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ファジィ微分方程式の解の吸引性について

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1 Introduction

There are many fruitful results on the representations of fuzzy numbers, differentials and integrals of fuzzy functions (see, e.g., in Goetschel-Voxman [8, 9], Dubois-Prade [3, 4, 5, 6], Puri-Ralescue [13], Furukawa [7], Kaleva [10, 11] etc). They establish fundamental results concerning differentials, integrals and fuzzy differential equations of fuzzy functions which map $\mathbf{R}$ to a set of fuzzy numbers. By using the results it seems to be difficult to apply all the practical and significant problems. In this studying we introduce the couple parametric representation(see [2]) corresponding to the representation of fuzzy numbers due to Goetschel-Voxman so that it is easy to solve fuzzy differential equations.

In Buckley [1], Kaleva [10, 11], Park [12] and Song [16], various types of conditions for the existence and uniqueness of solutions to fuzzy differential equations. By the couple representation some kinds of differential and integral of fuzzy functions can be easily treated in an analogous way with the real analysis as well as some type of fuzzy differential equations can be solved without difficulty.

In Section 2 we denote a fuzzy number x by (a, b), where a, b are endpoints of $\alpha-$cut set of the membership function μ_x. We give some kind of metric space which includes the set of fuzzy numbers as well as prove the continuity of a, b. In Section 3 we give definitions of differential and integral of fuzzy functions and sufficient conditions for fuzzy functions to be differentiable or integrable. In Section 4 we get basic results of existence and uniqueness of solutions for fuzzy differential equations by applying the contraction principle. In Section 5 we treat a fuzzy differential equation $x' = p(t)x$,where $p(t)$ is a fuzzy valued function, and we calculate the the exponential function e^x, where x is a fuzzy number. In the section we show the attractivity set, where all the solutions are approaching to the zero as the time increasing the infinity.

2 Parametric Representation of Fuzzy Numbers

In order to introduce a metric space which includes the set of fuzzy numbers, we define the following set.

$$X = \{x = (a, b) \in C(I) \times C(I)\}$$

where $I = [0, 1]$ and $C(I)$ is the set of continuous functions from I to $\mathbf{R}$. Denote a metric by $d(x, y) = \sup_{\alpha \in I}(|a(\alpha) - c(\alpha)| + |b(\alpha) - d(\alpha)|)$ for $x = (a, b), y = (c, d) \in X$. Then the metric space (X, d) is complete.

Definition 1 *Consider a set of fuzzy numbers with bounded supports as follows:*

$\mathcal{F}_{\mathbf{b}}^{st} = \{\mu : \mathbf{R} \to I$ *satisfying (i) – (iv) below.*$\}$

(i) There exist a unique $m \in \mathbf{R}$ *such that* $\mu(m) = 1$;

(ii) The set $supp(\mu) = cl(\{\xi \in \mathbf{R} : \mu(\xi) > 0\})$ *is bounded in* $\mathbf{R}$;

(iii) One of the following conditions holds;
(a) μ *is strictly fuzzy convex, i.e.,*

$$\mu(c\xi_1 + (1 - c)\xi_2) > \min[\mu(\xi_1), \mu(\xi_2)]$$

for $\xi_1, \xi_2 \in \mathbf{R}, 0 < c < 1$;
(b) $\mu(m) = 1$ *and* $\mu(\xi) = 0$ *for* $\xi \neq m$;

(iv) μ *is upper semi-continuous on* $\mathbf{R}$.

Remark 1 *The above condition (iiia) is stronger than one in the usual case. It follows that* $\mu(\xi)$ *is strictly increasing in* $\xi \in (-\infty, m)$ *and strictly decreasing in* $\xi \in (m, \infty)$. *This condition plays an important role in the proof of Theorem 1.*

We introduce the following parametric representation of $\mu \in \mathcal{F}_{\mathbf{b}}^{st}$,

$$\begin{aligned} a(\alpha) &= \min L_\alpha(\mu), \\ b(\alpha) &= \max L_\alpha(\mu) \end{aligned}$$

for $0 < \alpha \leq 1$ and

$$\begin{aligned} L_\alpha(\mu) &= \{\xi \in \mathbf{R} : \mu(\xi) \geq \alpha\}, \\ a(0) &= \min cl(supp(\mu)), \\ b(0) &= \max cl(supp(\mu)). \end{aligned}$$

See Figure 1.

Remark 2 *From the extension principle of Zadeh, it follows that*

$$\begin{aligned} & \mu_{x+y}(\xi) \\ = & \max_{\xi = \xi_1 + \xi_2} \min_{i=1,2} (\mu_i(\xi_i))) \\ = & \max\{\alpha \in I : \xi = \xi_1 + \xi_2, \xi_i \in L_\alpha(\mu_i)\} \\ = & \max\{\alpha \in I : \xi \in [a(\alpha) + c(\alpha), b(\alpha) + d(\alpha)]\}, \end{aligned}$$

where μ_1, μ_2 *are membership functions of* x, y, *respectively. Thus we get* $x + y = (a + c, b + d)$.

The following theorem is a basic result.

Theorem 1 *Denote* $\mu = (a, b) \in \mathcal{F}_{\mathbf{b}}^{st}$, *where* $a, b : I \to \mathbf{R}$. *The following properties(i)-(iii) hold:*

(i) a, b *are continuous on* I;

(ii) $\max a(\alpha) = a(1) = m$ *and* $\min b(\alpha) = b(1) = m$;

(iii) One of the following statements holds;
(a) a *is strictly increasing and* b *is strictly decreasing with* $a(\alpha) < b(\alpha)$;
(b) $a(\alpha) = b(\alpha) = m$ *for* $0 < \alpha < 1$.

Conversely, under the above conditions (i) -(iii), if we denote

$$\mu(\xi) = \sup\{\alpha \in I : a(\alpha) \le \xi \le b(\alpha)\}$$

then $\mu \in \mathcal{F}_{\mathbf{b}}^{st}$. *Moreover it follows that* $\mathbf{R} \subset \mathcal{F}_{\mathbf{b}}^{st}$ *and that* $\mathcal{F}_{\mathbf{b}}^{st}$ *is a closed convex cone in* X.

In the following example we illustrate typical three types of fuzzy numbers.

Example 1 *Consider the following* $L-R$ *fuzzy number* $x \in \mathcal{F}_{\mathbf{b}}^{st}$ *with a membership function as follows:*

$$\mu_x(\xi) = \begin{cases} L(\frac{m-\xi}{l})_+ & \text{for } \xi \le m \\ R(\frac{\xi-m}{r})_+ & \text{for } \xi > m \end{cases}$$

where $m \in \mathbf{R}, l > 0, r > 0$. L, R *are into mappings defined on* $\mathbf{R}^+ = [0, \infty)$. *Let* $L(\xi)_+ = \max(L(\xi), 0)$ *etc. We identify* μ_x *with* $x = (a, b)$ *Then we have* $a(\alpha) = m - L^{-1}(\alpha)l$ *and* $b(\alpha) = m + R^{-1}(\alpha)r$ *provided that* L^{-1} *and* R^{-1} *exist.*

Let $L(\xi) = -c_1\xi + 1$, *where* $c_1 > 0$. *We illustrate the following cases (a)-(c).*

(i) Let $R(\xi) = -c_2\xi + 1$, *where* $c_2 > 0$. *Then* $c_2 l(b - m) = c_1 r(m - a)$.

(ii) Let $R(\xi) = -c_2\sqrt{\xi}+1$, *where* $c_2 > 0$. *Then* $c_2 l(b - m)^2 = c_1 r^2(m - a)$.

(iii) Let $R(\xi) = -c_2\xi^2 + 1$, *where* $c_2 > 0$. *Then* $c_2^2 l^2(b - m) = c_1^2 r(a - m)^2$.

See Figure 2.

3 Differential and Integral of Fuzzy Valued Functions

Let an interval $J \subset \mathbf{R}$. We call a function $x : J \to \mathcal{F}_{\mathbf{b}}^{st}$ to be a fuzzy valued function. Denote

$$\begin{aligned} x(t) &= (a(t), b(t)) \\ &= \{(a(t,\alpha), b(t,\alpha))^T \in \mathbf{R}^2 : \alpha \in I\}. \end{aligned}$$

We define the continuiety and differentiability of fuzzy valued function as follows:

Definition 2 *A fuzzy valued function* $x : J \to \mathcal{F}_{\mathbf{b}}^{st}$ *is continuous at* $t \in J$ *if*

$$\lim_{h \to 0} d(x(t + h), x(t)) = 0.$$

Denote the set of all the continuous functions $x : J \to \mathcal{F}_{\mathbf{b}}^{st}$ *by* $\mathcal{C}(J)$.

Let the function $x : J \to \mathcal{F}_{\mathbf{b}}^{st}$ *by*

$$\begin{aligned} x(t) &= \{(a(t,\alpha), b(t,\alpha))^T \in \mathbf{R}^2 : \alpha \in I\} \\ &= (a(t,\cdot), b(t,\cdot)) = x(t,\cdot) \end{aligned}$$

for $t \in J$. *The function* x *is differentiable at* $t \in J$ *if for any* $\alpha \in I$ *there exist* $\frac{\partial a}{\partial t}(t,\alpha), \frac{\partial b}{\partial t}(t,\alpha)$ *such that* $\frac{\partial a}{\partial t}(t,\alpha) \le \frac{\partial b}{\partial t}(t,\alpha)$ *and* $\mu(t,\cdot) \in \mathcal{F}_{\mathbf{b}}^{st}$, *where* $\mu(t,\xi) = \sup\{\alpha \in I : \frac{\partial a}{\partial t}(t,\alpha) \le \xi \le \frac{\partial b}{\partial t}(t,\alpha)\}$. *The function* x *is differentiable on* J *if* x *is differentiable at any* $t \in J$. *Denote* $\frac{dx}{dt}(t) = x'(t) = (\frac{\partial a}{\partial t}(t,\cdot), \frac{\partial b}{\partial t}(t,\cdot))$ *and it is called to be the derivative of* $x(t)$.

We define an integral of an $\mathcal{F}_{\mathbf{b}}^{st}-$ valued function x.

Definition 3 *Let $x : J \to \mathcal{F}_{\mathbf{b}}^{st}$ be $x(t,\cdot) = (a(t,\cdot), b(t,\cdot))$ for $t \in J$. The function x is called to be integrable over $[t_1, t_2]$, since a, b are Riemann integrable over $[t_1, t_2]$. Then we define the integral as follows:*

$$\int_{t_1}^{t_2} x(s,\cdot)ds$$
$$= \{(\int_{t_1}^{t_2} a(s,\alpha)ds, \int_{t_1}^{t_2} b(s,\alpha)ds)^T \in \mathbf{R}^2 : \alpha \in I\}.$$

Remark 3 *Let $x(t) = (a(t,\cdot), b(t,\cdot)) \in \mathcal{F}_{\mathbf{b}}^{st}$ for $t \in J$.*

(i) If x is differentiable at t, we get the integral over $[t_1, t_2] \subset J$ as follows:

$$\int_{t_1}^{t_2} x'(s,\cdot)ds = x(t_2,\cdot) - x(t_1,\cdot).$$

(ii) If $x(t) \in \mathcal{F}_{\mathbf{b}}^{st}$ is integrable over $[t_1, t_2]$, then we have $\int_{t_1}^{t_2} x(s,\cdot)ds \in \mathcal{F}_{\mathbf{b}}^{st}$. And also we have

$$d(\int_{t_1}^{t_2} x(s,\cdot)ds, 0) \le \int_{t_1}^{t_2} d(x(s,\cdot), 0)ds.$$

4 Fuzzy Differential Equation I

Consider an initial value problem of a differential equation in the metric space X as follows:

$$x'(t) = f(t,x), \quad x(t_0) = x_0 \qquad (N)$$

where $t_0 \in \mathbf{R}, x_0 \in \mathcal{F}_{\mathbf{b}}^{st}$. Let $f : J_c \times X \to X$, where $J_c = [t_0, t_0 + c], c > 0$. We call the equation of (N) to be a fuzzy differential equation if $f(t,x)$ is a fuzzy valued function on a subset of $J_c \times X$. Moreover we assume that the following assumption. Denote the unit ball by $\mathcal{B} = \{x \in \mathcal{F}_{\mathbf{b}}^{st} : d(x,0) \le 1\}$. Define $r\mathcal{B} = \{rx \in \mathcal{F}_{\mathbf{b}}^{st} : x \in \mathcal{B}\}$ for $r > 0$.

Assumption (A) Let $r > 0$ and

$$\mathcal{B}(x_0, r) = x_0 + r\mathcal{B}.$$

The following conditions (i) and (ii) are satisfied.

(i) It follows that $f(t,x) \in \mathcal{F}_{\mathbf{b}}^{st}$ for any $(t,x) \in J_c \times \mathcal{B}(x_0,r)$, i.e., for any $x = \{(a(\alpha), b(\alpha)) : \alpha \in I\} \in \mathcal{F}_{\mathbf{b}}^{st}, t \in J_c$ the following properties (a)-(c) hold for $t \in J$:

  (a) $f_i(t, (a(\alpha), b(\alpha)), \alpha), i = 1, 2$, are contiunous in α;

  (b) For each $t \in J$ there exists a unique value $M(t) \in \mathbf{R}$ such that

$$\max_{\alpha} f_1(t, (a(\alpha), b(\alpha)), \alpha)$$
$$= f_1(t, (a(1), b(1)), 1) = M(t);$$
$$\min_{\alpha} f_2(t, (a(\alpha), b(\alpha)), \alpha)$$
$$= f_2(t, (a(1), b(1)), 1) = M(t);$$

  (c) One of the following statements holds;

  (a) $f_1(t, (a(\alpha), b(\alpha)), \alpha)$ is strictly increasing in α and $f_2(t, (a(\alpha), b(\alpha)), \alpha)$ is strictly decreasing in α;

  (b) $f_1(t, (a(\alpha), b(\alpha)), \alpha) = f_2(t, (a(\alpha), b(\alpha)), \alpha)$ for $0 < \alpha < 1$.

(ii) Function $f(t,x)$ is continuous on $(t,x) \in J_c \times \mathcal{B}(x_0, r)$.

In the same way in the theory of ordinary differential equations we give the following definition of solutions for initial value problems of fuzzy differential equations.

Definition 4 *Let J_1 be an interval in* $\mathbf{R}$ *and $t_0 \in J_1$. A function $x : J_1 \to \mathcal{F}_{\mathbf{b}}^{st}$ is a solution of (N) on J_1,if x satisfies the following conditions (i)-(iii).*

(i) $x(t_0) = x_0$;

(ii) $x(t) \in \mathcal{F}_{\mathbf{b}}^{st}$ *for* $t \in J_1$;

(iii) There exists $x'(t)$ and $x'(t) = f(t, x(t))$ for $t \in J_1$.

By applying the contraction principle we get the following theorem.

Theorem 2 *Suppose that the following conditions (i) and (ii) are satisfied under Assumption (A):*

(i) f is bounded, i.e., there exists an $M > 0$ such that

$$d(f(t,x),0) \leq M$$

for $(t, x) \in J_c \times \mathcal{B}(x_0, r)$;

(ii) f is Lipschitzian in x ,i.e., there exists an $L > 0$ such that

$$d(f(t,x), f(t,y)) \leq Ld(x,y)$$

for $(t,x),(t,y) \in J_c \times \mathcal{B}(x_0,r)$.

Then there exists a unique solution x for (N) such that

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s, \cdot))ds$$

for $t \in J_\rho = [t_0, t_0+\rho]$, where $\rho = \min(c, r/M)$.

We illustrate the above theorem by applying it to the following example.

Example 2 *Consider the following problem of fuzzy differential equation*

$$x' = p(t)x + q(t), \quad x(t_0) = x_0 \qquad (E)$$

$t \in \mathbf{R}, x_0, x(t) \in \mathcal{F}_{\mathbf{b}}^{st}$. Functions $p, q : \mathbf{R} \to \mathbf{R}$ are continuous.

Let $f(t,x) = p(t)x + q(t)$ and $p(t) \geq 0$. Since

$$\begin{aligned} d(f(t,x),0) &\leq p(t)d(x,0) + |q(t)|, \\ f(t,x) &\in f(t,y) + p(t)d(x,y)\mathcal{B}, \end{aligned}$$

it follows that f is bounded and Lipschitzian in x. From Theorem 2 we have a unique solution of (E) such that

$$x(t) = e^{\int_{t_0}^{t} p(s)ds} x_0 + \int_{t_0}^{t} e^{\int_s^t p(r)dr} q(s)ds$$

for $t \in \mathbf{R}$.

Let $p : \mathbf{R} \to (-\infty, 0]$ and $x(t) = (x_1(t), x_2(t))$. Then we have $x_1'(t) = p(t)x_2(t) + q(t), x_2' = p(t)x_1(t) + q(t)$, by denoting $x_0 = (a_0, b_0)$, so $x_1(t,\alpha)$ and $x_2(t,\alpha)$ satisfy

$$\begin{pmatrix} x_1(t,\alpha) \\ x_2(t,\alpha) \end{pmatrix} = \Phi(t,\alpha) \begin{pmatrix} a_0(t,\alpha) \\ b_0(t,\alpha) \end{pmatrix}$$
$$+\Phi(t,\alpha) \int_{t_0}^{t} \Phi^{-1}(s,\alpha) \begin{pmatrix} q(s,\alpha) \\ q(s,\alpha) \end{pmatrix} ds,$$

where $\Phi(\cdot,\cdot)$ is a fundamental matrix of

$$\begin{aligned} &\frac{d}{dt}(x_1(t,\alpha), x_2(t,\alpha))^T \\ &= (p(t,\alpha)x_2(t,\alpha), p(t,\alpha)x_1(t,\alpha))^T \end{aligned}$$

,i.e.,

$$\begin{aligned} \Phi(t,\alpha) &= \begin{pmatrix} \phi_{11}(t,\alpha) & \phi_{12}(t,\alpha) \\ \phi_{21}(t,\alpha) & \phi_{22}(t,\alpha) \end{pmatrix} \\ \phi_{11}(t,\alpha) &= \frac{e^{\int_{t_0}^{t} p(s,\alpha)ds} + e^{-\int_{t_0}^{t} p(s,\alpha)ds}}{2} \end{aligned}$$

$$\phi_{12}(t,\alpha) = \frac{e^{\int_{t_0}^{t} p(s,\alpha)ds} - e^{-\int_{t_0}^{t} p(s,\alpha)ds}}{2}$$
$$\phi_{21}(t,\alpha) = \frac{e^{\int_{t_0}^{t} p(s,\alpha)ds} - e^{-\int_{t_0}^{t} p(s,\alpha)ds}}{2}$$
$$\phi_{22}(t,\alpha) = \frac{e^{\int_{t_0}^{t} p(s,\alpha)ds} + e^{-\int_{t_0}^{t} p(s,\alpha)ds}}{2}$$

for $t \geq t_0, \alpha \in I$.

5 Fuzzy Differential Equation II

We consider the exponential function $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$, where $x \in \mathcal{F}_{\mathbf{b}}^{st}$, in the similar to the real analysis.

Let $x = (a, b)$, then $\alpha-$cut sets of membership function μ_x is $L_\alpha(\mu_x) = [a(\alpha), b(\alpha)]$ for $\alpha \in I$. We calculate $x^i, i = 2, 3, \cdots$, in the following cases $(a) - (c)$. By the extension principle of Zadeh, it follows that

$$\begin{aligned} & \mu_{x^i}(\xi) \\ = & \max_{\xi = \Pi_j \xi_j} \min_{1 \leq j \leq i} (\mu(\xi_j)) \\ = & \max\{\alpha \in I : \xi = \Pi_j \xi_j, \xi_j \in L_\alpha(\mu_x)\}. \end{aligned}$$

When $i = 2$, we find the following relations.

(a) If $0 \leq a(\alpha) \leq b(\alpha)$, then we get

$$L_\alpha(\mu_{x^2}) = [a(\alpha)^2, b(\alpha)^2],$$

thus $x^2 = (a^2, b^2) \in \mathcal{F}_{\mathbf{b}}^{st}$.

(b) If $a(\alpha) \leq b(\alpha) \leq 0$, then we have

$$L_\alpha(\mu_{x^2}) = [b(\alpha)^2, a(\alpha)^2],$$

thus $x^2 = (b^2, a^2) \in \mathcal{F}_{\mathbf{b}}^{st}$.

(c) If $a(\alpha) \leq 0 \leq b(\alpha)$, then it follows that

$$L_\alpha(\mu_{x^2}) = [a(\alpha)b(\alpha), c(\alpha)]$$

where $c(\alpha) = \max(a(\alpha)^2, b(\alpha)^2)$. Without loss of generality we denote μ_x by the membership function in *Example*1. Denote

$$\begin{aligned} L^{-1}(\alpha + h) &= L^{-1}(\alpha) + \Delta_1, \\ R^{-1}(\alpha + h) &= R^{-1}(\alpha) + \Delta_2, \end{aligned}$$

where $\Delta_1 < 0, \Delta_2 < 0$ for $h > 0$. Then we have

$$\begin{aligned} & a(\alpha + h)b(\alpha + h) - a(\alpha)b(\alpha) \\ = \ & a(\alpha)\Delta_2 r - b(\alpha)\Delta_1 l - \Delta_1\Delta_2 lr > 0. \end{aligned}$$

Since

$$\begin{aligned} a(\alpha + h) &= a(\alpha) - \Delta_1 l < 0, \\ b(\alpha + h) &= b(\alpha) - \Delta_2 r > 0, \end{aligned}$$

then we have

$$(a(\alpha) - \Delta_1 l)\Delta_2 r > 0,$$
$$(b(\alpha) - \Delta_2 r)\Delta_1 l < 0.$$

Thus ab is increasing in α and in the same way c is decreasing. And $x^2 = (ab, c) \in \mathcal{F}_{\mathbf{b}}^{st}$.

From the above discussion we give the following definition.

Definition 5 *Define*

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} \in \mathcal{F}_{\mathbf{b}}^{st}$$

for $x \in \mathcal{F}_{\mathbf{b}}^{st}$.

The following theorem shows the representation of e^x for $x \in \mathcal{F}_{\mathbf{b}}^{st}$.

Theorem 3 *Let* $x = (a, b) \in \mathcal{F}_{\mathbf{b}}^{st}$. *Then we get the following representation of the fuzzy number* e^x *as follows:*

(i) If $0 \le a(\alpha) \le b(\alpha)$*, then we have*

$$\begin{aligned} e^x &= (e^a, e^b) \\ &= \{(e^{a(\alpha)}, e^{b(\alpha)})^T \in \mathbf{R}^2 : \alpha \in I\}. \end{aligned}$$

(ii) If $a(\alpha) \le b(\alpha) \le 0$*, then we get*

$$e^x = \frac{e^a + e^b}{2} + (-\frac{e^{-a} - e^{-b}}{2}, \frac{e^{-a} - e^{-b}}{2}).$$

In some case the following property holds.

Theorem 4 *Let* $x = (a, b) \in \mathcal{F}_{\mathbf{b}}^{st}$ *and* $y = (c, d) \in \mathcal{F}_{\mathbf{b}}^{st}$ *with* $a(\alpha) \ge 0$ *and* $c(\alpha) \ge 0$ *for* $\alpha \in I$*. Then we have*

$$e^x e^y = e^{x+y}.$$

Example 3 *Consider behaviors of solutions of the following problem of a fuzzy differential equation*

$$x' = p(t)x, \quad x(t_0) = x_0 \qquad (E_0)$$

where $t \in \mathbf{R}, x_0$ *and* $x(t) \in \mathcal{F}_{\mathbf{b}}^{st}$*. Function* $p(t) = (p_1(t, \cdot), p_2(t, \cdot)) : \mathbf{R} \to \mathcal{F}_{\mathbf{b}}^{st}$ *is continuous.*

Remark 4 *Let* $T(x) = p(t)x$*. It follows that* T *is non-linear.*

In analyzing the ordinary differential equation $x' = a(t)x + b(t)$, where $a, b : \mathbf{R} \to \mathbf{R}$ are continuous, the condition that $\lim_{t\to\infty} \int^t a(s)ds = 0$ plays an important role in showing the property that $\lim_{t\to\infty} x(t) = 0$. Conserning fuzzy differetial equation (E_0), we get an extension result of asymptotic behaviors of ordinary linear differential equations as well as we observe a little different result as follows.

Theorem 5 *Consider Problem* (E_0)*. The following cases (i) and (ii) hold:*

(i) Let $p_1(t, \alpha) \ge 0$ *on* $\mathbf{R} \times I$ *and an initial condition* $x_0 = (a_0, b_0)$*. Then the solution* $x = (x_1, x_2)$ *of* (E_0) *is described by*

$$x(t, \alpha) = (e^{\int_\tau^t p_1(s,\alpha)ds} a(\alpha), e^{\int_\tau^t p_2(s,\alpha)ds} b(\alpha))$$

as long as $x_1(t, \alpha) \ge 0$ *for* $t \in J_1, \alpha \in I$*, where* J_1 *is an interval in* $[\tau, \infty)$ *and* $x(\tau) = (a(\alpha), b(\alpha))$*. It follows that*

$$x(t, \alpha) = (e^{\int_\tau^t p_2(s,\alpha)ds} a(\alpha), e^{\int_\tau^t p_2(s,\alpha)ds} b(\alpha))$$

as long as $x_1(t, \alpha) \le 0 \le x_2(t, \alpha)$ *and that*

$$x(t, \alpha) = (e^{\int_\tau^t p_2(s,\alpha)ds} a(\alpha), e^{\int_\tau^t p_1(s,\alpha)ds} b(\alpha))$$

as long as $x_2(t, \alpha) \le 0$ *for* $t \in J_1, \alpha \in I$*.*

(ii) Let $p_2(t, \alpha) \le 0$ *on* $\mathbf{R} \times I$*. As long as* $x_1(t, \alpha) \ge 0$ *or* $x_2(t, \alpha) \le 0$ *for* $t \ge \tau, \alpha \in I$*, it follows that*

$$\begin{aligned} & d(x(\tau, \cdot), 0) e^{\int_\tau^t p_1(s,\cdot)ds} \\ & \le d(x(t, \cdot), 0) \\ & \le d(x(\tau, \cdot), 0) e^{\int_\tau^t p_2(s,\cdot)ds} \end{aligned}$$

where $\tau \ge t_0, t \ge \tau, \alpha \in I$*. As long as* $x_1(t, \alpha) \le 0 \le x_2(t, \alpha)$ *for* $t \ge \tau, \alpha \in I$*, it follows that*

$$\begin{aligned} & d(x(\tau, \cdot), 0) e^{-\int_\tau^t p_2(s,\cdot)ds} \\ & \le d(x(t, \cdot), 0) \\ & \le d(x(\tau, \cdot), 0) e^{-\int_\tau^t p_1(s,\cdot)ds} \end{aligned}$$

$t \ge \tau, \alpha \in I$*.*

Example 4 *Consider Problem* (E_0) *with* $x(t_0) = (a_0, b_0) \in \mathcal{F}_{\mathbf{b}}^{st}$*. Suppose that* $\lim_{t\to\infty} \int_{t_0}^t p_2(t, \alpha) = -\infty$ *for* $t_0 \in \mathbf{R}, \alpha \in I$*.*

Seikkala [15] calculates the solution in case that $p(t) \equiv -1$. See Figure 3.

In the following theorem we show an attractivity set $\mathcal{A}^{E_0}(t_0)$ of (E_0) at t_0. Here $\mathcal{A}^{E_0}(t_0)$ is a subset of $\mathcal{F}_{\mathbf{b}}^{st}$ as follows:

Definition 6 *If $x_0 \in \mathcal{A}^{E_0}(t_0)$, then all the solutions x of (E_0) passing through $(t_0, x_0) \in \mathbf{R} \times \mathcal{F}_{\mathbf{b}}^{st}$ satisfies $\lim_{t\to\infty} d(x(t,\alpha), 0) = 0$ for $\alpha \in I$.*

It is clear that $x_0 = 0 \in \mathcal{A}^{E_0}(t_0)$ for any $t_0 \in \mathbf{R}$. In the case that $p_1(t,\alpha) \equiv p_2(t,\alpha) \leq 0$ and $\lim_{t\to\infty} \int_{t_0}^{t} p_1(s,\alpha)ds = -\infty$ for $t_0 \in \mathbf{R}, \alpha \in I$, it follows that $\mathcal{A}^{E_0}(t_0) = \mathbf{R}$ for $t_0 \in \mathbf{R}$.

When $p_1(t,\alpha) \not\equiv p_2(t,\alpha)$, we have the following theorem.

Theorem 6 *Consider Problem (E_0) with $p_1(t) \not\equiv p_2(t)$. Let $p_2(t,\alpha) \leq 0$ on $\mathbf{R} \times I$ and*

$$\lim_{t\to\infty} \int_{t_0}^{t} p_2(s,\cdot)ds = -\infty$$

for $t_0 \in \mathbf{R}$. Then we have $\mathcal{A}^{E_0}(t_0) = \{0 \in \mathbf{R}\}$ for any $t_0 \in \mathbf{R}$.

The above theorem is proved in [14]. Consider the following problem

$$x' = P_m(t)x, \quad x(t_0) = x_0 \qquad (P_m)$$

$P_m : \mathbf{R} \to \mathcal{F}_{\mathbf{b}}^{st}$ such that $P_m = (-m - q_1, -m + q_2)$ satisfying

$$m : \mathbf{R} \times I \to \mathbf{R}, m(t,\alpha) \geq 0,$$

$$q_i : \mathbf{R} \times I \to \mathbf{R},$$

$$0 \leq q_i(t,\alpha) \leq m(t,\alpha), i = 1, 2.$$

Theorem 7 *Suppose that for $\alpha \in I, t_0 \in \mathbf{R}$*

$$\lim_{t\to\infty} \int_{t_0}^{t} m(s,\alpha)ds = \infty$$

$$\lim_{t\to\infty} e^{-\int_{t_0}^{t} m(s,\alpha)ds} \int_{t_0}^{t} q(s,\alpha) e^{\int_{t_0}^{s}(2m(r,\alpha)+q(r,\alpha))dr} ds = 0$$

where $q(t,\alpha) = \max(q_1(t,\alpha), q_2(t,\alpha))$. Then for any solution $x = (x_1, x_2)$ of (P_m) it follows that

$$\lim_{t\to\infty} |x_1(t,\alpha) + x_2(t,\alpha)| = 0$$

for $\alpha \in I$.

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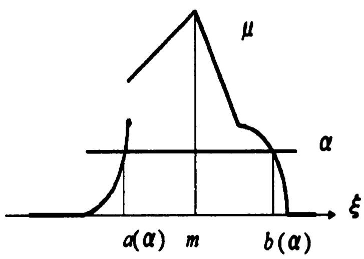

Figure 1: Fuzzy number $\mu = (a, b)$

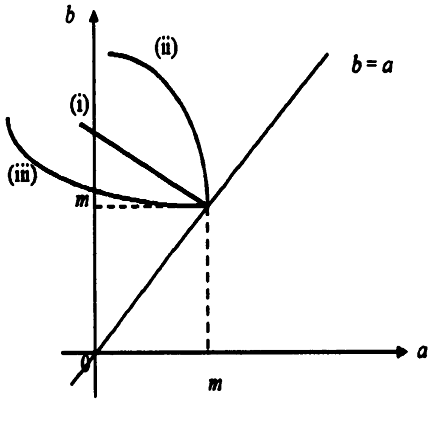

Figure 2: Fuzzy numbers $\mu = (a, b)$ in the following cases(a)-(c)

(i) $c_2 l(b-m) = c_1 r(m-a)$ (ii) $c_2 l(b-m)^2 = c_1 r^2(m-a)$ (iii) $c_2^2 l^2(b-m) = c_1^2 r(a-m)^2$

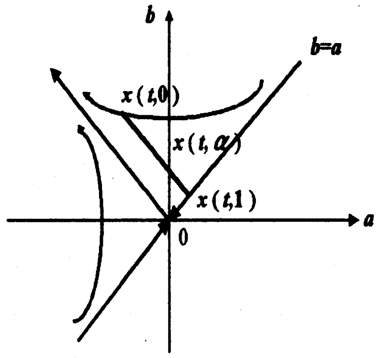

Figure 3: The solutions $x^{'}(t,\cdot) = -x(t,\cdot)$