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# On a paper of Hermite and Diophantine Approximation of Abelian Integrals

by

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## 0 Introduction

A careful study of a largely forgotten article of C. Hermite « Sur quelques équations différentielles linéaires » [HER], published in Crelle's Journal in 1875 gives us many interesting new results about simultaneous approximations of Abelian integrals.
Hermite's paper begins (as his title says) with a linear differential equation which is a particular case of the so-called Tissot Pochhammer differential equation. Later it deals with simultaneous approximations to logarithmic functions, both in the classical and hyperelliptic case.
However many proofs are not complete and it is often difficult to restore a correct version of them. Nevertheless many tools of modern mathematics are hidden in this paper of Hermite : "Padé approximation of the second kind, monodromy, Picard Lefchetz principle, computation of the determinants of periods of integrals", infortunately without arithmetic applications.
This paper was published in 1875, two years after the publication of the proof of the transcendence of $e$. If we carefully study Hermite's proof, we can see that both constructions are similar and uses Padé approximation of the second kind. It is curious and strange that Hermite uses linear differential equations to study simultaneous rational approximations of Abelian integrals at distinct points but one can recall for instance Siegel's $E$ and $G$ functions are also related to linear differential equations.
In the following we denote by

$$\begin{aligned}\Phi(x) &= X^{m+1} + a_1X^m + \cdots + a_{m+1} \in \mathbb{Q}[X] \\ &= \prod_{0\leq i\leq m} (X - e_i) \end{aligned} \tag{0.1}$$

with $e_i \neq e_j$ for $i \neq j$, $0 \leq i \leq m$, $0 \leq j \leq m$. $I$ denotes a quadratic imaginary field, $\mathbb{Z}(I)$ his ring of integers and for $i$, $1 \leq i \leq m$. For $k \geq 2$ and $k$ does not divide $m+1$, we set

$$f_i(x) = \Phi(x)^{1/k} \int_{e_0}^{x} t^{i-1}\Phi(t)^{-1/k}dt \tag{0.2}$$

(for a suitable choice of the branch of $\Phi(x)^{1/k}$ and for $e_0 \in \mathbb{Q}$).

The main result of this paper says that for $x \in I$ satisfying some arithmetic conditions the numbers $1, f_1(x), f_2(x), \cdots, f_m(x)$ are linearly independent over $\mathbb{Z}(I)$ and satisfy a linear independence measure condition. (see the theorems 2.1 and 2.2).

# 1 Tissot Pochhammer equation, cycloelliptic curves and Padé approximation

Let us consider the $k$-cyclic covering of the projective plane $\mathbb{P}_1 = \mathbb{P}_1(\mathbb{C})$

$$V_k(t,u) = \{(t,u) \in \mathbb{P}_1 \times \mathbb{P}_1 \mid u^k = \Phi(t)\} \tag{1.1}$$

(cycloelliptic curve).

Using the Riemann Hurwitz formula it is easy to compute the genus $g$ of $V_k(t,u)$.

For $m_i \in \mathbb{Z}^+$, $0 < m_i < k$, we set $\delta_i = m_i/k$ and

$$v(t) = \prod_{0 \le i \le m} (t-e_i)^{\delta_i - 1} \tag{1.2}$$

the branches of $v(t)$ are solutions of the linear differential equation of the first order

$$v'(t) + \{\sum_{0 \le i \le m} \frac{1-\delta_i}{t-e_i}\} v(t) = 0 \tag{1.3}$$

If $k=1$ (ie. $g=0$) we shall set $v(t) \equiv 1$. The differential forms

$$\omega_x(t) = \frac{v(t)dt}{x-t} \tag{1.4}$$

are differential forms of the third kind on $V_k(t,u)$.

The family of integrals

$$\varphi_j(x) = \int_{\gamma_j} \omega_x(t)dt = \int_{\gamma_j} \frac{v(t)dt}{x-t}, \quad 1 \le j \le m \tag{1.5}$$

where $\gamma_j$ denotes any path beginning at $e_0$ and ending at $e_j$ on $V_k(t,u)$ or a Pochhammer double loop encircling $e_0$ and $e_j$.

For the moment we assume that the paths $\gamma_j$ do not surround any point "above

x" on the Riemann surface $V_k(t,u)$.
For this choice of path, we can write for $|x| > 1$

$$\varphi_j(x) = \sum_{k=0}^{\infty} \Big( \int_{\gamma_j} v(t) t^k dt \Big) / x^{k+1} \tag{1.6}$$

ie. $\varphi_j(x)$ is holomorphic at infinity and has a zero there. Under the above assumptions we set

$$R_j(x) = \int_{\gamma_j} \frac{v(t)\Phi(t)^n dt}{(x-t)^{n+1}}, \quad 1 \le j \le m \tag{1.7}$$

then

$$R_j(x) = 1/x^{n+1} \sum_{k=0}^{\infty} \Big( \int_{\gamma_j} v(t)\Phi(t)^n t^k dt \Big) / x^k . \tag{1.8}$$

This family of functions on $V_k(t,u)$ gives an answer to the following question : "Find a polynomial $Q(x)$, not identically zero, of degree at most $N = nm$ such that for $j$, $1 \le j \le m$

$$Q(x)\varphi_j(x) - [Q(x)\varphi_j(x)]_{N-1} = O(1/x^{n+1}) \tag{1.9}$$

(where the bracket square $[\,]_{N-1}$ denotes the truncated series of order $N-1$) $R_j(x)$ has at the point at infinity a zero of order at least $n+1$ .

This problem has a solution, since it reduces to a system of $N = nm$ linear homogeneous equations in $N+1$ unknowns (the coefficients of $Q(x)$) and the solution is called Hermite-Padé approximation of the second kind in the neighborhood of infinity for the set of functions $\varphi_j(x)$.

Since for $j$, $1 \le j \le m$, $R_j(x) = O(1/x^{n+1})$, we can guess that for $1 \le j \le m$ the solution of this problem will be explicit. As in the Hermite's paper, we can give two proofs.

**First proof :**

A $n$ fold integration by parts gives

$$R_j(x) = (-1)^n/n! \int_{\gamma_j} \frac{D_t^{(n)}\{v(t)\Phi(t)^n\}dt}{x-t} \tag{1.10}$$

where $D_t$ denotes the derivation on the Riemann surface $V_k(t,u)$
If we set for $1 \le j \le m$

$$Q(t) = \frac{(-1)^n}{n!} v(t)^{-1} D_t^{(n)}\{v(t)\Phi(t)^n\} \tag{1.11}$$

$$P_j(x) = \int_{\gamma_j} \frac{Q(x)-Q(t)}{x-t} v(t) dt , \tag{1.12}$$

we see that $Q(t)$ is a polynomial of degree $mn$ and $P_j(x)$ is a polynomial of degree $mn-1$ and according to (1.11) and (1.12)

$$R_j(x) = Q(x)\varphi_j(x) - P_j(x) \tag{1.13}$$

**Remark 1.1** $Q(x)$ is a generalisation of Jacobi's polynomials.

**Second proof :**

In the remainder of this section, we assume that the path $\gamma_j = (e_0, e_j)$ begins at $e_0$ and ends at $e_j$, $(1 \le j \le m)$. On the Riemann surface $V_k(t,u)$, we can use "Hermite's trick" and deform this path to add to it a loop called a "vanishing cycle".
The new path $\gamma_j(x)$ is composed of the path $\gamma_j$, a line segment $\ell_{e_jx}$ in the positive sense, a small circle $C(x)$ of center $x$ and the segment $\ell_{e_jx}^{-1}$ in the opposite sense. We set $\gamma_x = \ell_{e_jx}^{-1} \circ C(x) \circ \ell_{e_jx}$. With these notations

$$\gamma_j(x) = \gamma_j + < \gamma_j \mid \gamma_x > \gamma_x \quad \text{(Picard Lefchetz Principle)} \tag{1.14}$$

where $< \gamma_j \mid \gamma_x >$ denotes the intersection index, $< \gamma_j \mid \gamma_x > = \pm 1$. Now we can rewrite the integral 1.7 with $\gamma_j(x)$ instead of $\gamma_j$. It follows that

$$\begin{aligned} \tilde{R}_j(x) &= \int_{\gamma_j(x)} \frac{v(t)\Phi(t)^n dt}{(x-t)^{n+1}} = \int_{\gamma_j} \frac{v(t)\Phi(t)^n dt}{(x-t)^{n+1}} \\ &+ 2i\pi \operatorname{res}\left\{\frac{v(t)\Phi(t)^n}{(x-t)^{n+1}}\right\}_{t=x}, \quad (< \gamma_j \mid \gamma_x > = 1) \end{aligned}$$

Hence

$$\tilde{R}_j(x) = R_j(x) + 2i\pi \frac{(-1)^n}{n!} D_x^{(n)}\{v(x)\Phi(x)^n\} \tag{1.15}$$

Similarly, writing (1.9) in the form

$$\tilde{R}_j(x) = \tilde{Q}(x) \int_{\gamma_j(x)} \frac{v(t)}{x-t} dt - \tilde{P}_j(x)$$

we get

$$\tilde{R}_j(x) = R_j(x) + 2i\pi v(x)\tilde{Q}(x)$$

and by identification with (1.15) we obtain $Q(x) = \tilde{Q}(x)$.

This second proof gives more. Namely that $R_{m+1}(x) = Q(x)v(x)$ is the $m+1$-th solution of the Tissot-Pochhammer linear differential equation of order $m+1$ satisfied by the $m$ linear independent solutions $R_1(x), R_2(x), \cdots, R_m(x)$. This equation, has the following form [INCE]

$$\begin{aligned} &(T.P.E) \quad \Phi(x)y^{(m+1)} - \mu\Phi'(x)y^{(m)} + \binom{\mu+1}{2}\Phi''(x)y^{(m-1)} - \\ &\Phi_0(x)y^{(m)} - (\mu+1)\Phi_0'(x)y^{(m-1)} + \binom{\mu+2}{2}\Phi_0''(x)y^{(m-2)} - \dots \end{aligned} \tag{1.16}$$

$$\Phi_0(x) = -\Phi(x)\sum_{j=0}^{m}(1-\delta_j)/(x-e_j) \tag{1.17}$$

$$\mu = -(n+1) \tag{1.18}$$

This differential equation is Fuchsian with singularities $e_0, e_1, \cdots, e_m, \infty$ (ramification's points of $V_k(t,u)$).

We can associate to (1.6) the Riemann scheme

$$\begin{pmatrix} e_0 & e_1 & \cdots\cdots & e_m & \infty \\ 0 & 0 & & 0 & n+1 \\ 1 & 1 & \cdots\cdots & 1 & n+2 \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ m-1 & m-1 & \cdots\cdots & m-1 & n+m \\ \delta_0-1 & \delta_1-1 & \cdots\cdots & \delta_m-1 & \mu' \end{pmatrix} \tag{1.19}$$

$\mu' = \sum_{j=0}^{m}(\delta_j - 1) + nm.$
which gives the roots (exponents) of the indicial equations of the T.P.E. at $e_0, e_1, \cdots, e_m, \infty$.

It is very useful to notice that $R_1(x), R_2(x), \cdots, R_m(x)$ belong to the exponents $n+1$ and that $R_{m+1}(x)$ belongs to the exponent $\mu'$ at infinity.

Furthermore, Cauchy's formula yields

$$R_{m+1}(x) = \frac{1}{2i\pi}\int_{\gamma_{m+1}} \frac{v(t)\phi(t)^n}{(x-t)^{n+1}}dt \tag{1.20}$$

where $\gamma_{n+1}$ denotes a closed Jordan curve containing $x$ in the interior of a domain where $v(t)$ is holomorphic.

Now if we choose the $m+1$ steepest descent paths $\tilde{\gamma}_j(x)$ homological to $\gamma_j$ such that the critical points of the integrands of $R_i(x)$, $1 \leq i \leq m+1$ are the roots of the family of equations

$$E_x : \quad (t-x)\Phi'(t) - \Phi(t) = 0\,. \tag{1.21}$$

If $\theta_1(x), \theta_2(x), \cdots \theta_m(x)$ denote the roots of (1.21), then for $1 \leq j \leq m+1$, $\theta_j(x) \in \tilde{\gamma}_j(x)$. With these choices, the new integrals

$$\tilde{R}_j(x) = \int_{\tilde{\gamma}_j(x)} \frac{v(t)\Phi(t)^n dt}{(x-t)^{n+1}}\,, 1 \leq j \leq m+1 \tag{1.22}$$

satisfy the equation (1.16). From this we can obtain the asymptotic behaviours of these integrals, namely

$$\lim_{n\to+\infty} 1/n \log|\tilde{R}_j(x)| = \log|\Phi'(\theta_j(x))| \tag{1.23}$$

**Remark 1.2** The relations $R_j(x) = O(1/x^{n+1}), 1 \leq k \leq m$ and $Q(x) = O(x^{mn})$ give

$$\varphi_j(x) - \frac{P_j(x)}{Q(x)} \underset{\infty}{\sim} 1/Q(x)^{1+1/m} \tag{1.24}$$

and if $|x| \to \infty$

$$Q(x) \underset{\infty}{\sim} R_j(x)^{-m} . \tag{1.25}$$

# 2 Arithmetic applications

## 2.1 The logarithmic case

In the section, we assume that $v(t) \equiv 1$ and we obtain simultaneous rational approximations of the integrals

$$\int_{e_0}^{e_j} \frac{dt}{x-t}, 1 \leq j \leq m . \tag{2.1}$$

For instance if we set $\Phi(t) = t^3 - t$, we can prove that for $q \in \mathbb{Z}^+$, $q \geq 4$ the three numbers

$$1, \log(1-\frac{1}{q}), \log(1+\frac{1}{q})$$

are linearly independent over $I$ and we find a very precise measure of linear independence for these numbers.

Following a method of M. Hata [HA] which uses Padé-type approximations, we can state the following estimate.

**Theorem 2.1** *For rational integers $b_0, b_1, b_2$ with $H = \max(|b_1|, |b_2|) \geq 2$*

$$|b_0 + b_1 \log 2 + b_2 \log 3| \geq H^{-10,101\cdots} \tag{2.2}$$

We outline only the proof.

One chooses $\phi(t) = t(t-1/2)(t-1/3)$, $D_n \in \mathbb{Z}^+$, such that for $i, j, k$, $0 \leq i \leq 2n$, $0 \leq j \leq 2n$, $0 \leq k \leq 2n$, $i+j+k-3n \neq 0$ and

$$D_n \binom{2n}{i}\binom{2n}{j}\binom{2n}{k} \Big/ (i+j+k-3n) \in \mathbb{Z} ,$$

then it is easy to verify that we have simultaneous approximations of $\log(2/3)$ and $\log(3/4)$, namely

$$\begin{cases} 6^{3n} D_n \int_0^{1/3} \frac{\phi(t)^{2n}}{(t-1)^{3n}} dt = a_n \log(2/3) - b_n^1 \\ 6^{3n} D_n \int_{1/3}^{1/4} \frac{\phi(t)^{2n}}{(t-1)^{3n}} dt = a_n \log(3/4) - b_n^2 \end{cases} \tag{2.3}$$

with $a_n \in \mathbb{Z}, b_n^1 \in \mathbb{Z}, b_n^2 \in \mathbb{Z}$ .

## 2.2 Abelian integrals

From now on we suppose that $g \geq 1$, $k \geq 2$ and $k$ does not divide $m+1$ (i.e the point at infinity is ramified on $V_k(t,u)$). We have to introduce some notations

- $v(t) = \Phi(t)^{-1+1/k} = u^{1-k}(t)$ (For a suitable determination of $u(t)$)
- $\tilde{d}$ a denominator of $a_1, a_2, \cdots, a_{m+1}, e_0$ ie. $\tilde{d}$ is such that $\tilde{d}a_j \in \mathbb{Z}, 1 \leq j \leq m+1$, resp. $\tilde{d}e_0 \in \mathbb{Z}$.
- $$E_k = \begin{cases} e^m \prod\limits_{\substack{p|k \\ p \text{ prime}}} p^{[\frac{m}{p-1}]} \text{ if } a_{m+1} \neq 0 \\ exp\left\{k/_{\varphi(k)} \sum\limits_{\substack{s=1 \\ (s,k)=1}}^{k} 1/s\right\} \prod\limits_{\substack{p|k \\ p \text{ prime}}} p^{[\frac{m}{p-1}]} \text{ if } a_{m+1} = 0 \end{cases}$$

  where $\varphi$ denotes Euler's $\varphi$ function.
- $\tilde{D}_k = kbE_k\tilde{d}$

In the sequel the roots $\theta_i(x)$, $1 \leq i \leq m+1$ of (1.21) are chosen such that

$$|\Phi'(\theta_1(X))| \leq |\Phi'(\theta_2(x)| \leq \cdots |\Phi'(\theta_m(x)| \leq |\Phi'(\theta_{m+1}(x)|$$

and we set $R(x) = |\Phi'(\theta_m(x)|$ ; $E(x) = |\Phi'(\theta_{m+1}(x))$. Then we have the main result of this paper.

**Theorem 2.2** *If $x = a/b \in I$ satisfies*

$$\tilde{D}_k^m R(x) < 1 . \tag{2.4}$$

*Then the numbers $1, f_1(x), f_2(x), \cdots, f_m(x)$ are linearly independent over $I$. Moreover, for arbitrary rational integers $a_0, a_1, a_2 \cdots a_m$ of $I$ with $H = \max(|a_1|, \cdots, |a_m|) \geq 2$, the linear form*

$$L = a_0 + \sum_{i=1}^{m} a_i f_i(x)$$

*admits the lower estimate $|L| \geq cH^{-C(x)}$ where $c$ is an effectively computable positive constant and*

$$C(x) = \log(\tilde{D}_k^m R(x)) \Big/ \log(\tilde{D}_k^m E(x)) \quad \textit{(Measure of linear independence)} \tag{2.5}$$

**Remark 2.1** Using (1.25), we see that if $|x| \to \infty$, $C(x) \to m$.
With the same notations as in the previous theorem we have

**Corollary 2.1** *If $x = \wp(u) = a/b$ satisfies $\tilde{D}_2^2 R(x) < 1$ ($\wp(u)$ denoting as usual the Weierstrass $\wp$ function) the numbers $\wp'(u)$, $u - \omega_1/2$, $\zeta(u) - \eta_1/2$ are linearly independent over $I$ and*

$$|b_0\wp'(u) + b_1(u - \omega_1/2) + b_2(\zeta(u) - \eta_1/2)| \geq CH^{-C(x)}$$

*where $\omega_1$, resp. $\eta_1$ denotes a period (resp.) a quasi-period of the elliptic curve.*

We give here only the ideas of the proofs of the main lemmas. The detailed proofs will appear in a forthcoming publication.

# 3 Sketch of the proofs

**Lemma 3.1** *Interchange of the argument and the parameter.*
*If we consider the integrals*

$$\int_{e_0}^{e_j} \frac{v(t)}{x-t} dt$$

*with* $v(t) = \phi(t)^{-1+1/k}$, *t is called the argument and x the parameter. Then this integral satisfies the following relation*

$$\Phi(x)\eta(x) \int_{\gamma_i} \frac{v(t)}{x-t} dt = \sum_{j=0}^{m-1} \left\{ \int_{\gamma_i} \left(v(t)t^j\right) dt \right\} \int_{e_0}^{x} \lambda_{m-1-j}(t)\eta(t)dt \qquad (3.1)$$

*$\eta(t)$ being a solution of the adjoint differential equation related to (3.1)*

$$-\phi(t)\eta'(t) + \left(\Phi_0(t) - \Phi'(t)\right)\eta(t) = 0$$

*$\lambda_{m-1-j}(t)$ is a polynomial of degree $m-1-j$ that depends on the differential equation T.P.E.*

We do not prove this relation here [HU]. It uses some tools about differential equations [INCE] or [SCH]. The reader can nevertheless verify the relation

$$\frac{\partial}{\partial x}\left\{ \frac{\sqrt{\phi(x)}}{(t-x)\sqrt{\phi(t)}} \right\} - \frac{\partial}{\partial t}\left\{ \frac{\sqrt{\phi(t)}}{(x-t)\sqrt{\phi(x)}} \right\} = \frac{U(x,t)}{\sqrt{\phi(x)}\sqrt{\phi(t)}} \qquad (3.2)$$

which is a particular case of (3.1) for the hyperelliptic case.
Where $U(x,t)$ denote the polynomial (anti-symmetric) of degree $m-1$.

$$U(x,t) = \frac{\frac{1}{2}(\phi'(x) + \phi'(t))(t-x) + \phi(x) - \phi(t)}{(t-x)^2} .$$

To obtain a relation in the hyperelliptic case it suffices to integrate this relation by respect to $t$.

**Lemma 3.2**

$$\int_{\gamma_i} \frac{Q(x) - Q(t)}{x-t} v(t)dt = \sum_{s=0}^{m-1} \Big( \int_{\gamma_i} t^s v(t)dt \Big) Q_{m-s}(x) . \qquad (3.3)$$

This lemma says that the module of differential forms $t^k v(t)dt$ with $k \in \mathbb{Z}^+$ is generated modulo exact differentials by $t^k v(t)$, $0 \le k \le m-1$ and the proof uses De Rham cohomology on $V_k(t,u)$.

Having disposed of these preliminary steps, we find $m$ relations that we can write for $1 \le i \le m$

$$R_i(x) = \sum_{j=1}^{m} \Big(\int_{\gamma_i} t^{j-1} v(t)dt\Big)\big\{Q(x)\big(\Phi(x)\eta(x)\big)^{-1}\int_{e_0}^{x} \lambda_{m-1-j}(t)\eta(t)dt - Q_{m-1}(x)\big\} \tag{3.4}$$

Now, we see that we must invert the matrix $\Delta = (\alpha_{ij})_{\substack{1\le i\le m\\ 1\le j\le n}}$ where $\alpha_{ij} = \int_{\gamma_i} t^{j-1}v(t)dt$.

The determinant of this matrix is the determinant of periods of integrals on the curve $V_k(t,u)$.
The proof that $\det(\Delta) \ne 0$ uses the properties of the differential equation (1.16) and in the particular case of elliptic curves we find $\det \Delta = \omega_1\eta_2 - \omega_2\eta_1 = i\pi/2$ (Legendre's relation).
The other lemmas are more classical applications of the theory of diophantine approximation [HU].

In the following, we set $v(x) = \phi(x)^{1-1/k}$.

**Lemma 3.3** *Let us set*

$$\tilde{E}_k = \tilde{E}_k(n,m) = \tilde{d}^{nm} k^{nm} \prod_{p|k} p^{[\frac{nm}{p-1}]} \ell cm(1, 1+k, \cdots, 1+knm)$$

*if* $a_{m+1} \ne 0$ *and*

$$\tilde{E}_k = \tilde{E}_k(n,m) = \tilde{d}^{nm} k^{nm} \prod_{p|k} p^{[\frac{nm}{p-1}]} \ell cm(1, 2, \cdots, nm)$$

*if* $a_{m+1} = 0$. *Then* $\tilde{E}_k Q(x)$ *resp.* $\tilde{E}_k Q_{m-i}(x) \in \mathbb{Z}[x]$, $1 \le i \le m$. *Furthermore*

$$\tilde{E}_k \le \tilde{D}_k^{nm} \quad \textit{(see 2.5)}$$

To finish the proof of the theorem (2.2), we are obliged to compute the determinant of the following lemma and to show that it is nonzero.

For $s = n, n+1, \cdots n+m$, we set

$$Q = Q_s, \quad R_i = R_{s,i}, \quad P_i = Q_{s,m-i}, \quad 0 \le i \le m$$

**Lemma 3.4** *The determinant*

$$\delta_m(x) = \begin{vmatrix} Q_n(x) & P_{n,1}(x) & \cdots & P_{n,m}(x) \\ Q_{n+1}(x) & P_{n+1,1}(x) & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ Q_{n+m}(x) & P_{n+m,1}(x) & \cdots & P_{n+m,m}(x) \end{vmatrix}$$

$= \phi(x)^{n(m+1)/2} w(x)/v(x)$ *where* $w(x)$ *is the wronskian of the Tissot-Pochhammer equation. In particular for* $x \ne \{e_1, e_2, \cdots, e_m\}$, $\Delta_m(x) \ne 0$.

This lemma shows that among the linear forms $b_0 Q_i(x) + \sum_{j=1}^{m} b_j P_{i,j}(x)$, there exists at least one form which is nonzero.

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