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Maskit surgery of entire functions

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1 Introduction

Rational functions have various kinds of finiteness. For instance, rational functions can be described by finitely many coefficients. Also, rational functions has a finite number of singularities, which are critical points in this case. And these two concepts of finiteness are essentially the same.

Finiteness of singularity has been generalized to the case of transcendental entire functions, but as finiteness in the target plane. We say that an entire function f is in the *Speiser class* or of *finite singular type* if the number of singular values is finite. Then, many properties of rational functions, or more precisely, of polynomials, still hold for functions in the Speiser class.

For instance, no wandering domains theorem and no Baker domains theroem still hold, and we have the finite-dimensional Teichmüller space of such a function. See [8] and [9].

Remark The Teichmüller theory states that the dimension of the Teichmüller space $T(f, F_f)$ of a function f in the Speiser class on the Fatou set F_f is finite-dimensional and parametrized by singular values in F_f. More precisely, we have

$$\dim_{\mathbb{C}} T(f, F_f) = N_{AC} - N_p,$$

where N_{AC} is the number of the foliated equivalence class of acyclic (non-periodic and non-preperiodic) singular values of f in F_f and N_p is the number of cycles of parabolic basins. Here we say that two singular values are *foliated equivalent* if the closures of the grand orbits of them are coincident with each other.

Now, when we discuss other kinds of issues such as the NILF conjecture, some difficulties appear. In particular, we do not necessarily know a natural representation space as in the case of Kleinian groups.

A natural concept on finiteness of entire function would be finiteness of singularities in number. One way to formulate this is the concept of *structural* *finiteness*; constructability from a finite number of building blocks representing *simple* singularities. Such a concept may be considered to correspond to finiteness of generators in number of a Kleinian group. Thus, we have new entries in some version of Sullivan's dictionary as follows.

Kleinian Groups	Entire Functions
generators	building blocks
finitely generated	structurally finite

Sullivan's Dictionary

2 Maskit surgery

We use two kinds of ***building blocks***; one is a ***quadratic block***

$$az^2 + bz + c : \mathbb{C} \to \mathbb{C} \qquad (a \neq 0)$$

and the other is an ***exponential block (exp-block)***

$$a \exp bz + c : \mathbb{C} \to \mathbb{C} \qquad (ab \neq 0).$$

The above two kinds of blocks are simplest in the sense that they are smooth covers of once-punctured $\mathbb{C}$ and, over the exception point, they has simplest singularities; a branch point and a logarithmic singurarity.

Remark The simplest covering structure is nothing but a similarity transformation

$$az + b : \mathbb{C} \to \mathbb{C} \qquad (a \neq 0),$$

which gives a universal covering structure of $\mathbb{C}$, and hence has no singularty. We call such one a $\mathbb{C}$-***block***.

Definition (Maskit surgery by connecting functions)

Let $f_j : \mathbb{C} \to \mathbb{C}$ $(j = 1, 2)$ be two entire functions, and A_j be the set of singular values of f_j.

Assume that there is a cross cut L in $\mathbb{C}$ such that

1. both of $L \cap A_1$ and $L \cap A_2$ are either empty or consist of a single point z_0, which is an isolated point of each A_j,

2. L separates $A_1 - \{z_0\}$ from $A_2 - \{z_0\}$, and

3. if $L \cap A_1 = L \cap A_2 = \{z_0\}$, then $\{z_0\}$ is a critical value of each f_j: for a small disk U with center z_0 such that $U \cap A_j = \{z_0\}$, $f_j^{-1}(U)$ has a relatively compact component W_j which contains a critical point for each f_j.

Then we say that an entire function $f : \mathbb{C} \to \mathbb{C}$ is constructed from f_1 and f_2 by a *Maskit surgery* with respect to L if the following assumptions are satisfied: Let D_j be the component of $\mathbb{C} - L$ containing $A_j - \{z_0\}$. Then there exist

1. components $\tilde{D}_1$ and $\tilde{D}_2$ of $f_1^{-1}(D_2)$ and $f_2^{-1}(D_1)$, respectively, such that $f_j : \tilde{D}_j \to D_{3-j}$ is biholomorphic and $\tilde{D}_j \cap W_j \neq \emptyset$ if $L \cap A_j$ are non-empty,
2. a cross cut $\tilde{L}$ in $\mathbb{C}$ such that f gives a homeomorphism of $\tilde{L}$ onto L, and
3. a conformal map ϕ_j of $\mathbb{C} - \tilde{D}_j$ onto U_j such that $f_j = f \circ \phi_j$ on $\mathbb{C} - \tilde{D}_j$, where U_1 and U_2 are components of $\mathbb{C} - \tilde{L}$.

Remark More presicely, the data for a Maskit surgery consists not only of two covering structures (blocks) f_1 and f_2, but also of the homotopy class of cross cut L modulo $\{z_0, \infty\}$ in the complement of $A_1 \cup A_2 - \{z_0\}$ and cut-off pieces $\tilde{D}_1$ and $\tilde{D}_2$. If $L \cap A_1 = L \cap A_2$ is empty, a Maskit surgery is quite simple, and we call such one a *Klein surgery.*

Definition We say that an entire function is *structurally finite* if it can be constructed from a finite number of building blocks by Maskit surgeries .

We say that a structurally finite function is *of type* (p, q) if it is made from p quadratic blocks and q exp-blocks.

Clearly, a structurally finite entire function belongs to the Speiser class. Typical examples of structurally finite entire function are polynomials and *decorated exponential functions*

$$P(z)e^{Q(z)}$$

with polynomials P and Q.

The fundamental properties of structurally finite entire functions are discussed in [15] and [16], some of which are gathered in §4. In this note, we discuss some of combinatorial features of Maskit surgeries and of structurally finite entire functions.

3 Configuration trees

To discribe a given structurally finite entire function, we use the following kind of configuration graph. We will define it in a general setting. I should remark that *the definition below is different from the old one in [14].*

Definition (Configuration tree) A *configuration tree* is a planar tree with the initial vertex (and hence edges have an orientation towards the initial vertex) and colored as follows:

1. There are two kind of vertices; white ones and black ones.

2. There are three kind of edges; white ones, black ones, and red ones.

3. Every connected component of the set of all white vertices and white edges is a subtree $\mathbb{R}$ with vertices $\mathbb{Z}$, which we call a $\mathbb{Z}$-*unit.*

4. Every edge not in any $\mathbb{Z}$-unit is colored black or red, according as it is towards the initial vertex from a black vertex or from a white vertex.

Also a configuration tree is associated with the *configuration data.*

1. *the singularity data*; the center locus attached to every $\mathbb{Z}$-unit and the decoration locus attached to every black edge, and

2. *a spider at* ∞, which assigns each of the singularity data a path to the infinity (cf. [5]).

Definition We call a pair of a red edge and the black vertex pointed by it a *reduction pair.* And when we change the initial vertex and the red edge in a reduction pair has the opposite orientation, we delete the pair, and attach a new pair to every white vertex a black edge now starting from.

We say that such a new configuration tree is obtained by a *change of the initial vertex.* Further, if a white vertex is the initial one, then we may attach a reduction pair and regard that *the newly attached black vertex is the initial one.* Thus we may always assume that *the initial vertex is black.*

We say that two configuration trees are *equivalent*, if, after suitable changes of the initial vertices of both, they are identical including colors.

Definition We say that a configuration tree T is *realizable* if there is an entire function f which gives a tree equivalent to T under the following injunctions;

1. a black edge and its starting black vertex represent a Maskit surgery attaching a quadratic block

2. a red edge and its starting $\mathbb{Z}$-unit represent a Maskit surgery attaching an exp-block.

We may call T a configuration tree of f (with respect to a suitable configuration data). And we also say that, in case 1), a black edge and the associated black vertex represent a $\mathbb{C}$-*decoration* and the decorated $\mathbb{C}$-block, respectively (cf. [9]).

Remark Realizability of a general configuration tree depends on the configuraiton data. Indeed, if the tree is not locally finite at a vertex and there are countably many edges to the vertex, then we can give a decoration loci to these edges so that the tree can not be realized.

Definition An *infinite end* of a locally finite tree is a sequence of decreasing components of the complements of compact subtrees exhausting the tree.

An infinite end determined by subtrees with white vertices only is called a *white end*, and one determined by subtrees with black vertices only is called a *black end*.

Proposition 1 (Danjoy-Carleman-Ahlfors Estimate) *Let T be a locally finite tree realizable by an entire function in the Speiser class and of order $\rho < +\infty$.*

Suppose that every infinite end is either white or black. Then the number of infinite ends of T is not greater than $\max\{2\rho, 1\}$. *In particular, the number of $\mathbb{Z}$-units is not greater than ρ*

Proof. Assume that there are (at least) $2m$ white ends and n black ends. Let B be a disk containing all singular values. Then we see that $f^{-1}(B)$ is connected and the complement has at least $m+n$ components. Each such component contains an asymptotic path for the singular value ∞, which defines a direct singularity. Also, every $\mathbb{Z}$-unit gives a direct finite singularity. Hence we have $2m+n$ direct singuralities. Then Danjoy-Carleman-Ahlfors theorem gives that $2m+n \leq \max\{2\rho, 1\}$, which shows the assertions. ∎

Remark Every transcendental entire function is represented by a configuration tree with an infinite end even if the order of it is 0.

In the case of meromorphic f of finite order, every indirect singularity is a limit of critical values ([3]).

Definition The *core* of a configuration tree is the smallest connected closed subtree containing all black vertices and non-white edges. And we call the tree is *virtually compact* if the core is compact.

A virtually compact tree is locally finite, and has only a finite number of ends. Moreover, we have the following

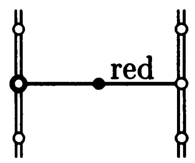

Figure 1: Configuration tree of $a \exp z^2 + b$

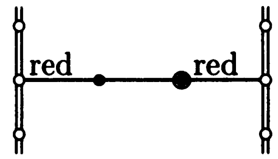

Figure 2: Another tree of $a \exp z^2 + b$

Theorem 2 (Virtual compactness) *The configuration tree of every structurally finite function has a virtually compact core.*

Conversely, every configuration tree with virtually compact core is realizable by a structurally finite entire function.

Thus the peripheral structure of a structurally finite entire function is represented by $\mathbb{Z}$-units. The proof is easily given from the definition.

Example 1 Figures 1 and 3 are configuration trees of $a \exp z^2 + b$ and $\mathrm{Cerf}(z) = a \int_0^z e^{t^2} dt + b$, respectively. Here the concentric circles indicate the initial vertices. Figures 2 and 4 are trees equivalent to those in Figures 1 and 3, respectively.

Example 2 Another typical example of configuration trees is a dual of a colored tree dessin of a Belyi function (cf. [12]).

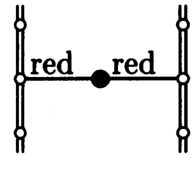

Figure 3: Configuration tree of $\mathrm{Cerf}(z) = a \int_0^z e^{t^2} dt + b$

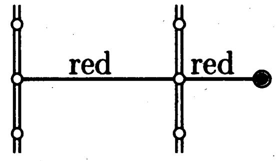

Figure 4: Another tree of Cerf$(z) = a\int_0^z e^{t^2}dt + b$

Here a *Belyi function* is a polynomial with only two critical values $0, 1$. Further, we assume that the tree dessin of f is collored so that every point in $f^{-1}(0)$ is a green vertex and every point in $f^{-1}(1)$ is a red one. We write this tree dessin as D_f.

Proposition 3 *For every Belyi function f, the configuraiton tree T_f can be constructed canonically from the colored tree dessin D_f when the initial vertex is given.*

Conversely, D_f can be obtained from the configuration tree T_f canonically.

Proof. The construction is as follows: First, we can consider each edge of the dessin D_f as a black vertex of the configuration tree T_f. Each black vertex neighboring to the initial vertex v_0 of T_f, which corresponds to an edge in D_f neighboring to the edge corresponding to v_0, are connected by a black edge of T_f with decoration locus determined by the color of the vertex of D_f, which are considered the first age of $\mathbb{C}$-decorations. Repeating this process, we have the configuration tree.

The converse can be defined also canonically; the set of black edges of T_f toward the same vertex is devided into two classes by the decoration loci, and each class corresponds to a single vertex in D_f. ∎

4 Properties of structurally finite entire functions

Definition Let f be a non-linear entire function. Then the *full deformation set* $FD(f)$ of f is the set of all entire functions g such that there is a quasiconformal self-map ϕ of $\mathbb{C}$ satisfiying the *qc-L^∞ condition*:

$$\|f - g\circ\phi\|_\infty = \sup_{\mathbb{C}} |f - g\circ\phi| < \infty.$$

Here we may assume that such a ϕ as above is always *normalized*, that is, fixes 0 and 1. Also it is clear that, if $g \in FD(f)$, then $f \in FD(g)$.

Definition For any two function f_1, f_2 in $FD(f)$, we set

$$d(f_1, f_2) = \inf \left(\log K(\phi_1 \circ \phi_2^{-1}) + \|f_1 \circ \phi_1 - f_2 \circ \phi_2\|_\infty \right),$$

where ϕ_1 and ϕ_2 move all normalized quasiconformal automorphisms of $\mathbb{C}$ satisfying the qc-L^∞ condition with f_1 and f_2, respectively.

Definition The pseudo-distance d is a distance, and $FD(f)$ with this distance is a complete metric space. We call this distance d on $FD(f)$ the *synthetic Teichmüller distance* on $FD(f)$. $FD(f)$, with the synthetic Teichmüller topology induced by d, is called the *full synthetic deformation space* of f and written as $FSD(f)$.

Theorem 4 (Inclusion Theorem) *For a structurally finite entire function f, the full deformation set $FD(f)$ contains all structurally finite entire functions of the same type. In particular, any function topologically equivalent to f belongs to $FD(f)$.*

Moreover, we can show that the family of all structurally finite entire function has an explicit representation.

Theorem 5 (Representation Theorem) *Every structurally finite entire function has the form*

$$\int^z P(t)e^{Q(t)}dt$$

with suitable polynomials P and Q.

Remark Such primitive functions have already appeared as typical examples in various contexts. See for instance, [1] [2] [11]. Also recall that Baker [1] first showed every structurally finite entire function has no wandering domains.

Now we can identify the set of all structurally finite entire function of type (p, q) with

$$SF_{p,q} = \left\{ \int_0^z (c_p t^p + \cdots + c_0)e^{a_q t^q + \cdots + a_1 t} dt + b \right\}$$

with $c_p a_q \neq 0$ if $q > 0$, and if $q = 0$ we regard that $SF_{p,0} = Poly_{p+1}$.

Definition For $f \in SF_{p,q}$, we set $SD(f) = SF_{p,q}$, and equip it with the synthetic Teichmüller topology, which we call the *synthetic deformation space* of f.

Here another natural topology on $SF_{p,q}$ is induced from the coefficient space, which we call the *coefficeint topology.*

Theorem 6 (Equivalence Theorem) *The synthetic Teichmüller topology on $SF_{p,q}$ is equivalent to the coefficient one.*

As for the Hausdorff dimension, we have the following results. Proofs will be given in [16].

Theorem 7 (Transcendental implies H-dim two) *For every transcendental structurally finite f, the Hausdorff dimension of $J(f)$ is two.*

Remark Compare with a theorem of Stallard ([13] II): For every transcendental entire function with bounded singular set, the Hausdorff dimension of $J(f)$ is greater than 1.

As for the area of the Julia set, we have the following

Theorem 8 (Hyperbolic implies area zero) *Let f be a (not necessarily transcendental) structurally finite entire function. If f is hyperbolic (cf. [7]), then $J(f)$ has vanishing area.*

Remark Devaney-Keen proved in [4] that, if the Schwarzian derivative of a meromorphic f is polynomial (such an f is structurally finite if f is entire) and f is hyperbolic, then the Julia set has vanishing area.

Corollary 1 (NILF) *Suppose that f is a structurally finite entire function and can be approximated by hyperbolic elements in $SD(f)$. Then the Julia set of f admits no f-invariant line fields.*

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