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Nonresonant Boundary Value Problems on a Half-line

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1 Introduction

We consider the boundary value problem (BVP):

$$u''(t)+f(t,u(t))=0 \quad \text{a.e. } t\in(0,+\infty), \qquad u(0)=\lim_{t\to+\infty}\frac{u(t)}{t}=0, \tag{1.1}$$

where $f:(0,+\infty)\times\mathbf{R}\to[-\infty,+\infty]$ is a Carathéodory function (i.e. $f(\cdot,u)$ is measurable for every $u\in\mathbf{R}$ and $f(t,\cdot)$ is continuous for a.e. $t\in(0,+\infty)$).

We first give some notations, which will be used below:

$$\begin{aligned}
AC[a,b] &= \{u \mid u \text{ is an absolutely continuous function on } [a,b]\};\\
AC_{loc}(\alpha,\beta) &= \{u \mid u_{|[a,b]}\in AC[a,b] \text{ for every compact interval } [a,b]\subset(\alpha,\beta)\};\\
L^1_{loc}(\alpha,\beta) &= \{u \mid u_{|[a,b]}\in L^1[a,b] \text{ for every compact interval } [a,b]\subset(\alpha,\beta)\};\\
C[\alpha,\beta] &= \{u\in C(\alpha,\beta) \mid {}^\exists\lim_{t\to\alpha}u(t)\in\mathbf{R},\ {}^\exists\lim_{t\to\beta}u(t)\in\mathbf{R}\};\\
AC[\alpha,\beta] &= \{u\in AC_{loc}(\alpha,\beta) \mid u'\in L^1(\alpha,\beta)\}\ (\subset C[\alpha,\beta]);\\
U &= \{u\in C[0,+\infty) \mid \frac{u}{1+(\cdot)}\in C[0,+\infty]\};\\
W &= \{u\in U \mid u\in AC_{loc}(0,+\infty),\ u'\in AC_{loc}(0,+\infty)\};\\
Z &= \{\psi\in L^1_{loc}(0,+\infty) \mid \|\psi\|_Z\equiv\int_0^{+\infty}\frac{t}{1+t}|\psi(t)|\,dt<+\infty\};\\
V &= \{\psi\in L^1_{loc}(0,+\infty) \mid \|\psi\|_V\equiv\int_0^{+\infty}t\,|\psi(t)|\,dt<+\infty\};\\
V_p &= \{\psi\in V \mid \psi(t)\ge 0 \text{ a.e. } t\in(0,+\infty),\ \int_0^{+\infty}t\,\psi(t)\,dt>0\};\\
Y &= \{v\in C[0,1]\cap C^1(0,1) \mid v'\in AC_{loc}(0,1)\};\\
X &= \{\phi\in L^1_{loc}(0,1) \mid \|\phi\|_X\equiv\int_0^1 s(1-s)\,|\phi(s)|\,ds<+\infty\};\\
X_p &= \{\phi\in X \mid \phi(s)\ge 0 \text{ a.e. } s\in(0,1),\ \int_0^1 s(1-s)\,\phi(s)\,ds>0\};
\end{aligned}$$

where $-\infty<a<b<+\infty$, $-\infty\le\alpha<\beta\le+\infty$.

Throughout this note we will make the following assumption on the Carathéodory function $f(t, u)$: **(A.F)** there exist $r_1 \in V$ and $r_2 \in Z$ such that

$$|f(t,u)| \le r_1(t)|u| + r_2(t) \quad \text{a.e. } t \in (0,+\infty) \quad {}^{\forall}u \in \mathbf{R}.$$

Further, we will assume that f satisfies a Dolph-type nonresonance condition with respect to the eigenvalue problem (EVP):

$$u''(t) + \lambda\, q(t)\, u(t) = 0 \quad \text{a.e. } t \in (0,+\infty), \qquad u(0) = \lim_{t\to+\infty} \frac{u(t)}{t} = 0, \tag{1.2}$$

where $q \in V_p$. A real number λ is called an eigenvalue of the EVP (1.2) (resp. EVP (1.5)) if there exists a nontrivial solution $u \in W$ (resp. $v \in Y$) of the EVP (1.2) (resp. EVP (1.5)), and the nontrivial solution u (resp. v) is said to be an eigenfunction corresponding to the eigenvalue λ. We shall show that the EVP (1.2) has an infinite but countable number of eigenvalues and they can be listed as

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \lambda_{n+1} < \cdots \to +\infty.$$

In the case where $q \in C[0,+\infty)$ and $q(t) > 0$ for $t \in (0,+\infty)$, similar results were known in Elbert, Kusano and Naito [1] and Kusano and Naito [2] (see also Kabeya [3]).

A solution of the BVP (1.1) (resp. BVP (1.4)) is a function $u \in W$ (resp. $v \in Y$) with $u(0) = \lim_{t\to+\infty} \frac{u(t)}{t} = 0$ (resp. $v(0) = v(1) = 0$) such that u (resp. v) satisfies the equation in (1.1) for a.e. $t \in (0,+\infty)$ (resp. (1.4) for a.e. $s \in (0,1)$).

Our main result is stated as follows:

Theorem 1.1 *Let $q \in V_p$. Assume that*

$$(\kappa_\infty - \lambda_n q) \in V_p \quad \textit{and} \quad (\lambda_{n+1} q - \kappa^\infty) \in V_p, \tag{1.3}$$

where

$$\kappa_\infty(t) \equiv \liminf_{|u|\to+\infty} \frac{f(t,u)}{u}, \quad \kappa^\infty(t) \equiv \limsup_{|u|\to+\infty} \frac{f(t,u)}{u}$$

for $t \in (0,+\infty)$, and λ_k is the k-th eigenvalue of the EVP (1.2). Then the BVP (1.1) has at least one solution $u \in W$.

The condition (1.3) is usually referred to as a Dolph-type nonresonance condition with respect to the EVP (1.2). Our method due to the transformation: $s = \dfrac{t}{1+t}$ and $v(s) = \dfrac{u(t)}{1+t}$. The transformation reduces the BVP (1.1) to the BVP:

$$v''(s) + F(s, v(s)) = 0 \quad \text{a.e. } s \in (0,1), \qquad v(0) = v(1) = 0, \tag{1.4}$$

where $F(s,v) = \dfrac{1}{(1-s)^3}\, f\Big(\dfrac{s}{1-s}, \dfrac{v}{1-s}\Big)$ for $s \in (0,1)$ and $v \in \mathbf{R}$. It also reduces the EVP (1.2) to the EVP:

$$v''(s) + \lambda\, a(s)\, v(s) = 0 \quad \text{a.e. } s \in (0,1), \qquad v(0) = v(1) = 0, \tag{1.5}$$

where $a(s) = \dfrac{1}{(1-s)^4}\, q\Big(\dfrac{s}{1-s}\Big)$ for $s \in (0,1)$. Then $q \in V$ is equivalent to $a \in X$. Moreover, $q \in V_p$ if and only if $a \in X_p$. The following was known in [12] (see also [4, Proposition 4.7]).

Lemma 1.2 *([12, Lemma 4.5]) Let $a \in X_p$. Then the EVP (1.5) has an infinite but countable number of eigenvalues and they can be listed as*

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \lambda_{n+1} < \cdots \to +\infty.$$

Moreover, for each $n \in \mathbf{N}$ the eigenfunction $v \in Y$ corresponding to λ_n is unique up to constant multiples.

To solve the reduced problem (1.4), we will use the following existence theorem in [4]:

Theorem 1.3 *([4, Theorem 5.1]) Let $a \in X_p$. Suppose that $F(s, v)$ is a Carathéodory function satisfying;*

$$|F(s,v)| \le b_1(s)|v| + b_2(s) \quad \text{a.e. } s \in (0,1) \quad {}^\forall v \in \mathbf{R}$$

for some $b_1, b_2 \in X$. Moreover, assume that $(\gamma_\infty - \lambda_n a) \in X_p$ and $(\lambda_{n+1} a - \gamma^\infty) \in X_p$, where

$$\gamma_\infty(s) \equiv \liminf_{|v| \to +\infty} \frac{F(s,v)}{v}, \quad \gamma^\infty(s) \equiv \limsup_{|v| \to +\infty} \frac{F(s,v)}{v}$$

for $s \in [0,1]$, and λ_k is the k-th eigenvalue of the EVP (1.5). Then the BVP (1.4) has at least one solution $v \in Y$.

The solvability of BVPs on semi-infinite intervals like (1.1) has been studied by Kurtz [5], Kiguradze and Shekhter [6], Chen and Zhang [7], O'Regan [8, 9] and others (see the references given in [5-9]). Although nonresonant type existence results for singular BVPs on compact intervals like (1.4) can be found in O'Regan [8, 9, 10], Kiguradze [11], Asakawa [4, 12], and others (see the references given in [8-11]), it seems that the nonresonant type of sufficient conditions for the solvability of BVPs like (1.1) is not studied so well.

2 Preliminaries

In this section we assume that $-\infty < a < b < +\infty$ and $-\infty < \alpha < \beta < +\infty$. We will consistently use the following well-known lemma (see for instance Rudin [13]):

Lemma 2.1 *Suppose that G is a function in $AC[a,b]$ with $G'(t) = g(t) \ge 0$ for a.e. $t \in (a,b)$, $G(a) = \alpha$ and $G(b) = \beta$, and that $F \in AC[\alpha, \beta]$. Then $F(G(\cdot)) \in AC[a,b]$,*

$$\frac{d}{dt}\Big[F(G(t))\Big] = f(G(t))\, g(t) \text{ for a.e. } t \in (a,b) \text{ and } \int_\alpha^\beta f(s)\, ds = \int_a^b f(G(t))\, g(t)\, dt,$$

where $f \equiv F' \in L^1(\alpha, \beta)$ and define $(\pm\infty) \cdot 0 = 0 \cdot (\pm\infty) = 0$.

We will need the following lemmas in the later sections (see [12] for more details).

Lemma 2.2 *Let G be a function in $AC[a,b]$ with $G'(t) > 0$ for a.e. $t \in (a,b)$, $G(a) = \alpha$ and $G(b) = \beta$. Suppose that M is a measurable subset of $[a,b]$, and that f and $\tilde{f}$ are measurable functions on $[a,b]$. (a) Then $G(M)$ is a measurable subset of $[\alpha, \beta]$ and $|G(M)| = \int_M G'(t)\, dt$. In particular, if $|M| = 0$, then $|G(M)| = 0$. (b) Then $f(G^{-1}(\cdot))$ is a measurable function on $[\alpha, \beta]$. Moreover, if $f(t) = \tilde{f}(t)$ for a.e. $t \in (a,b)$, then $f(G^{-1}(s)) = \tilde{f}(G^{-1}(s))$ for a.e. $s \in (\alpha, \beta)$.*

Lemma 2.3 *Let G be a function in $AC[a,b]$ with $G'(t)>0$ for a.e. $t\in(a,b)$, $G(a)=\alpha$ and $G(b)=\beta$. Then the inverse function G^{-1} of G is absolutely continuous on $[\alpha,\beta]$, and*

$$\frac{d}{ds}\Big[G^{-1}(s)\Big]=\frac{1}{G'(G^{-1}(s))}>0 \quad a.e.\ s\in(\alpha,\beta).$$

3 Green Operator

Let us define the functions $R[\psi](\cdot)$ and $T[\psi](\cdot)$ by

$$\begin{aligned} R[\psi](s) &= \frac{1}{(1-s)^4}\,\psi\Big(\frac{s}{1-s}\Big) \quad \text{for } \psi\in V,\\ T[\psi](s) &= \frac{1}{(1-s)^3}\,\psi\Big(\frac{s}{1-s}\Big) \quad \text{for } \psi\in Z \end{aligned}$$

for a.e. $s\in(0,1)$. An easy computation using Lemma 2.1, 2.2 and 2.3, shows that

Lemma 3.1 *The operator R is a bijective linear operator form V onto X and*

$$R^{-1}[\phi](t)=\frac{1}{(1+t)^4}\,\phi\Big(\frac{t}{1+t}\Big) \qquad (0<t<+\infty)$$

for every $\phi\in X$. Moreover, $\int_0^1 s(1-s)\,R[\psi](s)\,ds=\int_0^{+\infty} t\,\psi(t)\,dt$ for every $\psi\in V$. In particular, $\|R[\psi]\|_X=\|\psi\|_V$ for every $\psi\in V$, and $\psi\in V_p$ if and only if $R[\psi]\in X_p$.

Lemma 3.2 *The operator T is a bijective linear operator form Z onto X and*

$$T^{-1}[\phi](t)=\frac{1}{(1+t)^3}\,\phi\Big(\frac{t}{1+t}\Big) \qquad (0<t<+\infty)$$

for every $\phi\in X$. Moreover, $\int_0^1 s(1-s)\,T[\psi](s)\,ds=\int_0^{+\infty}\frac{t}{1+t}\,\psi(t)\,dt$ for every $\psi\in Z$.

For $\phi\in X$, define the function $L[\phi](\cdot)$ by

$$L[\phi](s)=(1-s)\int_0^s x\,\phi(x)\,dx+s\int_s^1(1-x)\phi(x)\,dx \qquad (0\le s\le 1).$$

The following lemma is the case $p\equiv 1$ in Lemma 3.3 of [12].

Lemma 3.3 *Let $\phi\in X$. Then the following two conditions are equivalent: (a) $v=L[\phi]$; (b) $v\in Y$ and v is a solution of the BVP:*

$$v''(s)+\phi(s)=0 \quad a.e.\ s\in(0,1), \qquad v(0)=v(1)=0. \tag{3.1}$$

Moreover, when either is the case, $v\in AC[0,1]$.

For a function $u\in U$, define the function $S[u](\cdot)$ by

$$S[u](s)=\frac{u(t)}{1+t}\ \ (\text{if } 0\le s<1),\ \ =\lim_{t\to+\infty}\frac{u(t)}{1+t}\ \ (\text{if } s=1),$$

where $t=\dfrac{s}{1-s}$. It is easy to see that S is a bijective linear operator from U onto $C[0,1]$

and that $S^{-1}[v](t) = \dfrac{v(s)}{1-s}$ $(0 \le t < +\infty)$ for every $v \in C[0,1]$, where $s = \dfrac{t}{1+t}$.

Lemma 3.4 *Let $u \in U$ and $\psi \in Z$. Suppose that $v = S[u]$ and $\phi = T[\psi]$.*
(a) Then $u \in W$ if and only if $v \in Y$. (b) Then u is a solution in W of the BVP:

$$u''(t) + \psi(t) = 0 \quad a.e.\ t \in (0,+\infty), \qquad u(0) = \lim_{t\to+\infty} \frac{u(t)}{t} = 0 \tag{3.2}$$

if and only if v is a solution in Y of the BVP (3.1).

Proof. For simplicity of notations, we denote by $'$ the differentiation with respect to t. Let $u \in W$ and set $v = S[u]$. Then $\left[\dfrac{u(t)}{1+t}\right]' = \dfrac{u'(t)(1+t) - u(t)}{(1+t)^2}$ for a.e. $t \in (0,+\infty)$. Using Lemma 2.1 we obtain $v \in C[0,1] \cap AC_{loc}(0,1)$ and

$$\frac{d}{ds}\Big[v(s)\Big] = \left[\frac{u(t)}{1+t}\right]' \frac{dt}{ds} = \frac{u'(t)(1+t) - u(t)}{(1+t)^2}\,\frac{1}{(1-s)^2} = u'(t)(1+t) - u(t)$$

for a.e. $s \in (0,1)$, where $t = \dfrac{s}{1-s}$. Again by Lemma 2.1, $\dfrac{dv}{ds} \in AC_{loc}(0,1)$, $v \in Y$ and

$$\frac{d^2}{ds^2}\Big[v(s)\Big] = \Big(u'(t)(1+t) - u(t)\Big)' \frac{dt}{ds} = (1+t)\, u''(t)\, \frac{1}{(1-s)^2} = u''\Big(\frac{s}{1-s}\Big)\, \frac{1}{(1-s)^3}$$

for a.e. $s \in (0,1)$. We further assume that u is a solution of the BVP (3.2). Then we have

$$\frac{d^2}{ds^2}\Big[v(s)\Big] = -\psi\Big(\frac{s}{1-s}\Big)\, \frac{1}{(1-s)^3} = -\phi(s) \quad \text{for a.e. } s \in (0,1).$$

It is clear that $v(0) = u(0) = 0$ and $v(1) = \lim_{t\to+\infty} \dfrac{u(t)}{t} = 0$. Thus, v is a solution of the BVP (3.1). Similar proof works for the converse implications. □

For $\psi \in Z$, define the function $K[\psi](\cdot)$ by

$$K[\psi](t) = \int_0^t y\,\psi(y)\,dy + t\int_t^{+\infty} \psi(y)\,dy \qquad (0 \le t < +\infty).$$

Lemma 3.5 *Let $\psi \in Z$. Then $u = K[\psi]$ if and only if $S[u] = L\big[T[\psi]\big]$.*

Proof. Let $\psi \in Z$ and set $\phi = T[\psi]$. Suppose that $u = K[\psi]$ and $v = L[\phi]$. Using Lemma 2.1 with $G(y) = \dfrac{y}{1+y}$ we obtain

$$\begin{aligned} v(s) &= (1-s)\int_0^s x\,\frac{1}{(1-x)^3}\,\psi\Big(\frac{x}{1-x}\Big)\,dx + s\int_s^1 (1-x)\frac{1}{(1-x)^3}\,\psi\Big(\frac{x}{1-x}\Big)\,dx \\ &= \frac{1}{1+t}\int_0^t y\,\psi(y)\,dy + \frac{t}{1+t}\int_t^{+\infty}\psi(y)\,dy = \frac{u(t)}{1+t} \quad (0 \le t < +\infty), \end{aligned}$$

where $s = \dfrac{t}{1+t}$. Thus $v = S[u]$, and $u = K[\psi]$ if and only if $v = L[\phi]$. This completes the proof. □

Lemma 3.5 together with Lemma 3.3 and Lemma 3.4 allow us to conclude that

Lemma 3.6 *Let $\psi \in Z$. Then the following two conditions are equivalent: (a) $u = K[\psi]$; (b) $u \in W$ and u is a solution of the BVP (3.2). Moreover, when either is the case,* $\frac{u}{1+(\cdot)} \in AC[0,+\infty]$.

4 Proof of Main Theorem

In this section we shall give a proof of Theorem 1.1. We first show that the BVP (1.1) is equivalent to the BVP (1.4) with $F(s,v)$ given by

$$F(s,v) = \frac{1}{(1-s)^3} f\left(\frac{s}{1-s}, \frac{v}{1-s}\right) \quad \text{a.e. } s \in (0,1) \quad u \in \mathbf{R}. \tag{4.1}$$

To do so, we will use the transformation: $s = \frac{t}{1+t}$ and $v(s) = \frac{u(t)}{1+t}$.

Lemma 4.1 *Suppose that $F : (0,1) \times \mathbf{R} \to [-\infty, +\infty]$ is the function defined by (4.1), where f is a Carathéodory function satisfying the condition (A.F). Then $F(s,v)$ is a Carathéodory function such that*

$$|F(s,v)| \leq b_1(s)|v| + b_2(s) \quad a.e.\ s \in (0,1) \quad {}^{\forall}v \in \mathbf{R}, \tag{4.2}$$

where $b_1 = R[r_1] \in X$ and $b_2 = T[r_2] \in X$.

Proof. Since f is a Carathéodory function, $f(\cdot, (1+(\cdot))v)$ is measurable on $(0,+\infty)$ for every $v \in \mathbf{R}$. It follows from (b) of Lemma 2.2 that $F(\cdot, v)$ is measurable. Using (a) of Lemma 2.2 we deduce that $f\left(\frac{s}{1-s}, \cdot\right)$ is continuous for a.e. $s \in (0,1)$. Hence $F(s,\cdot)$ is continuous for a.e. $s \in (0,1)$. Thus, $F(s,v)$ is a Carathéodory function. Using (a) of Lemma 2.2 it follows from (A.F) that

$$\left|f\left(\frac{s}{1-s}, \frac{v}{1-s}\right)\right| \leq r_1\left(\frac{s}{1-s}\right)\frac{|v|}{1-s} + r_2\left(\frac{s}{1-s}\right)$$

for a.e. $s \in (0,1)$ and for every $v \in \mathbf{R}$. This implies (4.2). □

Lemma 4.2 *Let $u \in U$ and let $v = S[u]$. Suppose that F is the Carathéodory function given by (4.1), where f is a Carathéodory function satisfying the condition (A.F). Then the following two assertions are equivalent: (a) u is a solution in W of the BVP (1.1); (b) v is a solution in Y of the BVP (1.4).*

Proof. Let $u \in U$ and set $v = S[u]$. It follows from (A.F) and Lemma 4.1 that $\psi(t) \equiv f(t, u(t)) \in Z$ and that $\phi(s) \equiv F(s, v(s)) \in X$. Moreover,

$$T[\psi](s) = \frac{1}{(1-s)^3} f\left(\frac{s}{1-s}, \frac{1}{(1-s)}\left[u\left(\frac{s}{1-s}\right)\left(1+\frac{s}{1-s}\right)^{-1}\right]\right) = \phi(s).$$

From (b) of Lemma 3.4 we see that (a) is equivalent to (b). This completes the proof. □

If $\lambda \in \mathbf{R}$ and $q \in V$, then $f(t,u) = \lambda q u$ is a Carathéodory function satisfying the condition (A.F) and $F(s,v) = \lambda \frac{1}{(1-s)^4} q\left(\frac{s}{1-s}\right) v = \lambda R[q](s)\, v$. As a direct consequence of Lemma 4.2 we have

Lemma 4.3 *Let $u \in U$, $\lambda \in \mathbf{R}$ and $q \in V$. Suppose that $v = S[u]$ and $a = R[q]$. Then the following two assertions are equivalent: (a) u is a solution in W of the EVP (1.2); (b) v is a solution in Y of the EVP (1.5).*

It follows from Lemma 4.3 that

Lemma 4.4 *Let $u \in U$, $\lambda \in \mathbf{R}$ and $q \in V$. Suppose that $v = S[u]$ and $a = R[q]$. (a) Then λ is an eigenvalue of the EVP (1.2) if and only if λ is an eigenvalue of the EVP (1.5). (b) Then u is an eigenfunction of the EVP (1.2) corresponding to λ if and only if v is an eigenfunction of the EVP (1.5) corresponding to λ.*

As we stated in Lemma 3.1, $q \in V_p$ is equivalent to $a \equiv R[q] \in X_p$. Combining Lemma 1.2 and Lemma 4.4 we obtain

Lemma 4.5 *Suppose that $q \in V_p$. Then the EVP (1.2) has an infinite but countable number of eigenvalues and they can be listed as*

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \lambda_{n+1} < \cdots \to +\infty.$$

Moreover, for each $n \in \mathbf{N}$ the eigenfunction $u \in W$ corresponding to λ_n is unique up to constant multiples, and the n-th eigenvalue λ_n of the EVP (1.2) is also the n-th eigenvalue of the EVP (1.5) with $a = R[q]$.

We have all the ingredients needed to prove Theorem 1.1.

PROOF OF Theorem 1.1 : We first solve the BVP (1.4) with the Carathéodory function $F(s, v)$ given by (4.1). Without loss of generality we can assume $r_1(t) \geq 0$ and $r_2(t) \geq 0$ for a.e. $t \in (0, +\infty)$. By Lemma 4.1,

$$|F(s, v)| \leq b_1(s)|v| + b_2(s) \quad \text{a.e. } s \in (0, 1) \quad {}^\forall v \in \mathbf{R},$$

where $b_1 = R[r_1] \in X$ and $b_2 = T[r_2] \in X$. Set $a = R[q]$. From Lemma 3.1 we have $a \in X_p$. It follows from (A.F) that $r_1(t) + \dfrac{r_2(t)}{|u|} \geq \dfrac{f(t, u)}{u} \geq -r_1(t) - \dfrac{r_2(t)}{|u|}$ for a.e. $t \in (0, +\infty)$ and for $u \neq 0$. From this we deduce that $\kappa_\infty \in V$ and $\kappa^\infty \in V$, where

$$\kappa_\infty(t) \equiv \liminf_{|u| \to +\infty} \frac{f(t, u)}{u} \quad \text{and} \quad \kappa^\infty(t) \equiv \limsup_{|u| \to +\infty} \frac{f(t, u)}{u}$$

for $t \in (0, +\infty)$. Then we have

$$\begin{aligned} \gamma_\infty(s) &\equiv \liminf_{|v| \to +\infty} \frac{F(s, v)}{v} = \frac{1}{(1 - s)^4} \liminf_{|v| \to +\infty} f\left(\frac{s}{1 - s}, \frac{v}{1 - s}\right) \frac{1 - s}{v} = R[\kappa_\infty](s), \\ \gamma^\infty(s) &\equiv \limsup_{|v| \to +\infty} \frac{F(s, v)}{v} = \frac{1}{(1 - s)^4} \limsup_{|v| \to +\infty} f\left(\frac{s}{1 - s}, \frac{v}{1 - s}\right) \frac{1 - s}{v} = R[\kappa^\infty](s) \end{aligned}$$

for a.e. $s \in (0, 1)$. Hence, we obtain

$$\gamma_\infty - \lambda_n a = R[\kappa_\infty - \lambda_n q] \quad \text{and} \quad \lambda_{n+1} a - \gamma^\infty = R[\lambda_{n+1} q - \kappa^\infty],$$

where λ_k is the k-th eigenvalue of the EVP (1.2). By Lemma 4.5, the λ_k is also k-th eigenvalue of the EVP (1.5). By assumption, $\kappa_\infty - \lambda_n q \in V_p$ and $\lambda_{n+1} q - \kappa^\infty \in V_p$. It follows from Lemma 3.1 that $\gamma_\infty - \lambda_n a \in X_p$ and $\lambda_{n+1} a - \gamma^\infty \in X_p$. By Theorem 1.3, there exists a solution $v \in Y$ of the BVP (1.4). Now, set $u(t) = S^{-1}[v](t) = (1+t)\, v\Big(\dfrac{t}{1+t}\Big)$. It follows from (b) of Lemma 4.2 that u is a solution in W of the BVP (1.1). This completes the proof. □

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