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On perturbations of rational maps and construction of semiconjugacies on the Julia sets
(有理写像の摂動と Julia 集合上の半共役の構成について)

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Abstract

In this note, we investigate perturbations of parabolic rational maps on the Riemann sphere, and dynamical stability of their Julia sets. A rational map f is called *parabolic* if every critical point is contained in the Fatou set. If a perturbation of f into another parabolic rational map is horocyclic, then we can construct a semiconjugacy on their Julia sets. This means that parabolic rational maps have weak J-stability.

1 J-stability

Let $f:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$ be a rational map of degree $d\geq 2$, and Rat_d the space of all rational maps of degree d. The topology of this space is defined by the uniform convergence on the sphere measured by the spherical distance $d_\sigma(\cdot,\cdot)$.

In this note, we discuss perturbations of a rational map (especially *parabolic* rational map) f within Rat_d, and study the dynamical stability of f on the Julia set $J(f)$: That is, structural stability of f restricted on the Julia set. Here a *perturbation* of f means a family of rational maps $\{f_\epsilon\in\mathrm{Rat}_d:\epsilon\in[0,1]\}$ satisfying $f_0=f$ and $d_\sigma(f_\epsilon,f)\to 0$ $(\epsilon\to 0)$. We represent this family as the form of convergence, $f_\epsilon\to f$.

About this, the result below is famous:

Theorem 1.1 (Mañé-Sad-Sullivan[10]) [1] *If f has a connected neighborhood $U\subset\mathrm{Rat}_d$ where the number of attracting cycles is locally constant, then*

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[1]The original theorem is much better.

for each $f_\epsilon \in U$ there exists a quasiconformal conjugacy $h_\epsilon : J(f_\epsilon) \to J(f)$, that is, $h_\epsilon \circ f_\epsilon = f \circ h_\epsilon$ on $J(f_\epsilon)$.

This means that the dynamics on the Julia set varies continuously for any perturbation. We say such a rational map f is *J-stable.* For example, hyperbolic rational maps are J-stable.

2 Parabolic bifurcation

Let a be a parabolic periodic point of period l with multiplier $(f^l)'(a) = \lambda$ such that $\lambda^q = 1$. Then we can take a local coordinate near a such that a is mapped to 0 and that

$$f^{lq}(z) = z + z^{p+1} + O(z^{p+2})$$

where p is a multiple of q and is unique for a. [2] We call p the *petal number* of a and is denoted by $p(a)$. Note that a has multiplicity $p+1$ as a fixed point of f^{lq} (See the left figure in Figure 2). Then by a perturbation of f, a parabolic cycle may split into $p+1$ cycles (maybe attracting, repelling, indifferent) with multiplicity, and the dynamics may change not only locally but also globally.

For example, let us consider perturbations of a quadratic polynomial $f(z) = z + z^2$, which has a parabolic fixed point with the petal number 1 at the origin.

(1) $f_\epsilon(z) = z + z^2 - \epsilon \ (\epsilon \searrow 0)$

(2) $f_\epsilon(z) = z + z^2 + \epsilon \ (\epsilon \searrow 0)$

Under the perturbation (1), the parabolic point 0 splits into an attracting fixed point $-\sqrt{\epsilon}$ and a repelling fixed point $\sqrt{\epsilon}$ (In this case, the Julia sets vary continuously). *Note that the number of attracting cycles is locally non-constant.*

Under the perturbation (2), the parabolic point 0 splits into a pair of repelling fixed points $\pm\sqrt{\epsilon}i$. (See Figure 1. In this case, the Julia sets vary discontinuously! [3])

Then let us consider:

Problem. For a rational map that has parabolic points, find a MSS-like theorem (or some kind of J-stability) by controlling the parabolic bifurcations.

[2] See [1, II.5] for basic properties of parabolic points.

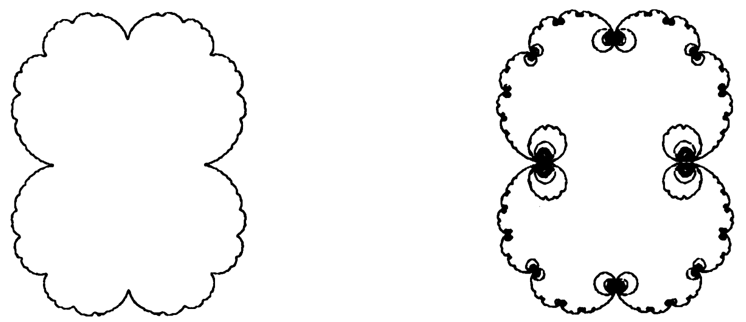

Figure 1: The Julia sets of $z + z^2$ and $z + z^2 + 0.001$.

3 Parabolic rational maps and horocyclic perturbation

For the problem, let us introduce the simplest class of rational maps that have parabolic points. f is called *parabolic* if all critical points of f are contained in $F(f)$. By Sullivan's classification of Fatou components and their properties, a parabolic rational map can have (super)attracting and parabolic basins, but no Siegel disks or Herman rings. Especially, hyperbolic rational maps are parabolic. Note that any orbit of $z \in F(f)$ is attracted to an attracting or parabolic cycle.

Next, to control the parabolic bifurcations, let us introduce some conditions for perturbations. Then we will be able to control the parabolic bifurcation so that the local dynamics near parabolic points change gently.

A perturbation is *horocyclic* if each parabolic point a of f satisfies following conditions:

(a) If a is period l and has p petals, its multiplier $(f^l)'(a) = \lambda$ is a primitive p-th root of unity;

(b) There are fixed points a_ϵ of f_ϵ^l with multipliers $(f_\epsilon^l)'(a_\epsilon) = \lambda_\epsilon$ satisfying $a_\epsilon \to a$ and $\lambda_\epsilon \to \lambda$; and

(c) If we set $\exp(L_\epsilon + i\theta_\epsilon) := \lambda_\epsilon/\lambda$, which tends to 1, then $\theta_\epsilon^2 = o(L_\epsilon)$.

Horocyclic perturbation is originally defined by C. McMullen as *horocyclic convergence* of rational maps[9, §7-9]. He defined it under more general conditions than the definition above. (For instance, under the original definition, we need not assume that the multiplier of a parabolic point of f with p petal is a primitive p-th root of unity.) Though we use the stronger conditions for simplicity, we can prove the main theorem in the next section under the original definition.

Let us consider the effects of horocyclic perturbations on the local dynamics near a, and their representation.

By the condition (b) of horocyclic perturbation, if ϵ is sufficiently small, a is perturbed into a periodic point a_ϵ of f_ϵ with the same period l and $a_\epsilon \to a$. Moreover, the multiplier $(f_\epsilon^l)'(a_\epsilon) = \lambda_\epsilon$ converges to λ.

As f_ϵ converges uniformly to f near a, for each f_ϵ, we can take a local coordinate which maps a_ϵ to 0 and converges uniformly to that of f. Thus we obtain a local representation of the convergence;

$$\begin{aligned} f_\epsilon^{lq}(z) &= \lambda_\epsilon z + A_\epsilon z^r + O(z^{r+1}) \\ \longrightarrow f^{lq}(z) &= z + z^{p+1} + O(z^{p+2}) \quad (\epsilon \to 0) \end{aligned}$$

where $2 \leq r \leq p$ and $A_\epsilon \to 0$.

Now let us consider a coordinate change by

$$\zeta = \phi_\epsilon(z) = z - B_\epsilon z^r \ , \quad B_\epsilon = \frac{A_\epsilon}{\lambda_\epsilon(\lambda_\epsilon^{r-1} - 1)}.$$

By the condition (a) of horocyclic convergence (the multiplier λ is a primitive p-th root of unity), we obtain $\lambda_\epsilon^{r-1} \neq 1$ for all $\epsilon \ll 1$. Thus we may suppose that $\phi_\epsilon \to id$ uniformly near the origin. For each ϵ, changing the coordinate by ϕ_ϵ, we obtain

$$\phi_\epsilon \circ f_\epsilon \circ \phi_\epsilon^{-1}(\zeta) = \lambda_\epsilon \zeta + O(\zeta^{r+1}).$$

So we can continue the discussion replacing r with $r+1$ until $r \leq p$. By composition of the finite number of coordinate changes, we obtain the normalized form of convergence:

$$f_\epsilon^{lp}(z) = \lambda_\epsilon^p z + z^{p+1} + O(z^{p+2}) \longrightarrow f^{lp}(z) = z + z^{p+1} + O(z^{p+2}). \tag{2.1}$$

This property is important to keep the symmetry of the dynamics for each petals.

Remark 3.a We can obtain the normalized form as (2.1) even if λ is not a primitive p-th root of unity: In fact, if $A_\epsilon/(\lambda_\epsilon^p - 1) = O(1)$ then B_ϵ does not diverges and thus we can apply this discussion. See [9, §7].

Next we consider the condition (c) of horocyclic perturbation. Let us set $\lambda_\epsilon/\lambda = \exp(L_\epsilon + i\theta_\epsilon)$. The geometric meaning of the relation $\theta_\epsilon^2 = o(L_\epsilon)$ is as follows: If we fix a pair of arbitrary small closed disks which are tangent to the imaginary axis at the origin for the both sides of the axis, then they contain $L_\epsilon + i\theta_\epsilon$ for all $\epsilon \ll 1$. By this relation, $L_\epsilon = 0$ implies $\theta_\epsilon = 0$. Equivalently, if $|\lambda_\epsilon/\lambda| = 1$ then a_ϵ is a parabolic point of f_ϵ with the same multiplier λ as a. Thus f_ϵ can not have irrationally indifferent periodic points.

By solving the equation $f_\epsilon^{lp}(z) = z$ near the origin, we obtain following three cases:

(1) a_ϵ is a parabolic point with p petals and the multiplier $\lambda_\epsilon = \lambda$; or

(2) a_ϵ is an attracting point, and there are p symmetrically arrayed repelling points near a_ϵ; or

(3) a_ϵ is a repelling point, and there are p symmetrically arrayed attracting points near a_ϵ.

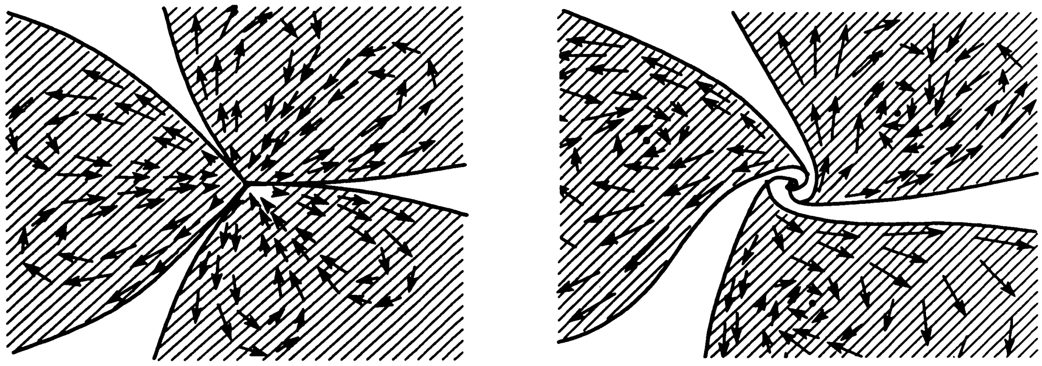

Figure 2: Horocyclic perturbation of a parabolic fixed point of f^{lp} with $p = 3$ petals. The left figure shows the case (1) and the right one shows the case (3).

For (2) and (3), if $p > 1$, these symmetrically arrayed periodic points have the same period lp and the multipliers $\approx \lambda_\epsilon^{-p^2}$. Moreover, they are contained in an open ball centered at a_ϵ with radius $O((1 - \lambda_\epsilon^p)^{1/p})$. We call them the *satellites* of a_ϵ and a_ϵ itself the *planet*. If $p = 1$, (2) and (3) are equivalent; that is, a splits into a pair of attracting and repelling points. Thus we formally define the satellite by attracting one and the planet by repelling one. For (1), we also call a_ϵ the planet, although it has no satellite.

Using these properties, we can obtain a key lemma of horocyclic perturbations. We define the *cycle* of a by the finite orbit of a, say

$$\alpha := \{a, f(a), \dots, f^{l-1}(a)\}.$$

Now we assume that a is parabolic and we call α a *parabolic cycle.*

Let us fix an $x \in \hat{\mathbb{C}}$ whose orbit accumulates to α. For an arbitrary small $\delta > 0$, set $\Delta = \Delta(\delta) := \bigcup_{a \in \alpha} B_\sigma(a, \delta)$. (Here $B_\sigma(a, \delta)$ is the open ball centered at a with radius δ measured by the spherical metric.) Then we can take $N_0 = N_0(x, \delta) \gg 0$ such that $f^n(x) \in \Delta$ for any $n \geq N_0$. The key lemma is:

Lemma 3.1 *If $f_\epsilon \to f$ horocyclically and $\epsilon \ll 1$, there exists an $N \geq N_0$ such that $f_\epsilon^n(x) \in \Delta$ for any $n \geq N$.*

This means that the change of local dynamics by the perturbation is controlled within Δ. This fact is essential for the construction of Ω_ϵ mentioned afterward. The proof is shown in [8].

4 Main results

Our main result is:

Theorem 4.1 (Weak J-stability) *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a parabolic rational map of degree $d \geq 2$ and $f_\epsilon \to f$ a horocyclic perturbation.*

If ϵ is sufficiently small, then we can construct a map $h_\epsilon = h : J(f_\epsilon) \to J(f)$ with following properties:

(i) h is continuous and surjective.

(ii) For any $x \in J(f_\epsilon)$, $f \circ h(x) = h \circ f_\epsilon(x)$.

(iii) If $|h^{-1}(y)| \geq 2$ for some $y \in J(f)$, then the forward orbit of y lands on a parabolic periodic point of f, say a, and $|h^{-1}(y)|$ corresponds to the petal number of a.

The properties (i) and (ii) mean that h gives a semiconjugacy between $J(f_\epsilon)$ and $J(f)$. The property (iii) means that the subset of $J(f_\epsilon)$ where h is *not* one-to-one is either countable or empty. If it is countable non-empty set, $h^{-1}(y)$ is consist of the preimages of repelling satellites of an attracting planet which is generated by the perturbation of a. If it is empty, none of parabolic points is perturbed into an attracting planet and h gives a topological conjugacy between the Julia sets.

Furthermore, we can conclude the Hausdorff convergence of the Julia sets. Let us fix an arbitrary small $r > 0$. If ϵ is sufficiently small, we can construct the semiconjugacy h to satisfy $\sup\{d_\sigma(h(x), x) : x \in J(f_\epsilon)\} < r$. Hence we obtain:

Corollary 4.2 *If f is parabolic and $f_\epsilon \to f$ horocyclically, then $J(f_\epsilon) \to J(f)$ in the Hausdorff topology.*

Remarks.

1. If a rational map f has no Siegel disks or Herman rings and $f_n \to f$ horocyclically, it is known that $J(f_n) \to J(f)$ in the Hausdorff topology [7], [9, Thm.9.1]. By this fact, we can obtain Corollary 4.2, because a parabolic rational map has no Siegel disks or Herman rings. However, the proofs are given by different ways.

2. We call a rational map f is *geometrically finite* if every critical point in $J(f)$ is eventually periodic. Note that parabolic rational maps are geometrically finite. G. Cui[2] showed that a geometrically finite rational map has a perturbation into a continuous family of geometrically finite rational maps with topological conjugacies on their Julia sets. To construct this perturbation, he used the technique of pinching deformation. For geometrically finite polynomials, P. Haïssinsky[6] gave another approach using qc-deformation. These results and our Theorem 4.1 partially solve the Goldberg-Milnor conjecture in [5].

5 Survey of the proof

In this section, we give a survey of the proof of Theorem 4.1. See [7] or [8] for more details.

Step1: Construction of Ω and ρ. Let f be a parabolic rational map and A the finite set of all parabolic points of f.

Proposition 5.1 *There exist a finitely connected compact set $\Omega \subset \hat{\mathbb{C}}$ and a piecewise continuous metric ρ with following properties:*

1. $\Omega \cap P_0(f) = A$.

2. $J(f) \subset \Omega$ *and* $f^{-1}(\Omega) \subset \mathrm{Int}(\Omega) \cup A$.

3. ρ *is defined on* $\mathrm{Int}(\Omega)$ *and small disk neighborhoods for each parabolic point of* f.

4. *For every C^1 curve $\eta \subset f^{-1}(\Omega)$,*

$$\mathrm{length}_\rho(f \circ \eta) > \mathrm{length}_\rho(\eta).$$

So f is expanding for ρ in the sense of this inequality.

Proof. To construct Ω, we need to remove the orbits of critical points of f from the sphere. First, remove small disks around the attracting cycles, and small attracting flowers around parabolic cycles. Next, remove finite number of disk-neighborhoods for the critical orbits which have not been removed. Since we can take such disks and flowers so that the images of them are strictly contained in themselves, the remained set satisfies the conditions of Ω.

One can find the details of construction of the metric ρ in [11, Step 4] (in the case of geometrically finite rational maps). See also [4, Exposé No.X] and [1, V.4.]. Here we only sketch the idea of the construction.

Let ρ_U be the Poincaré metric of $U := \mathrm{Int}(\Omega)$. Since this metric diverges near A, any curve in $f^{-1}(\Omega)$ terminating at A has infinite length with respect to ρ_U. So we need to modify ρ_U so that such curves have finite lengths.

For sufficiently small $\delta > 0$ and each $a \in A$, we set $\mathcal{D}_a := B_\sigma(a, \delta)$ and $\mathcal{D} := \bigcup_{a \in A} \mathcal{D}_a$. Note that $\Omega \cap \mathcal{D}$ is a finite union of narrow cusps near repelling directions. Thus on each $\mathcal{D}_a$, we can take a suitable local coordinate ζ_a such that f is expanding from the metric $|d\zeta_a|$ to the metric $|d\zeta_{f(a)}|$ on $\Omega \cap \mathcal{D}$. Furthermore, we take sufficiently large $M > 0$ and smaller δ if necessary, so that f is expanding from ρ_U to $M|d\zeta_a|$ on $f^{-1}(\Omega \cap \mathcal{D}_a) - \mathcal{D}$ for any $a \in A$. Then we can define the metric ρ by $\min\{\rho_U, M|d\zeta_a|\}$ on each $\mathcal{D}_a$ and by ρ_U otherwise. ■

Step2: Construction of Ω_ϵ and the "0-th" map h_0. Next we construct a compact set Ω_ϵ corresponding to Ω, and the correspondence is represented by the map $h_0 : \Omega_\epsilon \longrightarrow \Omega$.

Proposition 5.2 *For $\epsilon \ll 1$, there exist a compact set $\Omega_\epsilon \subset \hat{\mathbb{C}}$ and a continuous map $h_0 : \Omega_\epsilon \longrightarrow \Omega$ with following properties:*

1. *$\Omega_\epsilon \cap P_0(f_\epsilon)$ is the set of all parabolic points of f_ϵ.*
2. *$J(f_\epsilon) \subset \Omega_\epsilon$ and $f_\epsilon^{-1}(\Omega_\epsilon) \subsetneq \Omega_\epsilon$.*
3. *$h_0 : \Omega_\epsilon \longrightarrow \Omega$ is surjective.*
4. *If there exists $y \in \Omega$ such that $|h_0^{-1}(y)| \geq 2$ then y is a parabolic point and $|h_0^{-1}(y)| = p(y)$. By the perturbation, y splits into an attracting planet and $p(y)$ repelling satellites which coincide with $h_0^{-1}(y)$.*

Moreover, if we fix an arbitrary small $r > 0$, then we can make h_0 satisfy $\sup\{d_\sigma(h_0(x), x) : x \in \Omega_\epsilon\} < r$ for all $\epsilon \ll 1$.

We can construct Ω_ϵ by modification of Ω near the parabolic cycles. For this, we need a help of the key lemma of horocyclic perturbations. However, the construction is somehow complicated.

For Ω and Ω_ϵ, we set

$$\Omega_\epsilon^n := f_\epsilon^{-n}(\Omega_\epsilon)\ ;\ \ \Omega^n := f^{-n}(\Omega) \quad (n = 0, 1, 2, \ldots).$$

By the construction of these sets, $f_\epsilon : \Omega_\epsilon^{n+1} \longrightarrow \Omega_\epsilon^n$ and $f : \Omega^{n+1} \longrightarrow \Omega^n$ are covering maps. Moreover, $\{\Omega_\epsilon^n\}$ and $\{\Omega^n\}$ form decreasing sequences as below:

$$\Omega_\epsilon = \Omega_\epsilon^0 \supsetneq \Omega_\epsilon^1 \supsetneq \cdots \supsetneq \Omega_\epsilon^n \supsetneq \Omega_\epsilon^{n+1} \supsetneq \cdots \supsetneq J(f_\epsilon),$$
$$\Omega = \Omega^0 \supsetneq \Omega^1 \supsetneq \cdots \supsetneq \Omega^n \supsetneq \Omega^{n+1} \supsetneq \cdots \supsetneq J(f).$$

Step3: Construction of h_n. Let us set

$$A^- := \{y \in \Omega : |h_0^{-1}(y)| \geq 2\},$$

which is the finite set of all parabolic points of f perturbed into attracting planets of f_ϵ. Then we set $\Gamma^- := h_0^{-1}(A^-)$, which is the finite set of all repelling satellites generated by the perturbation of $A^- \subset A$. Note that A^- and Γ^- depend on ϵ.

In addition, we set

- $\Gamma_n^- := f_\epsilon^{-n}(\Gamma^-)$, $A_n^- := f^{-n}(A^-)$
- $\Gamma_\infty^- := \bigcup_{n=0}^\infty \Gamma_n^-$, $A_\infty^- := \bigcup_{n=0}^\infty A_n^-$.

Note that Γ_∞^- and A_∞^- have no critical point, and that $f^n(y)$ is a parabolic point of f for $y \in A_n^-$. We thus define the petal number of y by $p(y) := p(f^n(y))$.

First we construct $h_1 : \Omega_\epsilon^1 \to \Omega^1$ as the first lift of h_0:

Proposition 5.3 *For the map h_0, there exists a continuous map $h_1 : \Omega_\epsilon^1 \longrightarrow \Omega^1$ such that $f \circ h_1(x) = h_0 \circ f_\epsilon(x)$ for any $x \in \Omega_\epsilon^1$. If $\epsilon \ll 1$, h_1 is surjective and $h_1 : \Omega_\epsilon^1 - \Gamma_1^- \longrightarrow \Omega^1 - A_1^-$ is a homeomorphism.*

Proof. Fix an $x \in \Omega_\epsilon^1$, then $f_\epsilon(x) \in \Omega_\epsilon$ and $d_\sigma(f_\epsilon(x), h_0(f_\epsilon(x))) \leq r/2$. If $\epsilon \ll 1$ we may assume that $d_\sigma(f(x), f_\epsilon(x)) < r/2$, thus $d_\sigma(f(x), h_0(f_\epsilon(x))) < r$; that is, $h_0(f_\epsilon(x)) \in B_\sigma(f(x), r) \cap \Omega$. Since Ω is sufficiently far from the critical values, $B_\sigma(f(x), r)$ is evenly covered by f (If necessary, replace r with smaller one and repeat the argument). Thus we can take a branch of f^{-1} on $B_\sigma(f(x), r)$, say g, such that $g \circ f(x) = x$. This gives the map $h_1(x) := g \circ h_0 \circ f_\epsilon(x) \in \Omega^1$, which is clearly continuous. The last part of the statement is not difficult to prove. ■

Next we define $h_n : \Omega_\epsilon^n \longrightarrow \Omega^n$ inductively as following proposition. The proof is similar to the case of $n = 1$.

Proposition 5.4 *For $n \geq 1$, suppose that we have defined the continuous map $h_n : \Omega_\epsilon^n \to \Omega^n$ such that $d_\rho(h_{n-1}(x), h_n(x)) < r_\epsilon$ for any $x \in \Omega_\epsilon^n$. Then there exists a continuous map $h_{n+1} : \Omega_\epsilon^{n+1} \to \Omega^{n+1}$ such that $h_n \circ f_\epsilon = f \circ h_{n+1}$ and that $d_\rho(h_n(x), h_{n+1}(x)) < r_\epsilon$ for any $x \in \Omega_\epsilon^{n+1}$.*

Moreover, if $\epsilon \ll 1$, $h_n : \Omega_\epsilon^n \to \Omega^n$ is surjective and $h_n : \Omega_\epsilon^n - \Gamma_n^- \to \Omega^n - A_n^-$ is a homeomorphism for any n.

Where $d_\rho(\cdot, \cdot)$ is the distance measured by ρ, and r_ϵ is defined by

$$\sup\{d_\rho(h_0(x), h_1(x)) : x \in \Omega_\epsilon^1\}.$$

We can easily check that $r_\epsilon = O(r)$, thus we may suppose that it is sufficiently small if $\epsilon \ll 1$.

Step4: The function $\tau(s)$ and the proof of $h_n \to h$. In the proof of the convergence of h_n, the expanding property of f will play an important role. For instance, we can easily show the convergence when f is hyperbolic:

Proposition 5.5 *Suppose that f is hyperbolic. For $\epsilon \ll 1$, h_n converges uniformly to the limit h on $J(f_\epsilon)$.*

Proof. Since f has no parabolic point, we may use the Poincaré metric on $\mathrm{Int}(\Omega)$ as ρ without modification. Thus there is a constant κ such that $f^*\rho/\rho \geq \kappa > 1$ on Ω^1. By the definition of $\{h_n\}$, there exists a constant $C > 0$ such that

$$d_\rho(h_n(x), h_{n+1}(x)) < C/\kappa^{n+1}$$

for any $x \in J(f_\epsilon)$. Thus we can easily follow that h_n converges uniformly and rapidly to the limit h on $J(f_\epsilon)$.■

However in the case that f has parabolic points, f is not uniformly expanding and the convergence of h_n is very slow. To show the convergence, we will use the idea due to Douady-Hubbard[4, Exposé No.X] again. See also [11].

Let $s_0 > 0$ be the supremum of s such that $B_\rho(x, s)$ (an open ball with respect to ρ) is evenly covered by f for any $x \in \Omega^1$. We define a function $\tau : (0, s_0) \to \mathbb{R}^+$ by

$$\tau(s) := \sup_g \{d_\rho(g(x), g(y)) : x, y \in \Omega^1, d_\rho(x, y) \leq s\}.$$

Here g ranges over all branches of f^{-1}. Then τ has following properties:

(i) τ is an increasing and right-continuous function;

(ii) $s > \tau(s)$ for any s; and

(iii) the function $s \mapsto s - \tau(s)$ is also increasing.

(i) and (ii) are almost clear by the definition. (iii) can be followed by the fact that $\tau(s_1 + s_2) \leq \tau(s_1) + \tau(s_2)$.

By using this function, we can prove that:

Proposition 5.6 *For $\epsilon \ll 1$, h_n converges uniformly to the limit h on $J(f_\epsilon)$ where h satisfies $f \circ h = h \circ f_\epsilon$. Moreover, for arbitrary small $r > 0$,*

$$\sup\{d_\sigma(h(x), x) : x \in J(f_\epsilon)\} < r$$

if $\epsilon \ll 1$.

Proof. Fix an arbitrary L such that $0 < L < s_0$. Since $r_\epsilon = O(r)$, we may assume $\epsilon \ll 1$ such that

$$d_\rho(h_0(x), h_1(x)) < r_\epsilon \leq L - \tau(L)$$

for any $x \in \Omega_\epsilon^1$. We claim that $d_\rho(h_0(x), h_n(x)) < L$ on Ω_ϵ^n for any $n \geq 1$.

If $n = 1$, $d_\rho(h_0(x), h_1(x)) < L - \tau(L) < L$. For $n = k$, let us assume that $d_\rho(h_0(x), h_k(x)) < L$. Then for any $x \in \Omega_\epsilon^{k+1}$,

$$\begin{aligned} d_\rho(h_0(x), h_{k+1}(x)) &\leq d_\rho(h_0(x), h_1(x)) + d_\rho(h_1(x), h_{k+1}(x)) \\ &< d_\rho(h_0(x), h_1(x)) + \tau(d_\rho(h_0(f_\epsilon(x)), h_k(f_\epsilon(x)))) \\ &< L - \tau(L) + \tau(L) = L. \end{aligned}$$

We have thus proved the claim by induction on n.

Fix any $x \in J(f_\epsilon)$. For sufficiently large integer l, m,

$$\begin{aligned} d_\rho(h_l(x), h_{m+l}(x)) &< \tau^l\left(d_\rho(h_0(f_\epsilon^l(x)), h_m(f_\epsilon^l(x)))\right) \\ &< \tau^l(L) \to 0 \qquad (l \to \infty). \end{aligned}$$

Because x is arbitrary, h_n converges uniformly on $J(f_\epsilon)$. By the continuity of h_n, the limit h is also continuous. Since the topology of Ω^n defined by ρ is equivalent to that by the spherical metric σ, this convergence is also true with respect to σ.

The last part of the statement is easily followed by the construction of h_0 and the fact that $d_\rho(h_0(x), h(x)) < L$.■

Step5: Completing the Proof. Finally we complete the proof of Theorem 4.1 by the proposition below.

Proposition 5.7 *If $\epsilon \ll 1$, $h : J(f_\epsilon) \to J(f)$ has following properties.*

- *h is surjective.*
- *If $h(x) = h(x')$ for some different $x,\ x' \in J(f_\epsilon)$, then $x,\ x' \in \Gamma_\infty^-$.*
- *For $x,\ x'$ as above, there exists an integer N such that $f_\epsilon^N(x),\ f_\epsilon^N(x')$ are repelling satellites of an attracting planet a_ϵ which is generated by the perturbation of a point in A^-.*

Proof. Here we only show the proof of the surjectivity of h. Other properties are shown by using the expanding property of f with respect to the Poincaré metric of $\hat{\mathbb{C}} - P(f_\epsilon)$.

Fix any $y \in J(f)$. By the surjectivity of h_n, there is a sequence $x_n \in \Omega_\epsilon^n$ such that $h_n(x_n) = y$. For Ω_ϵ is compact, x_n has an accumulate point $x \in J(f_\epsilon)$ and we can take a subsequence x_{n_k} so that $x_{n_k} \to x$ $(k \to \infty)$. Because $h_n \to h$ uniformly and h is continuous, the inequality

$$d_\rho(y, h(x)) \leq d_\rho(h_{n_k}(x_{n_k}), h(x_{n_k})) + d_\rho(h(x_{n_k}), h(x))$$

implies $h(x) = y$.■

By this surjectivity of h and an fact that $h^{-1}(A_\infty^-) = \Gamma_\infty^-$, we obtain that h maps $J(f_\epsilon) - \Gamma_\infty^-$ to $J(f) - A_\infty^-$ homeomorphically.

References

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