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Oscillatory solutions of neutral differential equations

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1. INTRODUCTION AND MAIN RESULTS

We shall be concerned with the oscillatory behavior of solutions of the even order neutral differential equation

$$\frac{d^n}{dt^n}[x(t)+h(t)x(t-\tau)]+f(t,x(g(t)))=0. \tag{1.1}$$

Throughout this paper, the following conditions are assumed to hold: $n\geq 2$ is even; $\tau>0$; $h\in C(\mathbf{R})$; $g\in C[t_0,\infty)$, $\lim_{t\to\infty}g(t)=\infty$; $f\in C([t_0,\infty)\times\mathbf{R})$, $uf(t,u)\geq 0$ for $(t,u)\in[t_0,\infty)\times\mathbf{R}$, and $f(t,u)$ is nondecreasing in $u\in\mathbf{R}$ for each fixed $t\geq t_0$.

By a solution of (1.1), we mean a function $x(t)$ that is continuous and satisfies (1.1) on $[t_x,\infty)$ for some $t_x\geq t_0$.

A solution is said to be oscillatory if it has arbitrarily large zeros; otherwise it is said to be nonoscillatory. Equation (1.1) is said to be oscillatory if every solution of (1.1) is oscillatory.

Oscillation properties of even order neutral differential equations have been investigated by many authors. We refer the reader to [1–9, 11, 12, 18–24]. In particular, it has been shown by Zhang and Yang [24] that the odd order neutral differential equation

$$\frac{d^N}{dt^N}[x(t)-x(t-\tau)]+p(t)|x(t-\sigma)|^{\gamma-1}x(t-\sigma)=0$$

is oscillatory if and only the ordinary differential equation

$$x^{(N+1)}(t)+\tau^{-1}p(t)|x(t)|^{\gamma-1}x(t)=0$$

is oscillatory, where $N\geq 1$ is odd, $\gamma>0$, $\sigma\in\mathbf{R}$, $p\in C[t_0,\infty)$, $p(t)\geq 0$ for $t\geq t_0$. (See also Tang and Shen [23].) For even order neutral differential equations, recently, the following result has been established in [22].

Theorem A. *Let $c>0$. Then the even order neutral differential equation*

$$\frac{d^n}{dt^n}[x(t)+cx(t-\tau)]+f(t,x(g(t)))=0$$

is oscillatory if and only if the even order non-neutral differential equation

$$x^{(n)}(t)+\frac{1}{1+c}f(t,x(g(t)))=0 \tag{1.2}$$

is oscillatory.

The purpose of this paper is to generalize Theorem A with $c\neq 1$ for equation (1.1) with the following cases (H1) and (H2):

(H1) $0 \le \mu \le h(t) \le \lambda < 1$ for $t \in \mathbf{R}$;
(H2) $1 < \lambda \le h(t) \le \mu$ for $t \in \mathbf{R}$.

Here, μ and λ are constants. It is convenience only that the parts of μ and λ in (H1) and (H2) are opposite each other.

Throughout this paper we use the notation:

$$H_0(t) = 1; \quad H_i(t) = h(t)h(t-\tau)\cdots h(t-(i-1)\tau).$$

We define the function $S(t)$ on $\mathbf{R}$ by

$$S(t) = \begin{cases} \displaystyle\sum_{i=0}^{\infty}(-1)^i H_i(t) & \text{if (H1) holds,} \\ \displaystyle\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{H_i(t+i\tau)} & \text{if (H2) holds,} \end{cases} \qquad \text{for } t \in \mathbf{R}.$$

It is easy to see that $S(t)$ is converges uniformly on $\mathbf{R}$, and hence $S(t)$ is continuous on $\mathbf{R}$. In Section 2 we will show that

$$0 < \frac{1-\lambda}{1-\mu^2} \le S(t) \le \frac{1-\mu}{1-\lambda^2}, \quad t \in \mathbf{R}. \tag{1.3}$$

We note that if $\mu = \lambda = c \ne 1$, then

$$\frac{1-\lambda}{1-\mu^2} = \frac{1-\mu}{1-\lambda^2} = \frac{1}{1+c}, \quad \text{and} \quad S(t) = \frac{1}{1+c}.$$

Main result of this paper is the following theorem.

Theorem 1.1. *Suppose that* (H1) *or* (H2) *holds. Then equation* (1.1) *is oscillatory if and only if*

$$x^{(n)}(t) + f(t, S(g(t))x(g(t))) = 0 \tag{1.4}$$

is oscillatory.

The proof of Theorem 1.1 will be omitted for lack of space.

Theorem 1.1 means that equation (1.1) has a nonoscillatory solution if and only if equation (1.4) has a nonoscillatory solution.

Suppose that $h(t) \equiv c$, $c > 0$ and $c \ne 1$. Then $S(t) = (1+c)^{-1}$. Note that (1.2) is oscillatory if and only if

$$y^{(n)}(t) + f\left(t, (1+c)^{-1}y(g(t))\right) = 0$$

is oscillatory. Indeed, put $x(t) = (1+c)y(t)$. Hence, Theorem 1.1 is a generalization of Theorem A with $c \ne 1$.

Now we assume that

$$h(t+\tau) = h(t), \quad h(t) \ne 1 \quad \text{and} \quad h(t) \ge 0 \quad \text{for } t \in \mathbf{R}. \tag{1.5}$$

Then it is easy to verify that (H1) or (H2) holds, and $S(t) = [1+h(t)]^{-1}$. Consequently, from Theorem 1.1, we have the following result.

Corollary 1.1. *Suppose that* (1.5) *holds. Then equation* (1.1) *is oscillatory if and only if*

$$x^{(n)}(t) + f\left(t, \frac{x(g(t))}{1+h(g(t))}\right) = 0$$

is oscillatory.

The oscillatory behavior of solutions of non-neutral differential equations of the form

$$x^{(n)}(t) + f(t, x(g(t))) = 0 \tag{1.6}$$

has been intensively studied in the last three decades. We refer the reader to [3, 9, 13–16, 19] and the references cited therein. Combining Theorem 1.1 with the known oscillation results for non-neutral differential equations of the form (1.6), we can derive various oscillation results for neutral differential equations of the form (E). In Section 2 we obtain oscillation criteria for the linear neutral differential equation

$$\frac{d^n}{dt^n}[x(t) + h(t)x(t-\tau)] + p(t)x(t-\sigma) = 0, \tag{1.7}$$

and for the nonlinear neutral differential equation

$$\frac{d^n}{dt^n}[x(t) + h(t)x(t-\tau)] + p(t)|x(t-\sigma)|^{\gamma-1}x(t-\sigma) = 0, \tag{1.8}$$

where $\gamma > 0$, $\gamma \neq 1$ and the following conditions are assumed to hold:

$$\sigma \in \mathbf{R}; \quad p \in C[t_0, \infty),\ p(t) > 0 \text{ for } t \geq t_0. \tag{1.9}$$

It is possible to obtain oscillation results for more general equations such as (1.1). However, for simplicity, we have restricted our attention to equations (1.7) and (1.8). In Section 3 we prove that the function $u(t)$ behaves like the function $S(t)[u(t) + h(t)u(t-\tau)]$ as $t \to \infty$, under some conditions, which plays a crucial part in the proof of Theorem 1.1. We show the "if" part and the "only if" part of Theorem 1.1 in Sections 4 and 5, respectively.

Neutral differential equations find numerous applications in natural science and technology. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines. See Hale [10].

2. Oscillation criteria

In this section we establish oscillation criteria for neutral differential equations of the form (E).

First let us show that $S(t)$ satisfies (1.3).

Lemma 2.1. *If* (H1) *or* (H2) *holds, then* $S(t)$ *satisfies* (1.3).

Proof. Assume that (H1) holds. Let $t \in \mathbf{R}$. Then

$$S(t) = \sum_{j=0}^{\infty} H_{2j}(t)[1 - h(t + 2j\tau)].$$

We see that

$$S(t) \leq \sum_{j=0}^{\infty} \lambda^{2j}(1-\mu) = \frac{1-\mu}{1-\lambda^2},$$

and

$$S(t) \geq \sum_{j=0}^{\infty} \mu^{2j}(1-\lambda) = \frac{1-\lambda}{1-\mu^2}.$$

In the same way, the conclusion follows for the case (H2), by using

$$S(t) = \sum_{j=1}^{\infty} \frac{1}{H_{2j}(t+2j\tau)}[h(t+2j\tau)-1].$$

We need the following result which was obtained by Kusano and M. Naito [16].

Lemma 2.2. *If the differential inequality*

$$x^{(n)}(t) + f(t, x(g(t))) \leq 0$$

has an eventually positive solution, then the differential equation

$$x^{(n)}(t) + f(t, x(g(t))) = 0$$

has an eventually positive solution.

From Theorem 1.1, Lemmas 2.1 and 2.2, we have the following result.

Corollary 2.1. *Suppose that* (H1) *or* (H2) *holds. If*

$$x^{(n)}(t) + \frac{1-\lambda}{1-\mu^2} f(t, x(g(t))) = 0 \tag{2.1}$$

is oscillatory, then (1.1) *is oscillatory. If*

$$x^{(n)}(t) + \frac{1-\mu}{1-\lambda^2} f(t, x(g(t))) = 0 \tag{2.2}$$

has a nonoscillatory solution, then (1.1) *has a nonoscillatory solution.*

Proof. Assume that there exists a nonoscillatory solution of (1.1). Then Theorem 1.1 implies that (1.4) has a nonoscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t) > 0$ for all large t, since the case $x(t) < 0$ can be treated similarly. Put $y(t) = (1-\lambda)/(1-\mu^2)x(t)$. From Lemma 2.1 we see that

$$\begin{aligned}
-y^{(n)}(t) = -\frac{1-\lambda}{1-\mu^2}x^{(n)}(t) &= \frac{1-\lambda}{1-\mu^2} f(t, S(g(t))x(g(t))) \\
&\geq \frac{1-\lambda}{1-\mu^2} f\left(t, \frac{1-\lambda}{1-\mu^2}x(g(t))\right) \\
&= \frac{1-\lambda}{1-\mu^2} f(t, y(g(t)))
\end{aligned}$$

for all large t. From Lemma 2.2 it follows that (3.1) has a nonoscillatory solution.

Let $y(t)$ be an eventually positive solution of (3.2). Then Lemma 2.1 implies that $x(t) = (1-\lambda^2)/(1-\mu)y(t)$ is an eventually positive solution of

$$x^{(n)}(t) + f(t, S(g(t))x(g(t))) \leq 0,$$

and hence (1.1) has a nonoscillatory solution, by Lemma 2.2 and Theorem 1.1. This completes the proof.

Now let us derive oscillation criteria for (1.7) and (1.8).

The following oscillation result was obtained by Kitamura [15, Corollaries 5.1 and 3.1].

Lemma 2.3. *Assume that* (1.9) *holds. If*

$$\int^{\infty} t^{n-1-\varepsilon} p(t)dt = \infty \quad \text{for some } \varepsilon > 0, \tag{2.3}$$

then the equation

$$x^{(n)}(t) + p(t)x(t-\sigma) = 0 \tag{2.4}$$

is oscillatory. If

$$\int^{\infty} t^{n-1} p(t)dt < \infty, \tag{2.5}$$

then equation (3.4) *has a nonoscillatory solution.*

Lemma 2.4. *Assume that* $\gamma > 0$, $\gamma \neq 1$ *and* (1.9) *holds. Then the equation*

$$x^{(n)}(t) + p(t)|x(t-\sigma)|^{\gamma-1}x(t-\sigma) = 0$$

is oscillatory if and only if

$$\int^{\infty} t^{\min\{\gamma,1\}(n-1)} p(t)dt = \infty. \tag{2.6}$$

Combining Corollary 2.1 with Lemmas 2.3 and 2.4, we have the following oscillation criteria for equations (1.7) and (1.8).

Corollary 2.2. *If* (3.3) *holds, then* (1.7) *is oscillatory. If* (3.5) *holds, then* (1.7) *has a nonoscillatory solution.*

Corollary 2.3. *Equation* (1.8) *is oscillatory if and only if* (3.6) *holds.*

Remark 2.1. Corollary 2.2 with (H1) have been already established by Jaroš and Kusano [11, Theorems 3.1 and 4.1]. Corollary 2.2 with (H2) extends the results in [5, Theorem 1] and [8, Theorem 7].

Remark 2.2. Corollary 2.3 with (H1) has been obtained by Y. Naito [19] in the case where $h(t)$ is locally Lipschitz continuous.

3. Oscillation criteria

In this section we establish oscillation criteria for neutral differential equations of the form (E).

First let us show that $S(t)$ satisfies (1.3).

Lemma 3.1. *If* (H1) *or* (H2) *holds, then* $S(t)$ *satisfies* (1.3).

Proof. Assume that (H1) holds. Let $t \in \mathbf{R}$. Then

$$S(t) = \sum_{j=0}^{\infty} H_{2j}(t)[1 - h(t + 2j\tau)].$$

We see that

$$S(t) \leq \sum_{j=0}^{\infty} \lambda^{2j}(1-\mu) = \frac{1-\mu}{1-\lambda^2},$$

and

$$S(t) \geq \sum_{j=0}^{\infty} \mu^{2j}(1-\lambda) = \frac{1-\lambda}{1-\mu^2}.$$

In the same way, the conclusion follows for the case (H2), by using

$$S(t) = \sum_{j=1}^{\infty} \frac{1}{H_{2j}(t+2j\tau)}[h(t+2j\tau) - 1].$$

We need the following result which was obtained by Kusano and M. Naito [16].

Lemma 3.2. *If the differential inequality*

$$x^{(n)}(t) + f(t, x(g(t))) \leq 0$$

has an eventually positive solution, then the differential equation

$$x^{(n)}(t) + f(t, x(g(t))) = 0$$

has an eventually positive solution.

From Theorem 1.1, Lemmas 2.1 and 2.2, we have the following result.

Corollary 3.1. *Suppose that* (H1) *or* (H2) *holds. If*

$$x^{(n)}(t) + \frac{1-\lambda}{1-\mu^2} f(t, x(g(t))) = 0 \tag{3.1}$$

is oscillatory, then (1.1) *is oscillatory. If* (1.1) *is oscillatory, then*

$$x^{(n)}(t) + \frac{1-\mu}{1-\lambda^2} f(t, x(g(t))) = 0 \tag{3.2}$$

is oscillatory.

Proof of Corollary 2.1. It is sufficient to show the following (i) and (ii):

(i) equation (3.1) is oscillatory, then equation (1.4) is oscillatory;
(ii) equation (1.4) is oscillatory, then equation (3.2) is oscillatory.

We give the proof of (i) only. In exactly the same way, we can prove (ii). Let $x(t)$ be a nonoscillatory solution of (1.4). Without loss of generality, we may assume that $x(t) > 0$ for all large t, since the case $x(t) < 0$ can be treated similarly. Put $y(t) = (1-\lambda)/(1-\mu^2)x(t)$. Then Lemma 2.1 implies that

$$\begin{aligned} -y^{(n)}(t) = -\frac{1-\lambda}{1-\mu^2}x^{(n)}(t) &= -\frac{1-\lambda}{1-\mu^2}f(t, S(g(t))x(g(t))) \\ &\geq f\left(t, \frac{1-\lambda}{1-\mu^2}x(g(t))\right) \\ &\geq f(t, y(g(t))) \end{aligned}$$

for all large t. From Lemma 2.2 it follows that (3.1) has an eventually positive solution. This completes the proof.

Now let us derive oscillation criteria for (1.7) and (1.8). It is possible to obtain oscillation results for more general equations of the form (1.1). However, for simplicity, we have restricted our attention to equations (1.7) and (1.8).

The following oscillation result was obtained by Kitamura [15, Corollaries 5.1 and 3.1].

Lemma 3.3. *Assume that* (1.9) *holds. If*

$$\int^{\infty} t^{n-1-\varepsilon}p(t)dt = \infty \quad \text{for some } \varepsilon > 0, \tag{3.3}$$

then the equation

$$x^{(n)}(t) + p(t)x(t-\sigma) = 0 \tag{3.4}$$

is oscillatory. If

$$\int^{\infty} t^{n-1}p(t)dt < \infty, \tag{3.5}$$

then equation (3.4) *has a nonoscillatory solution.*

Lemma 3.4. *Assume that* $\gamma > 0$, $\gamma \neq 1$ *and* (1.9) *holds. Then the equation*

$$x^{(n)}(t) + p(t)|x(t-\sigma)|^{\gamma}\operatorname{sgn} x(t-\sigma) = 0$$

is oscillatory if and only if

$$\int^{\infty} t^{\min\{\gamma,1\}(n-1)}p(t)dt = \infty. \tag{3.6}$$

Combining Corollary 2.1 with Lemmas 2.3 and 2.4, we have the following oscillation criteria for equations (1.7) and (1.8).

Corollary 3.2. *If* (3.3) *holds, then* (1.7) *is oscillatory. If* (3.5) *holds, then* (1.7) *has a nonoscillatory solution.*

Corollary 3.3. *Equation* (1.8) *is oscillatory if and only if* (3.6) *holds.*

Remark 3.1. Corollary 2.2 with (H1) have been already established by Jaroš and Kusano [11, Theorems 3.1 and 4.1]. Corollary 2.2 with (H2) extends the results in [5, Theorem 1], [8, Theorem 7] and [21, Corollary 3].

Remark 3.2. Corollary 2.3 with (H1) was obtained by Y. Naito [19] in the case where $h(t)$ is locally Lipschitz continuous. Corollary 2.3 with (H2) is a improvement of the result in [21, Corollary 4].

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