k-j} \begin{pmatrix} \zeta_{k-\ell+j} \\ -i\zeta_{k-\ell+j} \end{pmatrix} \Big) + \begin{pmatrix} \zeta_k \\ -i\zeta_k \end{pmatrix} \Big\} \Bigg] \\ &= \begin{pmatrix} \mathrm{Re}\,\zeta \\ \mathrm{Im}\,\zeta \end{pmatrix}, \end{aligned}$$

where ζ is the complex number

$$\frac{\lambda^n}{1+(\ell\lambda^k+k\lambda^\ell)p^*e^{-i\theta}}\Big\{p^*e^{-i\theta}\Big(\sum_{j=0}^{k-1}\lambda^{\ell+k-j}\zeta_j+\sum_{j=0}^{\ell-1}\lambda^{\ell+k-j}\zeta_{k-\ell+j}\Big)+\zeta_k\Big\}.$$

From the real representation of $\mathbf{C}$, x_n^* is written as

$$\rho(\lambda)^n\Big(\rho(1)+(\ell\rho(\lambda)^k+k\rho(\lambda)^\ell)\rho(p^*e^{-i\theta})\Big)^{-1}$$
$$\Big\{\rho(p^*e^{-i\theta})\Big(\sum_{j=0}^{k-1}\rho(\lambda)^{\ell+k-j}z_j^0+\sum_{j=0}^{\ell-1}\rho(\lambda)^{\ell+k-j}z_{k-\ell+j}^0\Big)+z_k\Big\},$$

where $\rho:\mathbf{C}\setminus\{0\}\to\mathrm{GL}(2,\mathbf{R})$ sends a complex number $\alpha+i\beta$ into the matrix $\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix}$. Since $\rho(p^*e^{-i\theta})={}^tA$, so

$$x_n^*=R(n\omega)\Big(I_2+\big(\ell R(k\omega)+kR(\ell\omega)\big)\,{}^tA\Big)^{-1}$$
$$\Big[z_k^0+{}^tA\Big(\sum_{j=0}^{k-1}R((\ell+k-j)\omega)z_j^0+\sum_{j=0}^{\ell-1}R((\ell+k-j)\omega)z_{k-\ell+j}^0\Big)\Big],$$

obtaining the proof of Theorem 3.

Remark 1. If $\omega/\pi=(2\theta/\pi-1)/(2k+1)$, hence θ/π, is rational, then x_n^* is a periodic solution of (1); otherwise the ω-limit set of x_n^*, and therefore of x_n, is the circle at the center 0 with radius $\|\sum_{j=0}^k K_1(j)x_{-j}\|$.

We next consider cases (II) and (III), and simply give the statement of the results without proofs. In what follows we use the notation E_{ij}, meaning the 2×2 matrix with its (i,j)-component 1 and the others 0. In case (II) we get:

Theorem 4. *Suppose that $p=p_0^*$ in case* (II). *Then the solution x_n of* (1) *with initial values $x_{-k}, x_{-k+1},\cdots,x_0$ satisfies the following.*
(i) *If the second component of $x_0+p_0^*\big(\sum_{j=1}^{\ell}\lambda_0^{k+j}x_{-j}+\sum_{j=1}^{k}\lambda_0^{\ell+j}x_{-j}\big)$ is nonzero, then x_n diverges as $n\to\infty$, where $\lambda_0=e^{i\omega_0}$;*
(ii) *If the second component of $x_0+p_0^*\big(\sum_{j=1}^{\ell}\lambda_0^{k+j}x_{-j}+\sum_{j=1}^{k}\lambda_0^{\ell+j}x_{-j}\big)$ is*

$$x_n \to E_{11}R(n\omega_0)\sum_{j=0}^{k}K_2^{(1)}(j)x_{-j} + E_{22}R(n\omega_0)\sum_{j=0}^{k}K_2^{(2)}(j)x_{-j},$$

exponentially as $n \to \infty$, where

$$K_2^{(1)}(j) = \begin{cases} D\Big\{E_{11} + kD\Big(R(\ell\omega_0) + R(k\omega_0)\Big)E_{12}\Big\}, \quad j = 0; \\ D\Big(R(\ell\omega_0) + R(k\omega_0)\Big)R(j\omega_0) \\ \Big[p_0^* E_{11} + D\Big\{I_2 + p_0^*(k+j-1)\Big(R(\ell\omega_0) + R(k\omega_0)\Big)\Big\}E_{12}\Big], \\ \qquad\qquad j = 1, \cdots, \ell; \\ DR(j\omega_0)\Big[p_0^* R(\ell\omega_0)E_{11} + R(k\omega_0)E_{12} \\ +D\Big\{I_2 + p_0^*\Big((j-1)I_2 + k\Big(R(\ell\omega_0) + R(k\omega_0)\Big)R(\ell\omega_0) \\ +\ell R(k\omega_0)\Big)\Big\}E_{12}\Big], \qquad j = \ell+1, \cdots, k; \end{cases}$$

$$K_2^{(2)}(j) = \begin{cases} D^2 R(\omega_0)\Big\{I_2 + (p_0^* k - k + \ell)R(k\omega_0) + p_0^* kR(\ell\omega_0)\Big\}E_{22}, \\ \qquad\qquad j = 0; \\ p_0^* D^2\Big\{I_2 + (p_0^* k - k + \ell)R(k\omega_0) + p_0^* kR(\ell\omega_0)\Big\} \\ \Big(R(\ell\omega_0) + R(k\omega_0)\Big)R(j\omega_0)E_{22}, \qquad j = 1, \cdots, \ell; \\ p_0^* D^2\Big\{I_2 + (p_0^* k - k + \ell)R(k\omega_0) + p_0^* kR(\ell\omega_0)\Big\}R((\ell+j)\omega_0)E_{22}, \\ \qquad\qquad j = \ell+1, \cdots, k, \end{cases}$$

with

$$D = \Big(I_2 + p_0^*(\ell R(k\omega_0) + kR(\ell\omega_0))\Big)^{-1}.$$

Remark 2. Since ω_0/π is rational, x_n^* is a periodic solution. So, in case (II) with $p = p_0^*$, the system (1) has both unbounded solutions and asymptotic periodic solutions.

The following theorem gives explicitly asymptotic periodic solutions arised in case (III).

Theorem 5. *Suppose that $(p_1, p_2) \in \Gamma_2$ in case* (III). *Then the solution x_n, starting from initial values $x_{-k}, x_{-k+1}, \cdots, x_0$, satisfies the following:*
(i) *If $p_1 = p_0^*$ and $0 < p_2 < p_0^*$, then*

$$x_n \to 2E_{11}R(n\omega_0)\sum_{j=0}^{k} K_3(j)E_{11}x_{-j}, \text{ exponentially as } n \to \infty,$$

$$K_3(j) = \begin{cases} D, & j = 0; \\ p_0^* D\big\{R((k+j)\omega_0) + R((\ell+j)\omega_0)\big\}, & j = 1, \cdots, \ell; \\ p_0^* D R((\ell+j)\omega_0), & j = \ell+1 \cdots, k; \end{cases}$$

(ii) *If* $0 < p_1 < p_0^*$ *and* $p_2 = p_0^*$, *then*

$$x_n \to 2E_{22}R(n\omega_0)\sum_{j=0}^{k} K_3(j)E_{22}x_{-j}, \text{ exponentially as } n \to \infty;$$

(iii) *If* $p_1 = p_2 = p_0^*$, *then*

$$x_n \to 2R(n\omega_0)\sum_{j=0}^{k} K_3(j)x_{-j}, \text{ exponentially as } n \to \infty.$$

As an immedieate consequence of Theorem 5 with $\ell = k$, we have the following Corollary, obtaining asymptotic periodic solutions of the scalar equation (2) arised in the critical case.

Corollary 1. *Let* $p = 2\cos\{k\pi/(sk+1)\}$, *the critical value for the asymptotic stability of* (2) ([4,5]), *hold in the equation* (2). *Then the solution* u_n *with initial values* $u_{-k}, u_{-k+1}, \cdots, u_0 \in \mathbf{R}$ *satisfies*

$$u_n \to \frac{2}{1 + p^2k^2 + 2pk\cos k\omega_0}\Big\{(\cos(n-k)\omega_0)u_0 + \sum_{j=1}^{k}\Big(\cos(n+k+j)\omega_0 + pk\cos(n+j)\omega_0\Big)u_{-j}\Big\},$$

exponentially as $n \to \infty$.

References

[1] S. N. Elaydi, "An Introduction to Difference Equations", Springer-Verlag, New York, 1996.

[2] V. L. Kocic and G. Ladas, "Global Behavior of Nonlinear Difference Equations of Higher Order with Applications", Kluwer Academic Publishers, Dordrecht, 1993.

[3] S. A. Kuruklis, The asymptotic stability of $x_{n+1} - ax_n + bx_{n-k} = 0$, *J. Math. Anal. Appl.*, **188**, (1994), 719–731.

[4] S. A. Levin and R. M. May, A note on difference-delay equations, *Theor. Popul. Biol.*, **9**, (1976), 178–187.

[5] H. Matsunaga and T. Hara, The asymptotic stability of a two-dimensional linear delay difference equation, *Dynam. Contin. Discrete Impuls. Systems*, **6**, (1999), 465–473.

[6] Y. Nagabuchi, Asymptotic stability for a linear difference system with two delays, *Nonlinear Anal.*, to appear.

[7] V. G. Papanicolaou, On the asymptotic stability of a class of linear difference equations, *Math. Magagine*, **69**, (1996), 34–43.