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Counting functions for branched covers of elliptic curves and quasi-modular forms

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Abstract: We prove that each counting function of the m-simple branched covers with a fixed genus of an elliptic curve is expressed as a polynomial of the Eisenstein series E_2, E_4 and E_6. The special case $m=2$ was considered by Dijkgraaf.

1 Introduction

We consider the counting function

$$F_g^{(m)}(q)=\sum_{d\geq 1}N_{g,d}^{(m)}q^d$$

of the branched covers of an elliptic curve. Here, $N_{g,d}^{(m)}$ is the (weighted) number of isomorphism classes of branched covers, with genus $g(>1)$, degree d, and ramification index $(m,m,\ldots,m)$, of an elliptic curve. Such a cover is called an m-simple cover. Our aim is to prove that the formal power series $F_g^{(m)}$ converges to a function belonging to the graded ring of *quasi*-modular forms with respect to the full modular group $SL(2,\mathbf{Z})$, and hence can be expressed as a polynomial of the Eisenstein series E_2, E_4 and E_6 with rational coefficients.

For $m=2$, a 2-simple branched cover is usually referred as a simple branched cover. Dijkgraaf [3] has proved that the counting function $F_g^{(2)}(q)$ is a quasi-modular form with respect to $SL(2,\mathbf{Z})$. Our result is a generalization of this result for arbitrary $m\geq 2$.

The proof [3] for $m=2$ employs the 'Fermionic formula' [5] of the partition function,

$$\begin{aligned}&\exp\left(\sum_{g=1}^{\infty}F_g^{(2)}(q)\frac{X^{2g-2}}{(2g-2)!}\right)\\ =\ & q^{-1/24}\operatorname*{Res}_{z=0}\left(\prod_{p\in\frac{1}{2}+\mathbf{Z}_{\geq 0}}(1+zq^p\exp(p^2X/2))(1+z^{-1}q^p\exp(-p^2X/2))\frac{dz}{z}\right),\end{aligned}$$

whose quasi-modularity was proven by Kaneko and Zagier [7]. The quasi-modularity of the counting function $F_g^{(2)}$ supports the mirror symmetry for an elliptic curve. For $m \geq 3$, although the relation between the counting function $F_g^{(m)}$ and the theory of mirror symmetry has not yet been clarified, the quasi-modularity of the counting function is shown to hold.

The proof of our main theorem, Theorem 9, implies that all counting functions $F_q^{(m)}$ with $m \geq 2$ and $g > 1$ *live in* the infinite product

$$V(q, t_2, t_3, \dots) = \exp(-\sum_{j=1}^{\infty} \xi(-j) t_j) \times$$

$$\operatorname*{Res}_{z=0} \left(\prod_{p \in \frac{1}{2} + \mathbf{Z}_{\geq 0}} (1 + zq^p \exp(\sum_{k\geq 2} p^k t_k))(1 + z^{-1} q^p \exp(-\sum_{k\geq 2} (-p)^k t_k)) \frac{dz}{z} \right),$$

with the infinite set of variables $q = e^{t_1}, t_2, t_3, \dots$, where the renormalizing factor $\xi(-j)$ is the special value of a Hurwitz zeta function. To be more precise, we show that every $F_g^{(m)}$ is a linear combination of the Taylor coefficients of the function V. Then, the quasi-modularity of the counting function $F_g^{(m)}$ is derived from the corresponding property for V, which was established by Bloch-Okounkov [2]. The key step in the proof of our theorem is Proposition 4, which expresses certain character values of the symmetric group S_d in a way free of the degree d, thus enabling us to summing up the numbers of branched covers to form the generating function as indicated above.

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2 Counting functions

2.1 m-simple branched cover

We fix an elliptic curve E over $\mathbf{C}$ and an integer $m \geq 2$. A pair (f, C) consisting of a (smooth complex) curve C and a holomorphic map $f : C \to E$ is *an m-simple branched cover* if the following three conditions are safisfied:

(i) C is connected.

(ii) For any $P \in C$, the branching index $e(P) = 1$ or m.

(iii) If $P \neq P'$ and $e(P) = e(P') = m$, then $f(P) \neq f(P')$.

In the case $m = 2$, a 2-simple branched cover is usually called a 'simple branched cover'. An m-simple branched cover is a natural generalization of a simple branched cover. If f is of degree d and the curve C is of genus g, then the pair (f, C) is said to be of genus g and degree d.

Two m-simple branched covers (f, C) and (f', C') are isomorphic if there is an isomorphism $\varphi : C \to C'$ such that $f = f' \circ \varphi$. The group of automorphisms on (f, C) is denoted by $\mathrm{Aut}(f, C)$ [or simply by $\mathrm{Aut}(f)$]. We will see that this is a finite group.

By the Riemann-Hurwitz formula (see e.g., [6]), we have

$$2g(C) - 2 = d(2g(E) - 2) + \sum_{P \in C} (e(P) - 1).$$

Thus the number b of branch points and the genus g of the curve C always satisfy the relation $2g - 2 = (m-1)b$. Note that the genus g does not depend on the degree d. This relation implies that the number b of branch points should be even if m is even. If m is odd, the number of branch points is arbitrary. The case $g = 1$ corresponds to the case $b = 0$; that is, the cover $f : C \to E$ is unramified.

We choose b (distinct) points $P_1, \ldots, P_b \in E$. For $g = 1 + (m-1)b/2$, let $X_{g,d} = X^{(m)}_{g,d}$ be the set of isomorphism classes of m-simple branched covers of genus g and degree d such that the ramifications occur exactly over the points $P_1, \ldots, P_b$. We will see that $X_{g,d}$ is a finite set and does not depend on the choice of the set of branch points $P_1, \ldots, P_b$. In fact, $X_{g,d}$ can also be regarded as the fiber in the fibration

$$X_{g,d} \to \mathcal{M}_g(E, d) \to E_b,$$

where $\mathcal{M}_g(E, d)$ is the Hurwitz space of m-simple branched covers, and E_b is the configuration space of unordered b-points on E.

We count the (weighted) number of elements of $X_{g,d}$ so that

$$N_{g,d} = \sum_{f \in X_{g,d}} \frac{1}{|\operatorname{Aut}(f)|}.$$

Note that $N_{g,d} = 0$ unless $2(g-1) \in (m-1)\mathbf{Z}_{\geq 0}$. The generating functions F_g for $g > 1$ are now defined by

$$F_g(q) = F_g^{(m)}(q) = \sum_{d \geq 1} N_{g,d}\, q^d.$$

These functions are called the 'counting functions'.

It is necessary to define F_1 separately. This is because covers in the case of $g = 1$ are unbranched ($b = 0$). Note that neither $X_{1,d}$ nor $N_{1,d}$ depends on m. Then we employ the definition of $F_1(q)$ introduced for the case $m = 2$ [3, §2]:

$$F_1(q) = -\frac{1}{24} \log q + \sum_{d \geq 1} N_{1,d}\, q^d.$$

Here, the first term can be considered as the contribution of the constant map (the map of degree zero) which is not a stable map. Since $N_{1,d} = \sigma_1(d)/d$, where $\sigma_1(d)$ is the sum of all divisors of d, we have the expression

$$F_1(q) = -\log \eta(q),$$

where we denote the Dedekind eta function by

$$\eta(q) = q^{1/24} \prod_{n \geq 1} (1 - q^n).$$

Next, we introduce a two-variable partition function Z,

$$\begin{aligned} Z(q,X) = Z^{(m)}(q,X) &:= \exp\left(\sum_{g\geq 1} F_g(q)\frac{X^{(2g-2)/(m-1)}}{((2g-2)/(m-1))!}\right) \\ &= \exp\left(\sum_{b\geq 0} F_{1+(m-1)b/2}(q)\frac{X^b}{b!}\right), \end{aligned}$$

which is a formal power series in q and X. We see that

$$\begin{aligned} \eta(q)Z(q,X) &= \exp\left(\sum_{g\geq 2} F_g(q)\frac{X^{(2g-2)/(m-1)}}{((2g-2)/(m-1))!}\right) \\ &= \exp\left(\sum_{b\geq 1} F_{1+(m-1)b/2}(q)\frac{X^b}{b!}\right). \end{aligned} \tag{1}$$

In the definition of the counting function F_g, we restricted ourselves to connected covers. We also need to introduce the partition function $\hat{Z}$ of the counting functions of covers which are not necessarily connected. Let $\hat{X}_{g,d}$ be the set of isomorphism classes of m-simple branched covers, which are not necessarily connected, of genus g and degree d. In other words, for $\hat{X}_{g,d}$, we impose conditions (ii) and (iii), but drop condition (i). We define the corresponding (weighted) number of elements of $\hat{X}_{g,d}$ by

$$\hat{N}_{g,d} = \sum_{f\in\hat{X}_{g,d}} \frac{1}{|\operatorname{Aut}(f)|},$$

the modified counting function $\hat{F}_g$ for $g \geq 1$ by

$$\hat{F}_g(q) = \sum_{d\geq 1} \hat{N}_{g,d}\, q^d,$$

and its generating function $\hat{Z}$ by

$$\begin{aligned} \hat{Z}(q,X) &= \sum_{g\geq 1} \hat{F}_g(q)\frac{X^{(2g-2)/(m-1)}}{((2g-2)/(m-1))!} \\ &= \sum_{b\geq 0} \hat{F}_{1+(m-1)b/2}(q)\frac{X^b}{b!}. \end{aligned}$$

The relation between the functions Z and $\hat{Z}$ is given as follows.

Lemma 1 *We have the relation* $\hat{Z}(q,X) = q^{1/24}Z(q,X)$.

Proof: This follows from a standard argument [3]. □

2.2 Representations of the fundamental group

The weighted number $\hat{N}_{g,d}$ of covers which are not necessarily connected is expressed in terms of representations of the fundamental group of the punctured elliptic curve.

Let π_1^b be the fundamental group of the b-punctured curve $E \setminus \{P_1, \ldots, P_b\}$. It is known that the group π_1^b can be expressed in terms of the generators and relations as

$$\pi_1^b = \langle \alpha, \beta, \gamma_1, \ldots, \gamma_b \mid \gamma_1 \cdots \gamma_b = \alpha\beta\alpha^{-1}\beta^{-1} \rangle.$$

Here, we denote the simple curve around a point P_i by $\gamma_i \in \pi_1(E')$.

Let S_d be the symmetric group on d elements, and let $c^{(m)}$ be the conjugacy class of S_d of type $(m, 1^{d-m})$. In other words, the class $c^{(m)}$ consists of cycles of length m. We define

$$\Phi_{g,d} = \Phi_{g,d}^{(m)} = \{\varphi \in \mathrm{Hom}(\pi_1^b, S_d) \mid \varphi(\gamma_i) \in c^{(m)} \text{ for } i = 1, \ldots, b\},$$

where the symbol "Hom" represents the set of group homomorphisms. The symmetric group S_d acts on $\Phi_{g,d}$ by

$$\varphi^\sigma(\gamma) = \sigma^{-1}\varphi(\gamma)\sigma, \qquad \sigma \in S_d, \varphi \in \Phi_{g,d}.$$

Lemma 2 (i) *As a set, we have the bijection* $\hat{X}_{g,d} \cong \Phi_{g,d}/S_d$.

(ii) $\hat{N}_{g,d} = |\Phi_{g,d}|/|S_d|$.

Proof: (i) Let $E' = E \setminus \{P_1, \ldots, P_b\}$ be a punctured curve. Let us choose a base point $P_0 \in E'$ as a base point. Then the fundamental group $\pi_1(E') = \pi_1(E', P_0)$ is isomorphic to π_1^b. For an $f \in \hat{X}_{g,d}$, we construct the corresponding map $\varphi \in \Phi_{g,d}$. Let $f^{-1}(P_0) = \{Q_1, \ldots, Q_d\}$. Then we have the natural map

$$\varphi : \pi_1^b \cong \pi_1(E') \to \mathrm{Aut}(f^{-1}(P_0)) \cong S_d.$$

Conversely, for each $\varphi \in \Phi_{g,d}$, we construct a covering $f \in \hat{X}_{g,d}$. We denote the universal covering of E' by E'^{univ}. Let $C' = E'^{univ} \times_\varphi \{1, \ldots, d\} = E'^{univ} \times \{1, \ldots, d\}/\sim$, where $(x, i) \sim (\gamma x, \varphi(\gamma)i)$ when $\gamma \in \pi_1(E')$, $x \in E'^{univ}$ and $1 \le i \le d$. Then the natural projection $f' : C' \to E'^{univ}/\pi_1^b = E'$ is a covering of degree d. This extends to a ramified covering $f : C \to E$. It is easy to see that this construction gives the required bijection.

(ii) Under the bijection in (i), the group $\mathrm{Aut}(f)$ of automorphisms corresponds to the stabilizer subgroup of S_d at φ. This implies that

$$|\mathrm{Aut}(f)| = \#\{\sigma \in S_d \mid \varphi = \varphi^\sigma\}.$$

Then we have

$$\hat{N}_{g,d} = \sum_f \frac{1}{|\mathrm{Aut}(f)|} = \frac{1}{|S_d|} \sum_f \#\{\varphi^\sigma \mid \sigma \in S_d, \varphi \text{ corresponds to } f\} = |\Phi_{g,d}|/|S_d|.$$

□

2.3 Irreducible characters of symmetric group

The number of group homomorphisms appearing in the previous lemma is written as a sum over the irreducible representations of the symmetric group.

A partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)$ of d is a non-increasing sequence $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0$ of non-negative integers such that $\sum_{i=1}^{d} \lambda_i = d$. We denote by $\mathcal{P}_d$ the set of all partitions of d. It is known that the set of irreducible representations of the symmetric group S_d is parametrized by $\mathcal{P}_d$. For each $\lambda \in \mathcal{P}_d$, we denote by χ_λ the corresponding irreducible character. Since a character is a class function, the value $\chi_\lambda(c)$ is well-defined for each conjugacy class c of S_d. We introduce the modified character

$$f_\lambda(c) = \frac{|c| \cdot \chi_\lambda(c)}{\dim \lambda},$$

where $|c|$ is the number of elements in the conjugacy class c, and $\dim \lambda$ is the dimension of the representation λ, that is, the value of $\chi_\lambda(e)$ at the identity of S_d.

Lemma 3 *For $g = 1 + (m-1)b/2$, we have*

$$|\Phi_{g,d}^{(m)}|/|S_d| = \sum_{\lambda \in \mathcal{P}_d} f_\lambda(c^{(m)})^b.$$

Proof : We apply the formula in Lemma 4 of [3] with $G = S_d$, $R = \mathcal{P}_d$, $c_1 = \cdots = c_N = c^{(m)}$, $h = 1$ and $N = b$. □

2.4 Frobenius notation

Now we recall properties of Frobenius coordinates of partitions and shifted symmetric functions. Our Frobenius coordinates are parametrized by half-integers, not by integers, as is explained below.

For a partition $\lambda = (\lambda_1, \dots, \lambda_d) \in \mathcal{P}_d$, we define the shifted partition $\tilde{\lambda} = (\tilde{\lambda}_1, \dots, \tilde{\lambda}_d)$ by $\tilde{\lambda}_i = \lambda_i - i + \frac{1}{2}$. Let I be the set of positive half-integers, $I = \frac{1}{2} + \mathbf{Z}_{\geq 0} = \{\frac{1}{2}, \frac{3}{2}, \dots\}$. A partition λ gives us two subsets $P, Q \subset I$ such that

$$\begin{aligned} P &= \{\tilde{\lambda}_i \mid \tilde{\lambda}_i > 0, i = 1, \dots, d\}, \\ Q &= \{1/2, 3/2, \dots, (2d-1)/2\} \setminus \{-\tilde{\lambda}_i \mid -\tilde{\lambda}_i > 0, i = 1, \dots, d\} = \{\tilde{\lambda'}_i \mid \tilde{\lambda'}_i > 0, i = 1, \dots, d\}, \end{aligned}$$

where λ' is the conjugate partition of λ. Then the cardinality of P equals that of Q. Conversely, for a given pair of subsets $P, Q \subset I$ with $|P| = |Q|$, we have the corresponding partition $\lambda \in \mathcal{P}_d$ with $d = \sum_{p \in P} p + \sum_{q \in Q} q$.

We remark that our Frobenius coordinates (P, Q) are shifted by $1/2$ from the Frobenius coordinates $(\alpha_1, \dots, \alpha_r \mid \beta_1, \dots, \beta_r)$ introduced in Section I.1 of [8]. The precise relation is

$$P = \{\alpha_1 + \frac{1}{2}, \alpha_2 + \frac{1}{2}, \dots, \alpha_r + \frac{1}{2}\}, \qquad Q = \{\beta_1 + \frac{1}{2}, \beta_2 + \frac{1}{2}, \dots, \beta_r + \frac{1}{2}\}.$$

For $k \in \mathbf{Z}_{\geq 0}$ we define

$$\tilde{p}_k(\lambda) = \sum_{i=1}^{d} \left(\tilde{\lambda}_i^k - (-i + \frac{1}{2})^k \right).$$

This function is written as $p_k(\lambda)$ in (5.4) of [2]. For example, $\tilde{p}_0(\lambda) = 0$, $\tilde{p}_1(\lambda) = d$. From I.1.4 of [8] we have the relation

$$\sum_{i=1}^{d} (t^{\tilde{\lambda}_i} - t^{-i+\frac{1}{2}}) = \sum_{p \in P} t^p - \sum_{p \in Q} t^{-p},$$

where (P, Q) is the Frobenius coordinates of the partition λ. Applying $(t\frac{d}{dt})^k$ on both sides and letting $t = 1$, we have

$$\tilde{p}_k(\lambda) = \sum_{i=1}^{d} \left(\tilde{\lambda}_i^k - (-i + \frac{1}{2})^k \right) = \sum_{p \in P} p^k - \sum_{p \in Q} (-p)^k.$$

This is a power-sum symmetric functions in $\tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_d)$ plus some polynomial in d of degree $k + 1$. We now introduce two additional polynomials symmetric in the $\tilde{\lambda}_i$. Let $e_j(\tilde{\lambda})$ be the jth elementary symmetric function and $h_j(\tilde{\lambda})$ the jth complete symmetric function, defined by

$$\begin{aligned} e_j(\tilde{\lambda}) &= \sum_{1 \leq i_1 < \cdots < i_j \leq d} \tilde{\lambda}_{i_1} \cdots \tilde{\lambda}_{i_j}, \\ h_j(\tilde{\lambda}) &= \sum_{1 \leq i_1 \leq \cdots \leq i_j \leq d} \tilde{\lambda}_{i_1} \cdots \tilde{\lambda}_{i_j}. \end{aligned}$$

These two functions can be expressed as polynomials in power-sum symmetric functions, and thus as polynomials in $\tilde{p}_k(\lambda)$ and d.

2.5 Character formula

The character value $f_\lambda(c^{(m)})$ can be written in terms of $\tilde{p}_k(\lambda)$. Although the character depends strongly on the rank d of the symmetric group S_d, the following expression is independent of d. This is crucial for our calculation of the counting function.

Proposition 4 *There exists a polynomial $\phi_m(Y_1, \ldots, Y_m) \in \mathbf{Q}[Y_1, \ldots, Y_m]$ such that for all $d \geq 1$ and $\lambda \in \mathcal{P}_d$, we have*

$$f_\lambda(c^{(m)}) = \phi_m(\tilde{p}_1(\lambda), \ldots, \tilde{p}_m(\lambda)).$$

Proof: We consider a partition $\lambda = (\lambda_1, \ldots, \lambda_d)$. Let

$$\mu_i = \lambda_i + d - i = \tilde{\lambda}_i + d - \frac{1}{2},$$

and $\varphi(x) = \prod_{i=1}^{d}(x - \mu_i)$. Then, from Example I.7.7 in [8], we have

$$f_\lambda(c^{(m)}) = \frac{1}{m^2} \operatorname*{Res}_{x=\infty} \left(\frac{x(x-1)\cdots(x-m+1)\varphi(x-m)}{\varphi(x)} dx \right),$$

where the symbol "Res" denotes the residue. Since $\varphi(x+d-\frac{1}{2}) = \prod_{i=1}^{d}(x-\tilde{\lambda}_i)$, we obtain

$$\begin{aligned} & f_\lambda(c^{(m)}) \\ = \ & \frac{1}{m^2} \operatorname*{Res}_{x=\infty} \left((x+d-\frac{1}{2})(x+d-\frac{3}{2})\cdots(x+d-m+\frac{1}{2}) \frac{\varphi(x-m+d-\frac{1}{2})}{\varphi(x+d-\frac{1}{2})} dx \right) \\ = \ & -\frac{1}{m^2} \operatorname*{Res}_{y=0} \left((1+(d-\frac{1}{2})y)(1+(d-\frac{3}{2})y)\cdots(1+(d-m+\frac{1}{2})y) \frac{\prod_{i=1}^{d}(1-(m+\tilde{\lambda}_i)y)}{\prod_{i=1}^{d}(1-\tilde{\lambda}_i y)} \frac{dy}{y^{m+2}} \right) \end{aligned}$$

by changing coordinates. The products appearing here are generating functions of elementary (resp. complete) symmetric functions:

$$\begin{aligned} \prod_{i=1}^{d}(1-(m+\tilde{\lambda}_i)y) &= \sum_{j=0}^{d}(1-my)^{d-j}(-y)^j e_j(\tilde{\lambda}), \\ \prod_{i=1}^{d}(1-\tilde{\lambda}_i y)^{-1} &= \sum_{j=0}^{\infty} y^j h_j(\tilde{\lambda}), \end{aligned}$$

Then,

$$\begin{aligned} & f_\lambda(c^{(m)}) \\ = \ & -\frac{1}{m^2} \sum_{i=0}^{d} \sum_{j=0}^{\infty} e_i(\tilde{\lambda}) h_j(\tilde{\lambda}) \times \\ & \operatorname*{Res}_{y=0} \left((1+(d-\frac{1}{2})y)(1+(d-\frac{3}{2})y)\cdots(1+(d-m+\frac{1}{2})y)(1-my)^{d-i}(-y)^i y^j \frac{dy}{y^{m+2}} \right) \\ = \ & -\frac{1}{m^2} \sum_{i=0}^{d} \sum_{j=0}^{\infty} (-1)^i e_i(\tilde{\lambda}) h_j(\tilde{\lambda}) b_{ij}, \end{aligned}$$

where

$$b_{ij} = \operatorname*{Res}_{y=0} \left((1+(d-\frac{1}{2})y)(1+(d-\frac{3}{2})y)\cdots(1+(d-m+\frac{1}{2})y)(1-my)^{d-i} \frac{dy}{y^{m+2-i-j}} \right).$$

Lemma 5 *The value b_{ij} is written as a polynomial in d. Specifically, it is 0 if $i+j \geq m+2$ and a polynomial with rational coefficients of degree no greater than $m+1-i-j$ if $0 \leq i+j \leq m+1$.*

Proof: If $i+j \geq m+2$, then the function inside the summation is a polynomial in y, and thus it has no pole at $y=0$ and its residue b_{ij} is 0.

We consider the case $0 \leq i+j \leq m+1$. Since b_{ij} is the coefficient of $y^{m+1-i-j}$ in the polynomial

$$\begin{aligned}&(1+(d-\frac{1}{2})y)(1+(d-\frac{3}{2})y)\cdots(1+(d-m+\frac{1}{2})y)(1-my)^{d-i}\\ =& \sum_{s=0}^{d-i}\sum_{t=0}^{m} e_t(d-\frac{1}{2},d-\frac{3}{2},\ldots,d-m+\frac{1}{2})\binom{d-i}{s}(-m)^s y^{s+t},\end{aligned}$$

we have

$$b_{ij} = \sum_{s=0}^{m+1-i-j} e_{m+1-i-j-s}(d-\frac{1}{2},d-\frac{3}{2},\ldots,d-m+\frac{1}{2})\binom{d-i}{s}(-m)^s.$$

Then b_{ij} is a polynomial in d of degree no greater than $m+1-i-j$. □

We now return to the proof of Proposition 4. We have the finite sum expression

$$f_\lambda(c^{(m)}) = -\frac{1}{m^2}\sum_{i+j\leq m}(-1)^i e_i(\tilde{\lambda})h_j(\tilde{\lambda})b_{ij}.$$

This is a polynomial in e_i, h_j and d. We know that e_i and h_j are polynomials in power-sum symmetric functions $\tilde{p}_k(\lambda)$ and d. Then, since $d=\tilde{p}_1(\lambda)$, we have proved the existence of the function $\phi=\phi_m$. □

Example 6 *For $m=2,\ldots,5$, the polynomial ϕ_m is of the following form:*

$$\phi_2 = \frac{1}{2}Y_2, \quad \phi_3 = \frac{1}{3}Y_3 - \frac{1}{2}Y_1^2 + \frac{5}{12}Y_1, \quad \phi_4 = \frac{1}{4}Y_4 - Y_1Y_2 + \frac{11}{8}Y_2,$$

$$\phi_5 = \frac{1}{5}Y_5 - Y_3Y_1 + \frac{19}{6}Y_3 - \frac{1}{2}Y_2^2 + \frac{5}{6}Y_1^3 - \frac{15}{4}Y_1^2 + \frac{189}{80}Y_1.$$

This example suggests that the degree of the polynomial ϕ_m would be m if we consider the degree of Y_j to be j. The highest order term of ϕ_m would then be Y_m/m. Although it is not necessary to know the explicit form of the polynomial ϕ_m, it could be of an independent interest.

Lemma 7 (i) *For $i+j=m+1$, we have $b_{ij}=1$ and*

$$\sum_{i+j=m+1}(-1)^i e_i(\tilde{\lambda})h_j(\tilde{\lambda})b_{ij} = 0.$$

(ii) *For $i+j=m$, we have $b_{ij} = -\frac{m^2}{2}+mi$.*

For $m=2$, $\tilde{p}_2(\lambda)/2 = f_\lambda(c^{(2)})$ has a simple expression in terms of partitions. For a partition λ, we define $n(\lambda)=\sum_{i\geq 1}(i-1)\lambda_i$. We also define the content $c(x)$ as $c(x)=j-i$ for each box $x=(i,j)\in\lambda$, as in Section I.1 of [8]. Then

$$\tilde{p}_2(\lambda)/2 = f_\lambda(c^{(2)}) = n(\lambda') - n(\lambda) = \sum_{x\in\lambda} c(x).$$

3 Quasi-modular form

3.1 Eisenstein series

We give a brief summary of quasi-modular forms to fix the notation used here. (For the precise definition and further properties, see [7] and §3 of [2].) Let τ be a complex number with $\Im\tau > 0$ and $q = e^{2\pi\sqrt{-1}\tau}$. We denote the differential operator $\frac{1}{2\pi\sqrt{-1}}\frac{d}{d\tau} = q\frac{d}{dq}$ by D. For a subgroup Γ of the full modular group $SL(2,\mathbf{Z})$ of finite index, we denote the set of modular forms of weight k by $M_k(\Gamma)$ and the graded ring of modular forms by $M_*(\Gamma) = \oplus_{k\geq 0}M_k(\Gamma)$. Similarly, we denote the set of quasi-modular forms of weight k by $\mathrm{QM}_k(\Gamma)$ and the graded ring of quasi-modular forms by $\mathrm{QM}_*(\Gamma) = \oplus_{k\geq 0}\mathrm{QM}_k(\Gamma)$. The ring $M_*(\Gamma)$ is not closed under the differentiation D, but the ring $\mathrm{QM}_*(\Gamma)$ is closed under D. Examples of (quasi-)modular forms are provided by the Eisenstein series.

We denote the Bernoulli number by $B_k \in \mathbf{Q}$, which is defined by $\dfrac{x}{e^x-1} = \displaystyle\sum_{k=0}^{\infty} B_k\frac{x^k}{k!}$. For example, $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$ and $B_6 = \frac{1}{42}$.

We define the (normalized) Eisenstein series E_k for even $k \geq 4$ by

$$\begin{aligned} E_k(\tau) &= \frac{1}{2}\sum_{(c,d)=1}\frac{1}{(c\tau+d)^k} \\ &= 1-\frac{2k}{B_k}\sum_{n=1}^{\infty}(\sum_{d|n}d^{k-1})q^n = 1-\frac{2k}{B_k}\sum_{n=1}^{\infty}\frac{n^{k-1}q^n}{1-q^n}. \end{aligned}$$

(This is a convergent series in q.) Then E_k is a modular form of weight k for $SL(2,\mathbf{Z})$:

$$E_k(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^k E_k(\tau).$$

We also define

$$E_2(\tau) = 1-24\sum_{n=1}^{\infty}(\sum_{d|n}d)q^n.$$

Then E_2 is not a modular form, but a quasi-modular form of weight 2 for $SL(2,\mathbf{Z})$, so that

$$E_2(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^2E_2(\tau) + \frac{12}{2\pi\sqrt{-1}}c(c\tau+d).$$

The ring of quasi-modular forms for the full modular group $SL(2,\mathbf{Z})$ is $\mathrm{QM}_*(SL(2,\mathbf{Z})) = \mathbf{C}[E_2, E_4, E_6]$, and the operator D preserves this ring and increases the weight by 2:

$$D(E_2) = (E_2^2 - E_4)/12, \quad D(E_4) = (E_2E_4 - E_6)/3, \quad D(E_6) = (E_2E_6 - E_4^2)/2.$$

The following lemma is used for the proof of the main theorem.

Lemma 8 *If $\eta(q)A(q) \in \mathrm{QM}_k(SL(2,\mathbf{Z}))$, then $\eta(q)D^j(A(q)) \in \mathrm{QM}_{k+2j}(SL(2,\mathbf{Z}))$ for a positive integer j.*

Proof: Recall the definition of the Ramanujan delta, $\Delta(\tau) = \eta(q)^{24} = (E_4^3 - E_6^2)/1728$. Then we have $D \log \Delta(\tau) = E_2(\tau)$ and $D(\log \eta) = E_2/24$, and we obtain the formula

$$\eta(q) DA(q) = D(\eta(q)A(q)) - \frac{1}{24} E_2 \eta(q) A(q).$$

The condition $\eta(q)A(q) \in \mathrm{QM}_k(SL(2,\mathbf{Z}))$ implies $\eta(q)D(A(q)) \in \mathrm{QM}_{k+2}(SL(2,\mathbf{Z}))$. The assertion follows from induction on j. □

3.2 The character of the infinite wedge representation

We introduce the variables $t_1, t_2, t_3, \ldots$, and write $D_k = \frac{\partial}{\partial t_k}$ for $k \geq 1$. In what follows, the variable t_1 is related to q by $q = e^{t_1}$. In particular, for $k = 1$ we have $D = D_1 = q\frac{\partial}{\partial q}$. We define the infinite series

$$\begin{aligned} V'(q, t_2, t_3, \ldots) &= \sum_{d \geq 0} \sum_{\lambda \in \mathcal{P}_d} \exp(\tilde{p}_1(\lambda) t_1 + \tilde{p}_2(\lambda) t_2 + \tilde{p}_3(\lambda) t_3 + \cdots) & (2) \\ &= \sum_{d \geq 0} \sum_{\lambda \in \mathcal{P}_d} q^{\tilde{p}_1(\lambda)} \exp(\tilde{p}_2(\lambda) t_2 + \tilde{p}_3(\lambda) t_3 + \cdots). & (3) \end{aligned}$$

This expression appears in (0.10) of [2] as a character of the infinite wedge representation of an infinite dimensional Lie algebra (W_∞), and it is known to be a quasimodular form of weight $-\frac{1}{2}$ when suitably normalized. Let us explain this in more detail.

It is easy to see that V' is the coefficient of z^0 of an infinite product:

$$\begin{aligned} V' &= \operatorname*{Res}_{z=0} \left(\prod_{p \in \frac{1}{2} + \mathbf{Z}_{\geq 0}} (1 + z \exp(\sum_{k \geq 1} p^k t_k))(1 + z^{-1} \exp(-\sum_{k \geq 1} (-p)^k t_k)) \frac{dz}{z} \right) \\ &= \operatorname*{Res}_{z=0} \left(\prod_{p \in \frac{1}{2} + \mathbf{Z}_{\geq 0}} (1 + z q^p \exp(\sum_{k \geq 2} p^k t_k))(1 + z^{-1} q^p \exp(-\sum_{k \geq 2} (-p)^k t_k)) \frac{dz}{z} \right). \end{aligned}$$

To obtain a quasimodular form, we have to multiply a fractional power in e^{t_i}. Let $\xi(s) = \sum_{n \geq 1} (n - \frac{1}{2})^{-s} = (2^s - 1)\zeta(s)$, which is continued to a meromorphic function of s. The function $\xi(s)$ at negative integer values of s is well-defined, and $\xi(-2i) = 0$ for $i \in \mathbf{Z}_{>0}$. (For example, $\xi(-1) = 1/24$, $\xi(-3) = -7/960$.) We define

$$V(q, t_2, \ldots) = \exp(-\sum_{j=1}^{\infty} \xi(-j) t_j) \times V'(q, t_2, \ldots). \tag{4}$$

If we consider the case $t_2 = t_3 = \cdots = 0$, then the infinite product reduces to

$$q^{-\xi(-1)} \prod_{p \in \frac{1}{2} + \mathbf{Z}_{\geq 0}} (1 + z q^p)(1 + z^{-1} q^p) = \frac{\sum_{n \in \mathbf{Z}} z^n q^{n^2/2}}{\eta(q)}$$

since $\xi(-1) = 1/24$. Then

$$\eta(q)V(q, 0, 0, \dots) = 1. \tag{5}$$

Now, consider the Taylor expansion of V with respect to $(t_2, t_3, \dots)$

$$V(q, t_2, t_3, \dots) = \sum_K A_K(q) \frac{t^K}{K!}, \tag{6}$$

where $K = (k_2, k_3, \dots)$ with almost all $k_i = 0$, and $t^K/K! = t_2^{k_2} t_3^{k_3} \cdots / k_2! k_3! \cdots$ is multi-index notation. The relation (5) implies that $\eta(q)A_{(0,0,\dots)}(q) = 1$. It is shown in the proof of Theorem 4.1 of [2] that $\eta(q)A_K(q) \in \mathrm{QM}_*(SL(2,\mathbf{Z}))$ and is of weight $3k_2 + 4k_3 + \cdots = \sum_{i=2}^{\infty}(i+1)k_i$. By Lemma 8, we know that $\eta(q)D^j(A_K(q)) \in \mathrm{QM}_*(SL(2,\mathbf{Z}))$ and its weight is $2j + \sum_{i=2}^{\infty}(i+1)k_i$.

3.3 Main theorem

We arrive at the stage to state our main theorem.

Theorem 9 *The counting functions $F_g(q) = F_g^{(m)}(q)$ for $g \geq 2$ belong to the graded ring $\mathrm{QM}_*(SL(2,\mathbf{Z}))$ of quasimodular forms with respect to the full modular group $SL(2,\mathbf{Z})$. In particular, $F_g^{(m)}$ is a polynomial in E_2, E_4 and E_6 with rational coefficients.*

Proof: Summarizing Lemmas 1, 2 and 3, we obtain

$$\hat{Z}(q,X) = 1 + \sum_{b\geq 0}\sum_{d\geq 1}\sum_{\lambda\in\mathcal{P}_d} \frac{1}{b!} f_\lambda(c^{(m)})^b q^d X^b = 1 + \sum_{d\geq 1}\sum_{\lambda\in\mathcal{P}_d} \exp(f_\lambda(c^{(m)})X)q^d. \tag{7}$$

We can consider the term 1 as coming from the case $d = 0$, where $R_0 = \{\emptyset\}$, $f_\emptyset = 0$. From Proposition 4, we obtain

$$\begin{aligned}
& \exp(f_\lambda(c^{(m)})X)q^d \\
= & \left[\exp(\phi_m(\tilde{p}_1(\lambda), \tilde{p}_2(\lambda), \dots, \tilde{p}_m(\lambda))X)\exp(t_1\tilde{p}_1(\lambda) + t_2\tilde{p}_2(\lambda) + \cdots + t_m\tilde{p}_m(\lambda))\right]_{e^{t_1}=q, t_2=t_3=\cdots=0} \\
= & \left[\exp(\phi_m(D, D_2, \dots, D_m)X)\exp(t_1\tilde{p}_1(\lambda) + t_2\tilde{p}_2(\lambda) + \cdots + t_m\tilde{p}_m(\lambda))\right]_{e^{t_1}=q, t_2=\cdots=t_m=0}. \qquad (8)
\end{aligned}$$

Then by (7), (8) and (2), we have

$$\begin{aligned}
& \hat{Z}(q,X) \\
= & \left[\exp(\phi_m(D, D_2, \dots, D_m)X)\sum_{d\geq 0}\sum_{\lambda\in\mathcal{P}_d}\exp(t_1\tilde{p}_1(\lambda) + t_2\tilde{p}_2(\lambda) + t_3\tilde{p}_3(\lambda) + \cdots)\right]_{e^{t_1}=q, t_2=t_3=\cdots=0} \\
= & \left[\exp(\phi_m(D, D_2, \dots, D_m)X)V'(q, t_2, t_3, \dots)\right]_{t_2=t_3=\cdots=0} \\
= & \left[\exp(\phi_m(D, D_2, \dots, D_m)X)\exp(\sum_{j=1}^{\infty} t_j\xi(-j))V(q, t_2, t_3, \dots)\right]_{t_2=t_3=\cdots=0} \\
= & \; q^{1/24}\left[\exp(\phi_m(D + \xi(-1), D_2 + \xi(-2), \dots, D_m + \xi(-m))X)V(q, t_2, t_3, \dots)\right]_{t_2=t_3=\cdots=0}.
\end{aligned}$$

Here we have used (4) for the third equality and the last equality follows from the Leibniz rule. Then,

$$
\begin{aligned}
& \eta(q)Z(q,X) && (9)\\
= \;& \eta(q)q^{-1/24}\hat{Z}(q,X) \\
= \;& \eta(q)\left[\exp(\phi_m(D+\xi(-1),D_2+\xi(-2),\dots,D_m+\xi(-m))X)V(q,t_2,t_3,\dots)\right]_{t_2=t_3=\cdots=0} \\
= \;& \eta(q)\left[\exp(\phi_m(D+\xi(-1),D_2+\xi(-2),\dots,D_m+\xi(-m))X)\sum_K A_K(q)\frac{t^K}{K!}\right]_{t_2=t_3=\cdots=0}.
\end{aligned}
$$

The coefficient of X^b on the right-hand side of (9) is equal to the quantity

$$
\frac{1}{b!}\sum_K \eta(q)\left[\phi_m(D+\xi(-1),D_2+\xi(-2),\dots,D_m+\xi(-m))^b A_K(q)\frac{t^K}{K!}\right]_{t_2=t_3=\cdots=0}.
$$

This is a finite sum and belongs to $\mathrm{QM}_*(SL(2,\mathbf{Z}))$, by Lemma 8. Then the right-hand side of (9) is a formal power series in X with coefficients in $\mathrm{QM}_*(SL(2,\mathbf{Z}))$. Hence by (1), we have

$$
\sum_{b\geq 1} F_{1+(m-1)b/2}(q)X^b/b! = \log(\eta(q)Z(q,X)) = \sum_{l=1}^{\infty}(\eta(q)Z(q,X)-1)^j(-1)^{j-1}/j.
$$

This shows that $F_g(q)\in \mathrm{QM}_*(SL(2,\mathbf{Z}))$. □

The special case $m=2$ of our theorem has been considered by Dijkgraaf [3].

4 Concluding remarks

説明不足の点についていくつかコメントを付け加える．10 月の数理研および 1 月の岡山の集会での質問が参考になった．感謝したい．

1. 分岐点の個数 b と covering の曲線 C の種数 g は $2g-2=(m-1)b$ の関係にある．表示の簡単のため，g と b が同時に両方出てくるような式を書くことがあるが，g と b は独立ではなく常に上の関係にあるものとする．例えば $Z(q,X)$ の定義式 (page 4) は

$$
Z(q,X)=\exp\left(\sum_{g\geq 1}F_g(q)X^b/b!\right)
$$

と書くと見やすい．また，g か b のどちらかが整数にならないときは，その項は 0 であると約束する．

2. base となる曲線 E の種数が 1 でないときは covering の曲線 C の種数は分岐点の個数 b のみならず次数 d にも依存する (see page 3)．このときには母関数の作り方すら自明でない．

3. 分岐点 $P_1,\dots,P_b \in E$ の位置を変更しても m-simple branched coverings の同型類の集合 $X_{g,d}$ やその重み付き個数 $N_{g,d}$ は変わらない．点の配置空間上の smooth な family になるからである．従って，母関数 $F_{g,d}(q)$ を考えるときは分岐点の位置を気にしなくて良い．(see page 3)

4. §2.4 での説明はマヤ図形を用いてもわかりやすい．半整数で番号付けられた両側無限に伸びた箱が用意されていて玉が入っている．Frobenius notation で，P は正の番号の箱で玉が入っている場所を Q は負の番号の箱で玉が入っていない場所を表わす．$\lambda = \emptyset$ すなわち $P = Q = \emptyset$ を基底状態と見て，その状態からの励起 λ を関数 $\{\tilde{p}_k \mid k = 1, 2, \dots\}$ で測っている．infinite wedge representation との関係をつけるにはこの変数が都合が良い．

5. Proposition 4 は cycle type が 1 つのサイクルとなっている $(m, 1^{d-m})$ の形の共役類のときのものであるが，他のタイプの共役類に対してもこのような公式があるかもしれない．

6. V' の定義に現れる二項式の無限積で $t_2 = t_3 = \cdots = 0$ としたもの

$$\prod_{p\in\frac{1}{2}+\mathbf{Z}_{\geq 0}} (1 + zq^p)(1 + z^{-1}q^p)$$

は Jacobi triple product identity により

$$q^{1/24}\eta(q)^{-1}\sum_{n\in\mathbf{Z}} z^n q^{n^2/2}$$

に等しい．上の式は fermion による指標の表示，下の式は boson による指標の表示である．

7. この論文の位置関係をひとことで述べると

	無限積の保型性	counting function の保型性
$m = 2$	Kaneko-Zagier	Dijkgraaf
$m \geq 3$	Block-Okounkov	O-.

8. $V(q, t_2, \dots)$ の保型性の由来は Virasoro algebra を拡張した $W_{1+\infty}$ algebra の表現 ([2]) であることから導かれていると見られる．考えている counting function F_g が保型性を持つことは，この V と関係をつけることで証明される．これがこの論文の主定理 (Theorem 9, page 12) である．しかし，保型性の理由（由来）はもっと別のところにあるはずであろう．実際 $m = 2$ のときは，$F_g(q)$ が保型性を持つことから，変数 q は形式変数ではなく $H/SL(2,\mathbf{Z})$ に意味を持つことになり，このことが‘ある意味で’ 1 次元の場合のミラー対称性を表わしているとみなすことができる [3].

9. $m = 2$ の場合は $\hat{Z}$ 自身が無限積表示 (の z^0 の係数) で書けるが，$m \geq 3$ の場合はそのような表示はない．ϕ_m が線型でないことが関係している．また，ϕ_m が $m \geq 3$ の場合は斉次でもないので F_g は quasi-modular form ではなく，(異なる次数の)quasi-modular form の有限和で表わされている．各次数の成分が何を意味するか，あるいは一つの次数を取り出すように $X_{g,d}$ を分割するなどということができるかどうかはわからない．

10. F_g は E_2, E_4, E_6 の多項式として（一意に）書ける事は証明したが，その多項式の具体形はわかっていない．[7] では $m = 2$ の場合に，E_2 に関する最高次の係数を与えている．この部分は $m \geq 3$ でも拡張できる可能性がある．なお，その論文では next term も原理的には同様の方法を続けることができるが計算は急速に unmanageable になるとの言明がある．

11. この論文は **math-ph/9909023** においてある.

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