数理解析研究所講究録 1216 巻 2001 年 33-44

Permanence of an SIR Epidemic Model with Distributed Time Delays

静岡大学 馬万彪 (Wanbiao Ma, Shizuoka University)
静岡大学 竹内康博 (Yasuhiro Takeuchi, Shizuoka University)
大阪府立大学 原惟行 (Tadayuki Hara, Osaka Prefecture University)
Edoardo Beretta (University Di Urbino)

1 Introduction

In this paper, we shall consider the following SIR epidemic model with distributed time delays,

$$\begin{cases} \dot{S}(t) = & -\beta S(t)\int_0^h I(t-s)d\eta(s) - \mu_1 S(t) + b \\ \dot{I}(t) = & \beta S(t)\int_0^h I(t-s)d\eta(s) - \mu_2 I(t) - \lambda I(t) \\ \dot{R}(t) = & \lambda I(t) - \mu_3 R(t). \end{cases} \tag{1.1}$$

In model (1.1), $S(t)+I(t)+R(t) \equiv N(t)$ denotes the number of a population at time t; $S(t)$, $I(t)$ and $R(t)$ denote the numbers of the population susceptible to the disease, of infective members and of members who have been removed from the possibility of infection through full immunity, respectively. It is assumed that all newborns are susceptible. The positive constants μ_1, μ_2 and μ_3 represent the death rates of susceptibles, infectives and recovered, respectively. It is natural biologically to assume that

$$\mu_1 \leq \min\{\mu_2,\ \mu_3\}.$$

The positive constants b and λ represent the birth rate of the population and the recovery rate of infectives, respectively. The positive constant β is the average number of contacts per infective per day. The nonnegative constant h is the time delay. The function $\eta(s)$: $[0,h] \to \mathcal{R} = (-\infty,+\infty)$ is nondecreasing and has bounded variation such that

$$\int_0^h d\eta(s) = \eta(h) - \eta(0) = 1.$$

The term $\beta S(t)\int_0^h I(t-s)d\eta(s)$ can be considered as the force of infection at time t. For the detailed biological meanings we refer to [1-4], [10] and [12-13].

The initial condition of (1.1) is given as

$$S(\theta) = \varphi_1(\theta), \quad I(\theta) = \varphi_2(\theta), \quad R(\theta) = \varphi_3(\theta), \quad (-h \leq \theta \leq 0), \tag{1.2}$$

where $\varphi = (\varphi_1, \varphi_2, \varphi_3)^T \in C$ such that $\varphi_i(\theta) = \varphi_i(0) \geq 0$ $(-h \leq \theta \leq 0,\ i = 1,3)$, $\varphi_2(\theta) \geq 0$ $(-h \leq \theta \leq 0)$, and C denotes the Banach space $C([-h,0],\mathcal{R}^3)$ of continuous functions mapping the interval $[-h,0]$ into $\mathcal{R}^3$. By a biological meaning, we further assume that $\varphi_i(0) > 0$ for $i = 1,2,3$.

From Lemma 1 in the following section, the solution $(S(t), I(t), R(t))$ of (1.1) with the initial condition (1.2) exists for all $t \geq 0$ and is unique. Furthermore, $S(t) > 0$, $I(t) > 0$ and $R(t) > 0$ for all $t \geq 0$. Note that there are no time delay in the state variables $S(t)$ and $R(t)$ of (1.1). In the phase space C, the solution $(S(t), I(t), R(t))$ can also be denoted in the form of (S_t, I_t, R_t) for $t \geq 0$. Here $S_t = S(t+\theta) = S(t)$, $I_t = I(t+\theta)$ and $R_t = R(t+\theta) = R(t)$ for $t \geq 0$ and $-h \leq \theta \leq 0$.

For any parameters h, β, b, λ, and μ_i $(i = 1,2,3)$, (1.1) always has a disease free equilibrium (i.e. boundary equilibrium) $E_0 = (S_0,\ 0,\ 0)$, here $S_0 = b/\mu_1$. Furthermore, if

$$\frac{b}{\mu_1} > S^* \equiv \frac{\mu_2 + \lambda}{\beta}, \tag{1.3}$$

then (1.1) also has an endemic equilibrium (i.e. interior equilibrium) $E_+ = (S^*, I^*, R^*)$, where

$$S^* = \frac{\mu_2 + \lambda}{\beta}, \quad I^* = \frac{b - \mu_1 S^*}{\beta S^*}, \quad R^* = \frac{\lambda(b - \mu_1 S^*)}{\mu_3 \beta S^*}.$$

We see that the model (1.1) is factually a natural generalization of the following well-known SIR model without time delay, which was first proposed and studied in [1] and [10],

$$\begin{cases} \dot{S}(t) = & -\beta S(t)I(t) - \mu S(t) + \mu \\ \dot{I}(t) = & \beta S(t)I(t) - \mu I(t) - \lambda I(t) \\ \dot{R}(t) = & \lambda I(t) - \mu R(t), \end{cases} \tag{1.4}$$

where β, μ and λ are positive constants. In (1.4), it is assumed that the total number of population $N(t)$ is constant (i.e. $N(t) = 1$ for all $t \geq 0$) and that the birth and the death rates of population are the same. It is shown in [1] and [10] that the condition

$$\delta \equiv \frac{\beta}{\lambda + \mu} > 1$$

is the threshold of (1.4) for an epidemic to occur, that is, if $\delta \leq 1$, the disease will eventually disappear and all population will become susceptibles (i.e. the disease free equilibrium $E_0 = (1,0,0)$ of (1.4) is globally asymptotically stable), if $\delta > 1$, the disease always remains endemic and the numbers of the susceptibles, infectives and removed will eventually tend to some positive constants, respectively (i.e. the endemic equilibrium

$$E_+ = (\frac{1}{\delta}, \frac{\mu(\delta - 1)}{\beta}, \frac{\lambda(\delta - 1)}{\beta})$$

of (1.4) is globally asymptotically stable).

Recently, in [3-4] and [12], it is tried to show such the threshold phenomenon as for (1.4) still remains true for the model (1.1) with time delay h, i.e. the following conjecture may be true.

Conjecture: *For any time delay h,* (1.3) *is the threshold of* (1.1) *for an epidemic to occur.*

It is shown in [4] that, if $b/\mu_1 < S^*$ (or $b/\mu_1 = S^*$), the disease free equilibrium E_0 is globally asymptotically stable (or globally attractive, respectively) for any time delay h. If $b/\mu_1 > S^*$ (i.e. (1.3) is valid), the disease free equilibrium E_0 becomes unstable and the endemic equilibrium E_+ is locally asymptotically stable for any time delay h. In fact, in [4], it is also shown that the endemic equilibrium E_+ is also globally asymptotically stable for some samll time delay h. For a class of simpler model than (1.1), [12] condisered the global asymptotic stability of the endemic equilibrium E_+ under some *stronger conditions than* (1.3). It is not difficult to see that the results given in [12] still remain true for the model (1.1).

The purpose of the paper is to give a complete answer to the conjecture in some sense. In fact, we shall show that, for any time delay h, (1.3) is necessary and sufficient for the permanence of (1.1). In biology, our result says that (1.3) is the threshold for an endemic to occur for any time delay h. To prove our result, some analysis techniques on limit sets of differential dynamical systems developed in [5], [7] and [9] have been used.

2 Main result

Definition. (1.1) *is said to be permanent if there are positive constants ν_i and M_i $(i = 1, 2, 3)$ such that*

$$\nu_1 \leq \liminf_{t\to+\infty} S(t) \leq \limsup_{t\to+\infty} S(t) \leq M_1,$$

$$\nu_2 \leq \liminf_{t\to+\infty} I(t) \leq \limsup_{t\to+\infty} I(t) \leq M_2,$$

$$\nu_3 \leq \liminf_{t\to+\infty} R(t) \leq \limsup_{t\to+\infty} R(t) \leq M_3$$

hold for any solution of (1.1) *with the initial condition* (1.2). *Here ν_i and M_i $(i = 1, 2, 3)$ are independent of* (1.2).

The following is our main result of the paper.

Theorem. *For any time delay h,* (1.3) *is necessary and sufficient for the permanence*

Not that the desease free equilibrium E_0 of (1.1) is globally asymptotically stable or globally attractive if (1.3) is not valid. We only need to prove the sufficiency. Let us first show the following Lemmas 1-4.

Lemma 1. *The solution* $(S(t), R(t), I(t))$ *of* (1.1) *with* (1.2) *exists and is positive for* $t \geq 0$. *Further,*

$$\limsup_{t\to+\infty} N(t) \leq \frac{b}{\mu_1}. \tag{2.1}$$

Proof of Lemma 1. Note that the right hand side of (1.1) is completely continuous and locally Lipschitizian on C. It follows from [9] and [11] that the solution $(S(t), I(t), R(t))$ of (1.1) exists and is unique on $[0, \alpha)$ for some $\alpha > 0$. It is easy to see that $S(t) > 0$ for all $t \in [0, \alpha)$. In fact, this follows from that $\dot{S}(t) = b > 0$ for any $t \in [0, \alpha)$ when $S(t) = 0$. Let us show that $I(t) > 0$ for all $t \in [0, \alpha)$. In fact, assume that there exists some $t_1 \in (0, \alpha)$ such that $I(t_1) = 0$ and $I(t) > 0$ for $t \in [0, t_1)$, integrating the second equation of (1.1) from 0 to t_1, we see that

$$I(t_1) = I(0)e^{-(\mu_2+\lambda)t_1} + \beta \int_0^{t_1} (S(u) \int_0^h I(u-s)d\eta(s))e^{-(\mu_2+\lambda)(t_1-u)}du > 0,$$

which contradicts to $I(t_1) = 0$. From (1.1), we also have that $R(t) > 0$ for all $t \in [0, \alpha)$. Thus, for $t \in [0, \alpha)$,

$$\dot{N}(t) \leq -\mu_1 N(t) + b, \tag{2.2}$$

which implies that $(S(t), I(t), R(t))$ is uniformly bounded on $[0, \alpha)$. It follows from [9] and [11] that $(S(t), I(t), R(t))$ exists and is unique and positive for $t \geq 0$. From (2.2), we also have (2.1). This completes the proof of Lemma 1.

Remark 1. For any nonnegative initial function $\varphi \in C$, by a similar method as that used in Lemma 1, we can show that the following (i), (ii) and (iii) are true.

(i) The solution $(S(t), I(t), R(t))$ of (1.1) exists and $S(t) > 0$ $(t > 0)$, $I(t) \geq 0$ and $R(t) \geq 0$ $(t \geq 0)$.

(ii) If $\varphi_1(0) > 0$ and $\varphi_2(0) + \int_0^h \varphi_2(-s)d\eta(s) > 0$, then, the solution $(S(t), I(t), R(t))$ of (1.1) exists and $S(t) > 0$ $(t \geq 0)$, $I(t) > 0$ and $R(t) > 0$ $(t > 0)$.

(iii) If $\varphi_2(\theta) = \varphi_3(0) = 0$ for any $\theta \in [-h, 0]$, then, the solution $(S(t), I(t), R(t))$ of (1.1) exists and $S(t) > 0$ $(t > 0)$ and $I(t) = R(t) = 0$ $(t \geq 0)$.

In fact, let the solution $(S(t), I(t), R(t))$ exist and be unique on $[0, \alpha)$ for some $\alpha > 0$. It is easy to show that $S(t) > 0$ for $t \in (0, \alpha)$. From the proof of Lemma 1 and the continuity of the solution $(S(t), I(t), R(t))$ of (1.1) with respect to the initial function φ, we easily show that $I(t) \geq 0$ and $R(t) \geq 0$ for $t \in (0, \alpha)$. Thus, (i) of Remark 1 holds.

If $\varphi_2(0) + \int_0^h \varphi_2(-s)d\eta(s) > 0$, from (1.1), we have that $\dot{I}(0+0) > 0$. This implies that $I(t) > 0$ for small $t > 0$, from which we can further show that $I(t) > 0$ for all $t \in (0, \alpha)$.

Furthermore, from (1.1), we also have that $R(t) > 0$ for $t \in (0, \alpha)$. This shows that (ii) of Remark 1 holds.

If $\varphi_2(\theta) = \varphi_3(0) = 0$ for all $\theta \in [-h, 0]$, it is clear that

$$S(t) = (\varphi_1(0) - \frac{b}{\mu_1})e^{-\mu_1 t} + \frac{b}{\mu_1} > 0,$$

and $I(t) = R(t) = 0$ for all $t \geq 0$. This shows that (iii) of Remark 1 holds.

Lemma 2. *The solution* $(S(t), R(t), I(t))$ *of* (1.1) *with* (1.2) *satisfies*

$$\liminf_{t \to +\infty} S(t) \geq \frac{\mu_1 b}{b\beta + \mu_1^2} \equiv \nu_1. \tag{2.3}$$

Proof of Lemma 2. For any sufficiently small positive constant ε, it follows from Lemma 1 that there is some sufficiently large $t_1 > 0$ such that for $t \geq t_1$, $I(t) \leq b/\mu_1 + \varepsilon$. Thus, from (1.1) we have that for $t \geq t_1 + h$,

$$\dot{S}(t) \geq -[\beta(\frac{b}{\mu_1} + \varepsilon) + \mu_1]S(t) + b,$$

which implies that

$$\liminf_{t \to +\infty} S(t) \geq \frac{b\mu_1}{\beta(b + \mu_1\varepsilon) + \mu_1^2}.$$

Note that ε may be arbitrarily small, we see that (2.3) holds. This proves Lemma 2.

Lemma 3. *The set Q is positively invariant for* (1.1) *and attracts all solutions of* (1.1). *Here*

$$Q = \{\varphi \mid \varphi = (\varphi_1, \varphi_2, \varphi_3) \in C, \quad \frac{b}{\mu_0} \leq \varphi_1(\theta) + \varphi_2(\theta) + \varphi_3(\theta) \leq \frac{b}{\mu_1},$$
$$\nu_1 \leq \varphi_1(\theta) \leq \frac{b}{\mu_1}, \quad \varphi_1(\theta) = \varphi_1(0), \quad \varphi_2(\theta) \geq 0,$$
$$\varphi_3(\theta) = \varphi_3(0) \geq 0 \ (-h \leq \theta \leq 0)\},$$

and $\mu_0 = \max\{\mu_1, \ \mu_2, \ \mu_3\}$.

Proof of Lemma 3. By Lemmas 1-2 and the fact $\dot{N}(t) \geq b - \mu_0 N(t)$, it is enough to show that Q is positively invariant for (1.1).

For any initial function $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in Q$, let $(S(t), I(t), R(t))$ be the solution of (1.1). From Remark 1, we have that $(S(t), I(t), R(t))$ is nonnegative for all $t \geq 0$.

Now, let us show that $S(t) \leq b/\mu_1$ for all $t \geq 0$. If not, there exists some $t_1 > 0$ such that $S(t_1) > b/\mu_1$ and $\dot{S}(t_1) > 0$ by the mean value theorem. Thus, it follows from (1.1) that

$$\dot{S}(t_1) = -\beta S(t_1) \int_0^h I(t_1 - s) d\eta(s) - \mu_1 S(t_1) + b < 0,$$

which is a contradiction. Moreover, note that for any $t \geq 0$,

$$-\mu_0 N(t) + b \leq \dot{N}(t) \leq -\mu_1 N(t) + b.$$

Hence, we see that $b/\mu_0 \leq N(t) \leq b/\mu_1$ for any $t \geq 0$.

Let us show that $S(t) \geq \nu_1$ for all $t \geq 0$. If not, we can find some $t_2 \geq 0$ such that $S(t_2) = \nu_1$, $S(t) \geq \nu_1$ for all $-h \leq t \leq t_2$ and $\dot{S}(t_2) \leq 0$. On the other hand, it follows from (1.1) that

$$\begin{aligned}
\dot{S}(t_2) &= -\beta S(t_2) \int_0^h I(t_2 - s) d\eta(s) - \mu_1 S(t_2) + b \\
&\geq -\beta \nu_1 (\frac{b}{\mu_1} - \nu_1) - \mu_1 \nu_1 + b \\
&= \beta \nu_1^2 \\
&> 0.
\end{aligned}$$

Note that the first inequality of the above is true since we have $I(t) \leq b/\mu_1 - \nu_1$ for $t - h < t < t_2$ because of $N(t) \leq b/\mu_1$, $S(t) \geq \nu_1$ and $I(t) \leq N(t) - S(t)$ for $t - h < t < t_2$. Thus, we again have a contradiction. These shows that Q is positively invariant for (1.1). The proof of Lemma 3 is completed.

Lemma 4. *If* (1.3) *holds, then the solution* $(S(t), I(t), R(t))$ *of* (1.1) *with* (1.2) *satisfies*

$$\liminf_{t \to +\infty} I(t) \geq \nu_2. \tag{2.4}$$

Here ν_2 *is some positive constant which does not depend on the initial function* φ.

Proof of Lemma 4. From Lemma 3, we see that it is enough to consider the solution (S_t, I_t, R_t) $(t \geq 0)$ with the initial function $\varphi \in Q$ such that $\varphi_2(0) > 0$ and $\varphi_3(0) > 0$. From Lemma 1, we see that the omega limit set $\omega(\varphi)$ of (S_t, I_t, R_t) $(t \geq 0)$ is nonempty, compact, invariant and $\omega(\varphi) \subset Q$ ([15]).

We will first show that $\liminf_{t \to +\infty} I(t) > 0$ by proving the following two cases are impossible.

The case (i): $\lim_{t \to +\infty} I(t) = 0$. For this case, it follows from (1.1) and Lemma 1 that

$$\lim_{t \to +\infty} S(t) = \frac{b}{\mu_1}, \quad \lim_{t \to +\infty} R(t) = 0.$$

Thus, $E_0 = (b/\mu_1, 0, 0) = \omega(\varphi)$. For sufficiently small positive constant ε, there exists a positive time sequence $\{t_n\}$: $t_n \to +\infty$ $(n \to +\infty)$ such that

$$S(t_n) \geq \frac{b}{\mu_1} - \varepsilon, \quad \dot{I}(t_n) \leq 0, \quad I(t) \geq I(t_n) \ (t_n - h \leq t \leq t_n).$$

It follows from (1.1) and (1.3) that

$$\dot{I}(t_n) \geq [\beta(\frac{b}{\mu_1} - \varepsilon) - (\mu_2 + \lambda)] I(t_n) > 0 \tag{2.5}$$

for sufficiently small ε. This is a contradiction to $\dot{I}(t_n) \leq 0$ and shows that the case (i) is impossible.

The case (ii): $\limsup_{t\to+\infty} I(t) > \liminf_{t\to+\infty} I(t) = 0$. For this case also, there exists a positive time sequence $\{t_n\}$: $t_n \to +\infty$ $(n \to +\infty)$ such that

$$\lim_{n\to+\infty} I(t_n) = 0, \quad \dot{I}(t_n) = 0, \quad I(t) \geq I(t_n) \ (t_n - h \leq t \leq t_n).$$

Define the continuous functions sequence $\{f_n(\theta)\} = \{I(t_n + \theta)\}$ for $-h \leq \theta \leq 0$. From (1.1) and Lemma 1 we see that $\{f_n(\theta)\}$ is uniformly bounbed and equi-continuous on $-h \leq \theta \leq 0$. Hence, it follows from Ascoli's theorem that $f_n(\theta)$ converges to some continuous function $\tilde{\varphi}_2(\theta)$ uniformly on $-h \leq \theta \leq 0$ as $n \to +\infty$. Here, we note that $\tilde{\varphi}_2(0) = 0$. Furthermore, without loss of generality, from Lemma 1, we can also assume that $\lim_{n\to+\infty} S(t_n) = \rho_1$ and $\lim_{n\to+\infty} R(t_n) = \rho_2$ for some constants $\rho_1 : \nu_1 \leq \rho_1 \leq b/\mu_1$ and $\rho_2 : 0 \leq \rho_2 \leq b/\mu_1$. If $\rho_1 = b/\mu_1$, then $\tilde{\varphi}_2(\theta) = \rho_2 = 0$ for all $-h \leq \theta \leq 0$ by Lemma 3. As done in (2.5), we can get a contradiction.

If $\rho_1 < b/\mu_1$, we see that

$$E_\rho = (\rho_1, \tilde{\varphi}_2, \rho_2) \in \omega(\varphi).$$

Now, let us consider the solution $(\tilde{S}_t, \tilde{I}_t, \tilde{R}_t)$ of (1.1) through E_ρ. From the invariance of $\omega(\varphi)$, we have that $(\tilde{S}_t, \tilde{I}_t, \tilde{R}_t) \in \omega(\varphi)$ for all $t \in \mathcal{R}$.

Let $\tilde{\gamma}^+ : (\tilde{S}_t, \tilde{I}_t, \tilde{R}_t)$ $(t \geq 0)$ and $\tilde{\gamma}_1^- : (\tilde{S}_t, \tilde{I}_t, \tilde{R}_t)$ $(t < 0)$ be the positive and negative semi-orbits of (1.1), respectively. It follows from Remark 1 and $\tilde{\varphi}_2(0) = 0$ that $\int_0^h \tilde{I}_t d\eta(s) + \tilde{I}(t) = 0$ for all $t < 0$. Hence, from (1.1), we have that $\tilde{I}(t) = 0$ for all $t \leq 0$. It follows that $\tilde{\varphi}_2(\theta) = 0$ for all $-h \leq \theta \leq 0$ and $\tilde{I}(t) = 0$ for all $t \geq 0$. From (1.1), we have that, on the full orbit $\tilde{\gamma}^- \cup \tilde{\gamma}^+$: $(\tilde{S}_t, \tilde{I}_t, \tilde{R}_t)$ $(t \in \mathcal{R})$, $\tilde{S}_t = \tilde{S}(t) = \tilde{g}_1(t)$, $\tilde{I}_t = 0$, and $\tilde{R}_t = \tilde{R}(t) = \tilde{g}_2(t)$, here

$$\tilde{g}_1(t) = -(\frac{b}{\mu_1} - \rho_1)e^{-\mu_1 t} + \frac{b}{\mu_1}, \quad \tilde{g}_2(t) = \rho_2 e^{-\mu_3 t}.$$

It is clear that the negative semi-orbit $\tilde{\gamma}_1^-$ cannot be completely included in $\omega(\varphi)$ for $\rho_1 < b/\mu_1$. This is a contradiction. This shows that case (ii) is also impossible.

The above analyses show that

$$\liminf_{t\to+\infty} I(t) = \eta > 0 \tag{2.6}$$

for some constant η which may be dependent on the initial function φ.

In the remaining part of the proof, let us show that (2.4) holds.

For any initial functions sequence $\{\varphi_n\} = \{(\varphi_1^{(n)}, \varphi_2^{(n)}, \varphi_3^{(n)})\} \in Q$, let $(S_t^{(n)}, I_t^{(n)}, R_t^{(n)})$ be the solution of (1.1) with the initial function φ_n. Let $\omega_n(\varphi_n)$ be the omega limit set of $(S_t^{(n)}, I_t^{(n)}, R_t^{(n)})$. By a completely similar argument as that used in [5] and [9], we have that there exits some compact and invariant set $\omega^* \in Q$ such that $\mathrm{dist}(\omega_n(\varphi_n), \omega^*) \to 0$ as $n \to +\infty$. Here, $\mathrm{dist}(\omega_n(\varphi_n), \omega^*)$ means Hausdorff distance.

If (2.4) does not hold, for some initial function sequence $\{\varphi_n\} = \{(\varphi_1^{(n)}, \varphi_2^{(n)}, \varphi_3^{(n)})\} \in Q$ such that $\varphi_2^{(n)}(0) > 0$ and $\varphi_3^{(n)}(0) > 0$, we have that there is some $\bar{\varphi} = (\bar{\varphi}_1, \bar{\varphi}_2, \bar{\varphi}_3) \in \omega^*$ such that $\bar{\varphi}_2(\theta_0) = 0$ for some $-h \leq \theta_0 \leq 0$. Now, let $(\bar{S}(t), \bar{I}(t), \bar{R}(t))$ be the solution of (1.1) with the initial function $\bar{\varphi}$. Then, by the invariance of ω^*, we have that $(\bar{S}(t), \bar{I}(t), \bar{R}(t)) \in \omega^*$ for all $t \in \mathcal{R}$. Note that Remark 1 and $\bar{\varphi}_2(\theta_0) = 0$, we easily have that $\int_0^h \bar{I}(t-s)d\eta(s) + \bar{I}(t) = 0$ for all $t \leq \theta_0$. Hence, it follows from (1.1) that $I(t) = 0$ for all $t \leq 0$. This imples that $\bar{\varphi}_2(\theta) = 0$ for all $-h \leq \theta \leq 0$. It follows from Remark 1 and (1.1) that $\bar{S}_t = \bar{S}(t) = \bar{g}_1(t)$, $\bar{I}_t = \bar{I}(t) = 0$ and $\bar{R}_t = \bar{R}(t) = \bar{g}_2(t)$ for all $t \in \mathcal{R}$, where

$$\bar{g}_1(t) = -(\frac{b}{\mu_1} - \bar{\varphi}_1(0))e^{-\mu_1 t} + \frac{b}{\mu_1}, \quad \bar{g}_2(t) = \bar{\varphi}_3(0)e^{-\mu_3 t}.$$

If $\bar{\varphi}_1(\theta) = \bar{\varphi}_1(0) < b/\mu_1$ for $-h \leq \theta \leq 0$ or $\bar{\varphi}_3(\theta) = \bar{\varphi}_3(0) > 0$ for $-h \leq \theta \leq 0$, we see that the negative semi-orbit $(\bar{S}(t), \bar{I}(t), \bar{R}(t))$ $(t \leq 0)$ is unbounded. This is a contradiction.

If $\bar{\varphi}_1(\theta) = b/\mu_1$ for $-h \leq \theta \leq 0$, we have that $\bar{\varphi}_2(\theta) = \bar{\varphi}_3(\theta) = 0$ for $-h \leq \theta \leq 0$. Hence, $\bar{\varphi} = (b/\mu_1, 0, 0) = E_0 \in \omega^*$.

Let us show that E_0 is factually isolated (see [5] or [9]). That is, there exists some neighborhood U of E_0 in Q such that E_0 is the largest invariant set in U.

In fact, let us choose

$$U = \{\varphi| \ \ \varphi = (\varphi_1, \ \varphi_2, \ \varphi_3) \in Q, \ \ ||\varphi - E_0|| < \varepsilon\}$$

for some sufficiently small positive constant ε. We shall show that E_0 is the largest invariant set in U for some ε.

If not, for any sufficiently small ε, there exists some invariant set W $(W \subset U)$ such that $W \backslash E_0$ is not empty. Let $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in W \backslash E_0$ and (S_t, I_t, R_t) be the solution of (1.1) with the initial function φ. Then, $(S_t, I_t, R_t) \in W$ for all $t \in \mathcal{R}$.

If $\varphi_2(0) + \int_0^h \varphi_2(-s)d\eta(s) = 0$, by the invariance of W and Remark 1, we also have the contradiction that $\varphi = E_0$ or that the negative semi-orbit (S_t, I_t, R_t) $(t < 0)$ of (1.1) through φ is unbounded.

If $\varphi_2(0) + \int_0^h \varphi_2(-s)d\eta(s) > 0$, from Remark 1 we see that $I(t) > 0$ for all $t \geq 0$. Now, let us consider the continuous function

$$P(t) = I(t) + \rho \int_0^h \int_{t-\tau}^t I(u)dud\eta(\tau) \tag{2.7}$$

for some constant $\rho > 0$. We see that for $t \geq 0$, the time derivative of $P(t)$ along the solution $(S(t), I(t), R(t))$ satisfies

$$\begin{aligned}
\dot{P}(t) &= \dot{I}(t) + \rho(I(t) - \int_0^h I(t-\tau)d\eta(\tau)) \\
&= [\rho - (\mu_2 + \lambda)]I(t) + [\beta S(t) - \rho] \int_0^h I(t-\tau)d\eta(\tau) \\
&\geq [\rho - (\mu_2 + \lambda)]I(t) + [\beta(\frac{b}{\mu_1} - \varepsilon) - \rho] \int_0^h I(t-\tau)d\eta(\tau) \\
&= [\frac{\beta b}{\mu_1} - (\mu_2 + \lambda) - \beta\varepsilon]I(t),
\end{aligned} \tag{2.8}$$

for $t \geq 0$. Here, we chose $\rho = \beta(b/\mu_1 - \varepsilon) > 0$ and used the inequality $S(t) \geq b/\mu_1 - \varepsilon$ for all $t \in \mathcal{R}$. From (2.6), we have that $I(t) \geq \eta/2$ for all large $t \geq t_1 > 0$, which, together with (2.8) and (1.3), shows that for some sufficiently small ε,

$$\dot{P}(t) \geq \frac{\eta}{2}[\frac{\beta b}{\mu_1} - (\mu_2 + \lambda) - \beta\varepsilon] > 0,$$

for all $t \geq t_1$. Thus, $P(t) \to +\infty$ as $t \to +\infty$. This contradicts to Lemma 1. This shows that E_0 is isolated.

We easily see that the semigroup defined by the solution of (1.1) satisfies the conditions of Lemma 4.3 in [9] with $M = E_0$. Thus, by Lemma 4.3 in [9], we have that there is some $\xi = (\xi_1, \xi_2, \xi_3)$ such that $\xi \in \omega^* \cap (W^s(E_0)\backslash E_0)$. Here, $W^s(E_0)$ denotes the stable set of E_0.

If $\xi_2(0) + \int_0^h \xi_2(-s)d\eta(s) = 0$, again by the invariance of W and Remark 1, we also have the contradiction that $\xi = E_0$ or that the negative semi-orbit $\hat{S}_t, \hat{I}_t, \hat{R}_t)$ $(t < 0)$ of (1.1) through ξ is unbounded.

If $\xi_2(0) + \int_0^h \xi_2(-s)d\eta(s) > 0$, from Remark 1, we see that $\hat{S}(t) > 0$, $\hat{I}(t) > 0$ and $\hat{R}(t) > 0$ for all $t > 0$. It follows from $\xi \in \omega^* \cap (W^s(E_0)\backslash E_0)$ that

$$\lim_{t\to+\infty} \hat{S}(t) = \frac{b}{\mu_1}, \quad \lim_{t\to+\infty} \hat{I}(t) = \lim_{t\to+\infty} \hat{R}(t) = 0,$$

which contradicts to (2.6). This completes the proof of Lemma 4.

Proof of Theorem. Note that (2.4), from (1.1) we easily have that

$$\liminf_{t\to+\infty} R(t) \geq \frac{\lambda\nu_2}{\mu_2} \equiv \nu_3 > 0.$$

Thus, (1.1) is permanent by Lemmas 1, 2 and 4. This proves our theorem.

3 Conclusion

In this paper, we considered permanence of (1.1). In biology, our theorem together with results in [3-4] and [12] show that, for any time delay h, the condition (1.3) is the threshold of (1.1) for an endemic to occur. On the other hand, the simulations for (1.1) given below suggest that the condition (1.3) maybe also implies *the global asymptotic stability of the endemic equilibrium* E_+ *of* (1.1) *for any time delay* h. Unfortunately, we cannot give a complete proof to the problem. We can only show that the endemic equilibrium E_+ of (1.1) is globally asymptotically stable for *small time delay* h [4].

Example. Note that the first two equations of (1.1) are independent of the state variable $R(t)$ and that the third equation of (1.1) is linear with respect to $I(t)$ and $R(t)$.

We consider the following sub-systems (3.1) and (3.2) with discrete and distributed time delays, respectively.

$$\begin{cases} \dot{S}(t) = & -0.1S(t)I(t-h) - 0.1S(t) + 0.5 \\ \dot{I}(t) = & 0.1S(t)I(t-h) - \alpha I(t), \end{cases} \tag{3.1}$$

$$\begin{cases} \dot{S}(t) = & -0.1S(t)\int_0^h(\frac{e^{-s}}{1-e^{-h}})I(t-s)ds - 0.1S(t) + 0.5 \\ \dot{I}(t) = & 0.1S(t)\int_0^h(\frac{e^{-s}}{1-e^{-h}})I(t-s)ds - \alpha I(t)\ , \end{cases} \tag{3.2}$$

where $\alpha > 0$ and $h > 0$. It is clear that the condition (1.3) is reduced to

$$0 < \alpha < \frac{1}{2}. \tag{3.3}$$

There exists the disease free equilibrium $E_0 = (5, 0)$ for (3.1) and (3.2). If (3.3) holds, there also exists the endemic equilibrium $E_+ = (10\alpha, (0.5-\alpha)/\alpha)$ for (3.1) and (3.2).

The following figures 1-2 illustrate our theorem and further suggest that, *for large time delay h, the endemic equilibrium E_+ of* (1.1) *is also globally asymptotically stable if and only if* (1.3) *holds.*

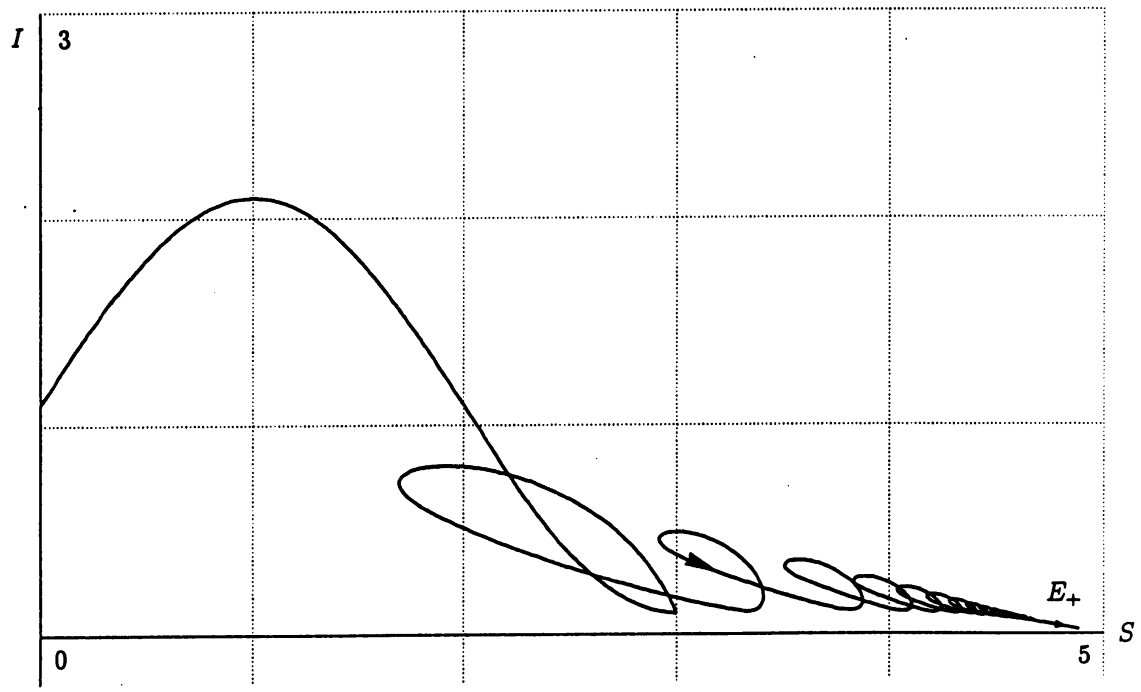

Figure 1. The graph of the trajectory of (3.1) with $\alpha = 0.49$, $h \doteq 30$ and the initial function $\varphi_1(\theta) = 0.1\theta + 3$ and $\varphi_2(\theta_2) = 1.1 - \cos(0.05\pi\theta)$ for $\theta \in [-h, 0]$.

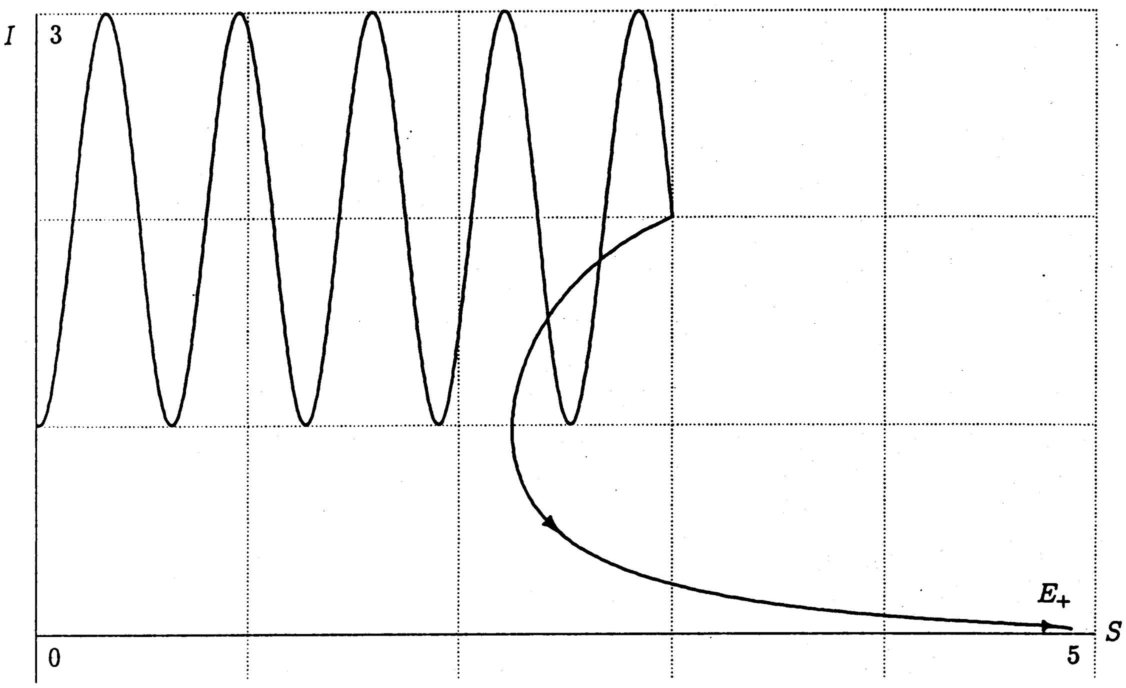

Figure 2. The graph of the trajectory of (3.2) with $\alpha = 0.49$, $h = 30$ and the initial function $\varphi_1(\theta) = 0.1\theta + 3$ and $\varphi_2(\theta_2) = 2 - \sin(\theta)$ for $\theta \in [-h, 0]$.

References

[1] R. M. Anderson, and R. M.May, Population biology of infectious diseases: Part I, *Nature* **280** (1979), 361-367.

[2] E. Beretta, V. Capasso and F. Rinaldi, Global stability results for a generalized Lotka-Volterra system with distributed delays: Applications to predator-prey and epidemic systems, *J. Math. Biol.* **26** (1988), 661-668.

[3] E. Beretta, and Y. Takeuchi, Global stability of an *SIR* epidemic model with time delays, *J. Math. Biol.* **33** (1995), 250-260.

[4] E. Beretta, T. Hara, W. Ma and Y. Takeuchi, Global asymptotic stability of an *SIR* epidemic model with distributed time delay, (to appear)

[5] G. Butler, H. I. Freedman, and P. Waltman, Uniformly persistent systems, *Proc. AMS Math. Soc.* **96** (1986), 425-430.

[6] K. L. Cooke, Stability analysis for a vector disease model, *Rocky Mount. J. Math.* **9** (1979), 31-42.

[7] H. I. Freedman, and S. Ruan, Uniform persistence in functional differential equations, *J. Differential Equations* **115** (1995), 173-192.

[8] J. K. Hale, "Theory of Functional Differential Equations," Springer-Verlag, New York, 1977.

[9] J. K. Hale, and P. Waltman, Persistence in infinite-dimensional systems, *SIAM J. Math. Anal.* **20** (1989), 388-395.

[10] H. W. Hethcote, Qualitative analyses of communicable disease models, *Math. Biosci.* **7** (1976), 335-356.

[11] Y. Kuang, "Delay Differential Equations with Applications in Population Dynamics," San Diego, Academic Press, 1993.

[12] Y. Takeuchi, W. Ma, and E. Beretta, Global asymptotic properties of a delay SIR epidemic model with finite incubation times, *J. Nonl. Anal.*, **42** (2000), 931-947.

[13] H. L. Smith, Subharmonic bifurcation in an S-I-R epidemic model, *J. Math. Biology* **17** (1983), 163-177.

[14] H. Thieme, Persistence under relaxed point-dissipativity (with applications to an epdemic model), *SIAM J. Math. Anal.*, **24** (1993), 407-435.

[15] T. Yoshizawa, "Stability Theory by Liapunov's Second Method," The Mathematical Society of Japan, Tokyo, 1966.