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Local diffeomorphisms with positive entropy and chaos in the sense of Li-Yorke

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Abstract

We show that if f is a C^2-local diffeomorphism with positive entropy on a n-dimensional closed manifold ($n \geq 2$) then f is chaotic in the sense of Li-Yorke.

1 Introduction

We study chaotic properties of dynamical systems with positive entropy. Notions of chaos have been given by Li and Yorke [15], Devaney [5] and others. It is well known that if a continuous map of an interval has positive entropy, then the map is chaotic according to the definition of Li and Yorke (cf. [2]). For invertible maps the following holds: let f be a C^2-diffeomorphism of a closed C^∞-manifold. If the topological entropy of f is positive, then f is chaotic in the sense of Li-Yorke [31].

In this paper we show the following:

Theorem A *Let f be a C^2-local diffeomorphism of a closed C^∞-manifold. If the topological entropy of f is positive, then f is chaotic in the sense of Li-Yorke.*

From this theorem we obtain the following corollary.

Corollary B *Let f be a C^2-local diffeomorphism of a closed C^∞-manifold. If f is not invertible, then f is chaotic in the sense of Li-Yorke.*

First we shall explain here the definitions and notations used above. Let X be a compact metric space with metric d and let $f : X \to X$ be a continuous map. A subset S of X is a *scrambled set* of f if there is a positive number $\tau > 0$ such that for any x, $y \in S$ with $x \neq y$,

1. $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > \tau$,
2. $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$.

If there is an uncountable scrambled set S of f, then we say that f is *chaotic in the sense of Li-Yorke* . Li and Yorke showed in [15] that if $f: [0,1] \to [0,1]$ is a continuous map with a periodic point of period 3, then f is chaotic in this sense. Note that any scrambled set contains at most one point x which does not satisfy the following: for any periodic point $p \in X$,

$$\limsup_{n\to\infty} d(f^n(x), f^n(p)) > 0.$$

For another sufficient condition for the chaos in the sense of Li-Yorke, the readers may refer to [4], [7], [8], [9], [10], [11], [19], [20], [34].

Concerning the chaos in the sense of Li-Yorke, Kato introduced the notion of "$*$-chaos" as follows: let F be a closed subset of X. A map $f : X \to X$ is *$*$-chaotic* on F (in the sense of Li-Yorke) if the following conditions are satisfied:

1. there is $\tau > 0$ with the property that for any nonempty open subsets U and V of F with $U \cap V = \emptyset$ and for any natural number N, there is $n \geq N$ such that $d(f^n(x), f^n(y)) > \tau$ for some $x \in U$, $y \in V$, and

2. for any nonempty open subsets U, V of F and any $\varepsilon > 0$ there is a natural number $n \geq 0$ such that $d(f^n(x), f^n(y)) < \varepsilon$ for some $x \in U$, $y \in V$.

Such a set F is called a *$*$-chaotic set.* If S is a scrambled set, then the closure of S, $\bar{S}$, is a $*$-chaotic set. In [10] Kato showed that the converse is true. This is stated precisely as follows:

Lemma 1 ([10], Theorem 2.4) *Let X be a compact metric space and let F be a closed subset of X. If $f : X \to X$ is continuous and is $*$-chaotic on F, then there is an F_σ-set $S \subset F$ such that S is a scrambled set of f and $\overline{S} = F$. If F is perfect (i.e. F has no isolated points), we can choose S as a countable union of Cantor sets.*

By this lemma, to show the existence of uncountable scrambled sets it suffices to show the existence of perfect $*$-chaotic sets.

To obtain Theorem A we consider the inverse limit system of f. Let M be a closed C^∞-manifold and let d be the distance for M induced by a Riemannian metric $\|\cdot\|$ on TM. Let $M^{\mathbf{Z}}$ denote the product topological space $M^{\mathbf{Z}} = \{(x_i) : x_i \in M, i \in \mathbb{Z}\}$. Then $M^{\mathbf{Z}}$ is compact. We define a compatible metric $\tilde{d}$ for $M^{\mathbf{Z}}$ by

$$\tilde{d}((x_i),(y_i)) = \sum_{i=-\infty}^{\infty} \frac{d(x_i, y_i)}{2^{|i|}} \quad ((x_i),(y_i) \in M^{\mathbf{Z}}).$$

For $f\colon M \to M$ a continuous surjection, we let

$$M_f = \{(x_i) : x_i \in M \text{ and } f(x_i) = x_{i+1}, i \in \mathbb{Z}\}.$$

Then M_f is a closed subset of $M^{\mathbf{Z}}$. The space M_f is called the ***inverse limit space*** constructed by f. A homeomorphism $\tilde{f} : M_f \to M_f$, which is defined by

$$\tilde{f}((x_i)) = (f(x_i)) \text{ for all } (x_i) \in M_f,$$

is called the ***shift map*** determined by f. We denote as $P^0 : M_f \to M$ the projection defined by $(x_i) \mapsto x_0$. Then $P^0 \circ \tilde{f} = f \circ P^0$ holds. Remark that f is chaotic in the sense of Li-Yorke if and only if so is $\tilde{f}$.

We can show that the topological entropy, $h(f)$, of f coincides with that of $\tilde{f}$. Indeed, for an f-invariant probability measure ν, we can find an $\tilde{f}$-invariant probability measure μ such that $\nu(A) = P^0_*\mu(A)(= \mu((P^0)^{-1}A))$ for any Borel set $A \subset M$ ([18] Lemma IV 8.3). Let us denote as $h_\nu(f)$ and $h_\mu(\tilde{f})$ the metric entropy of (M, f, ν) and $(M_f, \tilde{f}, \mu)$ respectively. Then we have $h_\nu(f) = h_{P^0_*\mu}(f) = h_\mu(\tilde{f})$ ([25] Lemma 5.2). Therefore, the conclusion is obtained by the variational principle ([32] Theorem 8.6).

We say that a differentiable map $f\colon M \to M$ is a ***local diffeomorphism*** if for $x \in M$ there is an open neighborhood U_x of x in M such that $f(U_x)$ is open in M and $f|_{U_x} : U_x \to f(U_x)$ is a diffeomorphism. Since M is connected, then the cardinal number of $f^{-1}(x)$ is constant. This constant is called the *covering degree* of f. If the covering degree of f is greater than one, (M_f, M, C, P^0) is a fiber bundle where C denotes the Cantor set (see [1] Theorem 6.5.1).

Let μ be a Borel probability measure on M_f and let $\mathcal{B}$ be the Borel σ-algebra on M_f completed with respect to μ. For ξ a measurable partition of M_f and $\tilde{x} \in M_f$ we denote as $\xi(\tilde{x})$ the element of the partition ξ which contains the point $\tilde{x}$. Then there exists a family $\{\mu^\xi_{\tilde{x}} | \tilde{x} \in M_f\}$ of Borel probability measures satisfying the following conditions:

1. for $\tilde{x}$, $\tilde{y} \in M_f$ if $\xi(\tilde{x}) = \xi(\tilde{y})$ then $\mu^\xi_{\tilde{x}} = \mu^\xi_{\tilde{y}}$,

2. $\mu^\xi_{\tilde{x}}(\xi(\tilde{x})) = 1$ for μ-almost all $\tilde{x} \in M_f$,

3. for $A \in \mathcal{B}$ a function $\tilde{x} \mapsto \mu^\xi_{\tilde{x}}(A)$ is measurable and $\mu(A) = \int_{M_f} \mu^\xi_{\tilde{x}}(A) d\mu(\tilde{x})$.

The family $\{\mu^\xi_{\tilde{x}} | \tilde{x} \in M_f\}$ is called a ***canonical system of conditional measures*** for μ and ξ (see [26] for more details).

To prove Theorem A it suffices to show the following theorem.

Theorem C *Let f be a C^2-local diffeomorphism of a closed C^∞-manifold M and let μ be an $\tilde{f}$-invariant ergodic Borel probability measure on M_f.*

If the metric entropy of μ is positive, then there exists a measurable partition η of M_f such that $supp(\mu_{\tilde{x}}^{\eta})$ is a perfect $*$-chaotic set for μ-almost all $\tilde{x} \in M_f$.

Here the *support* $\mathrm{supp}(\nu)$ of a finite measure ν is the smallest closed set C with $\nu(C) = \nu(M_f)$. Equivalently, $\mathrm{supp}(\nu)$ is the set of all $\tilde{x} \in M_f$ with the property that $\nu(U) > 0$ for any open U containing $\tilde{x}$.

Let us see how Theorem A follows from Theorem C. We know that $h(\tilde{f}) = \sup\{h_\mu(\tilde{f}) : \mu \in \mathcal{M}_e(\tilde{f})\}$ where $\mathcal{M}_e(\tilde{f})$ is the set of all $\tilde{f}$-invariant ergodic probability measures (cf.[27]). Thus, if $h(\tilde{f}) = h(f) > 0$, then we can choose $\mu \in \mathcal{M}_e(\tilde{f})$ with $h_\mu(\tilde{f}) > 0$. Therefore, by Theorem C and Lemma 1, f is chaotic in the sense of Li-Yorke.

2 Key Lemmas

In this section we prepare some lemmas which need to prove Theorem C. Let f be a C^2-local diffeomorphism of a closed C^∞-manifold M and μ be an $\tilde{f}$-invariant ergodic Borel probability measure on M_f with $h_\mu(\tilde{f}) > 0$. As in the previous section we denote as $\mathcal{B}$ the Borel σ-algebra on M_f completed with respect to μ. For μ-almost all $\tilde{x} = (x_i) \in M_f$, there exist a splitting of the tangent space $T_{x_0}M = \oplus_{i=1}^{s(x_0)} E_i(\tilde{x})$ and real numbers $\lambda_1(x_0) < \lambda_2(x_0) < \cdots < \lambda_{s(x_0)}(x_0)$ such that

- (a) the maps $\tilde{x} \mapsto E_i(\tilde{x})$, $\lambda_i(x_0)$ and $s(x_0)$ are measurable, moreover $E_i(\tilde{f}(\tilde{x})) = D_{x_0}f(E_i(\tilde{x}))$ and $\lambda_i(x_0)$, $s(x_0)$ are f-invariant ($i = 1, \cdots, s(x_0)$),
- (b) $\displaystyle\lim_{n\to\pm\infty} \frac{1}{n} \log \|(D_{x_0}f^{|n|})^{\pm 1}(v)\| = \lambda_i(x_0) \quad (0 \neq v \in E_i(\tilde{x}),\ i = 1, \cdots, s(x_0))$ and
- (c) $\displaystyle\lim_{n\to\pm\infty} \frac{1}{n} \log |\det(D_{x_0}f^{|n|})^{\pm 1}| = \sum_{i=1}^{s(x_0)} \lambda_i(x_0) \dim E_i(\tilde{x})$

([21], [33], [29], [30]). The numbers $\lambda_1(x_0), \cdots, \lambda_{s(x_0)}(x_0)$ are called *Lyapunov exponents* of f at x_0. Since μ is ergodic, we can put $s = s(x_0)$, $\lambda_i = \lambda_i(x_0)$ and $m_i = \dim E_i(\tilde{x})$ ($i = 1, \cdots, s$) for μ-almost all $\tilde{x} = (x_i) \in M_f$.

A well-known theorem of Margulis and Ruelle [28] says that entropy is always bounded above by the sum of positive Lyapunov exponents; i.e. $h_{P^0_*\mu}(f) \leq \sum_{\lambda_i > 0} \lambda_i m_i$. Since $\tilde{f}$ has positive entropy, we have $0 < h_\mu(\tilde{f}) = h_{P^0_*\mu}(f) \leq \max\{\lambda_i\} = \lambda_s$. Fix $0 < \lambda < \min\{\lambda_i : \lambda_i > 0\}$. From [24], [29] and [30] there are measurable functions $\tilde{\beta} > \tilde{\alpha} > 0$ and $\tilde{\gamma} > 1$ with the following properties: For $\tilde{x} = (x_i) \in M_f$ we put

$$\tilde{W}^u_{loc}(\tilde{x}) = \{\tilde{y} = (y_i) \in M_f : d(x_0, y_0) \leq \tilde{\alpha}(\tilde{x}),\ d(x_i, y_i) \leq \tilde{\beta}(\tilde{x}) e^{-i\lambda}\ (i \geq 1)\}.$$

Then

- (a) the map P^0 restricted to $\tilde{W}^u_{loc}(\tilde{x})$ is injective,
- (b) $P^0(\tilde{W}^u_{loc}(\tilde{x}))$ is a C^2-submanifold of the ball $\{y \in M : d(x_0, y) \leq \tilde{\alpha}(\tilde{x})\}$,
- (c) $T_{x_0}P^0(\tilde{W}^u_{loc}(\tilde{x})) = \oplus_{\lambda_i > 0} E_i(\tilde{x}) (\neq \{0\})$ for μ-almost all $\tilde{x} \in M_f$,
- (d) $d(y_i, z_i) \leq \tilde{\gamma}(\tilde{x}) d(y_0, z_0) e^{-i\lambda}$ for $(y_n), (z_n) \in \tilde{W}^u_{loc}(\tilde{x})$.

For the case when f is invertible we may refer to [6], [22] and [23].

Let ξ and η be measurable partitions of M_f. Put $\tilde{f}^n\xi = \{\tilde{f}^n C : C \in \xi\}$ for $n \in \mathbb{Z}$ and then $(\tilde{f}^n\xi)(\tilde{x}) = \tilde{f}^n(\xi(\tilde{f}^{-n}(\tilde{x})))$ for $\tilde{x} \in M_f$. $\eta \leq \xi$ means that for μ-almost all $\tilde{x} \in M_f$ one has $\xi(\tilde{x}) \subset \eta(\tilde{x})$.

Lemma 2 *Let f and μ be as above. Then there exists a measurable partition ξ of M_f such*

(a) $\xi \leq \tilde{f}^{-1}\xi$,

(b) *for μ-almost all $\tilde{x} \in M_f$, $\xi(\tilde{x}) \subset \tilde{W}^u_{loc}(\tilde{x})$ and $\xi(\tilde{x})$ contains a neighborhood of $\tilde{x}$ open in $\tilde{W}^u_{loc}(\tilde{x})$,*

(c) $\bigvee_{n=0}^{\infty} \tilde{f}^{-n}\xi$ *is the partition into points.*

This lemma is similar to [13] Proposition 3.1, [16] Proposition 5.2 and [17] Lemma 2.2. So we omit the proof.

Let $\mathcal{C}$ denote the family of all nonempty closed subsets of M_f and define a metric d_H by

$$d_H(A,B) = \max\{\sup_{b\in B} d(A,b),\ \sup_{a\in A} d(a,B)\} \quad (A,B \subset \mathcal{C})$$

where $d(A,b) = \inf\{d(a,b) : a \in A\}$. Then it is known that $(\mathcal{C}, d_H)$ is a compact metric space (cf.[12]). If ξ is a measurable partition, then $\tilde{x} \mapsto \overline{\xi(\tilde{x})} \in \mathcal{C}$ is measurable. Indeed, this follows from [3] Theorems III.2, III.9, III.22 and the fact that $\{(\tilde{x}, \xi(\tilde{x})) : \tilde{x} \in M_f\}$ is a Borel subset of $M_f \times M_f$. For $A \subset M_f$ we put $\operatorname{diam}(A) = \sup\{d(\tilde{x},\tilde{y}) : \tilde{x},\tilde{y} \in A\}$. Then we have $\operatorname{diam}(A) = \operatorname{diam}(\bar{A})$. Since $\tilde{x} \mapsto \overline{\xi(\tilde{x})} \in \mathcal{C}$ is measurable, $\tilde{x} \mapsto \operatorname{diam}(\xi(\tilde{x}))$ is also a measurable function. By Lemma 2 (c) we have that for μ-almost all $\tilde{x} \in M_f$

$$\operatorname{diam}((\tilde{f}^{-n}\xi)(\tilde{x})) \to 0 \tag{1}$$

as $n \to \infty$.

Let ξ and η be measurable partitions of M_f and let $\{\mu^{\xi}_{\tilde{x}} | \tilde{x} \in M_f\}$ be a canonical system of conditional measures for μ and ξ. The *mean conditional entropy of η with respect to ξ* is defined by

$$H_\mu(\eta|\xi) = \int -\log \mu^{\xi}_{\tilde{x}}(\eta(\tilde{x}))d\mu(\tilde{x})$$

(see [27] for details).

Lemma 3 *Let f and μ be as above and let ξ be as in Lemma 2. Then,*

$$h_\mu(\tilde{f}) = H_\mu(\tilde{f}^{-1}\xi|\xi).$$

For the case when f is invertible this lemma is proved by Ledrappier and Young [14]. We recall that if the covering degree of f is greater than one, then (M_f, M, C, P^0) is a fiber bundle where C denotes the Cantor set. In view of this fact, the above lemma can be proved by almost the same arguments as the proof of [14] Corollary 5.3 and [16] Corollary 7.1 with some slight modifications. Here we omit the proof.

By Lemma 2(a) we have that $\xi \geq \tilde{f}\xi \geq \tilde{f}^2\xi \geq \cdots$. Let us introduce a measurable partition defined by $\eta = \bigwedge_{i=0}^{\infty} \tilde{f}^i\xi$. Then we have $\tilde{f}\eta = \eta$. For simplicity put

$$\mu_{\tilde{x}} = \mu^{\eta}_{\tilde{x}} \quad \text{and} \quad \mu^n_{\tilde{x}} = \mu^{\tilde{f}^n\xi}_{\tilde{x}} \quad (n \in \mathbb{Z}).$$

By Doob's theorem it follows that for a μ-integrable function $\psi : M_f \to \mathbb{R}$

$$\int \psi d\mu_{\tilde{x}} = \lim_{n\to\infty} \int \psi d\mu^n_{\tilde{x}} \tag{2}$$

for μ-almost all $\tilde{x}$. Since $\tilde{f}\eta = \eta$ and $\tilde{f}_*\mu = \mu$, by the uniqueness of a canonical system of conditional measures (cf.[26]) we have that for μ-almost all $\tilde{x}$

$$\tilde{f}_*\mu_{\tilde{x}} = \mu_{\tilde{f}\tilde{x}} \quad \text{and} \quad \tilde{f}_*\mu^n_{\tilde{x}} = \mu^{n+1}_{\tilde{f}\tilde{x}} \quad (n \in \mathbb{Z}). \tag{3}$$

Here $(\tilde{f}_*\nu)(A) = \nu(\tilde{f}^{-1}A)$ for a Borel probability measure ν on M_f and $A \in \mathcal{B}$.

Let $C(M_f)$ be the Banach space of continuous real-valued functions of M_f with the sup norm $|\cdot|_\infty$, and let $\mathcal{M}(M_f)$ be a set of all Borel probability measures on M_f with the weak topology. Since $C(M_f)$ is separable, there exists a countable set $\{\varphi_1, \varphi_2, \cdots\}$ which is dense in $C(M_f)$. For ν, $\nu' \in \mathcal{M}(M_f)$ define

$$\rho(\nu, \nu') = \sum_{n=1}^{\infty} \frac{|\int \varphi_n d\nu - \int \varphi_n d\nu'|}{2^n |\varphi_n|_\infty}.$$

Then ρ is a compatible metric for $\mathcal{M}(M_f)$ and $(\mathcal{M}(M_f), \rho)$ is compact (cf.[18]). Since (2) holds for $\{\varphi_i\}$, we have

$$\mu_{\tilde{x}} = \lim_{n\to\infty} \mu_{\tilde{x}}^n \tag{4}$$

for μ-almost all $\tilde{x}$. For $\nu \in \mathcal{M}(M_f)$ and a measurable partition ξ, by the definition of conditional measures $\{\nu_{\tilde{x}}^{\xi}\}$, the map

$$M_f \ni \tilde{x} \mapsto \int \varphi_n d\nu_{\tilde{x}}^{\xi}$$

is measurable for $n \geq 1$ and thus $\tilde{x} \mapsto \nu_{\tilde{x}}^{\xi} \in \mathcal{M}(M_f)$ is measurable.

Lemma 4 *Let f, μ and $\{\mu_{\tilde{x}} | \tilde{x} \in M_f\}$ be as above. Then for $\varepsilon > 0$ there exists a closed set F_ε with $\mu(F_\varepsilon) \geq 1 - \varepsilon$ satisfying the map*

$$F_\varepsilon \ni \tilde{x} \mapsto \mu_{\tilde{x}} \in \mathcal{M}(M_f)$$

is continuous.

Proof. Let $\{\varphi_1, \varphi_2, \cdots\}$ be as above and let $\varepsilon > 0$. Since $\tilde{x} \mapsto \int \varphi_i d\nu_{\tilde{x}}^{\xi}$ is measurable for $i \geq 1$, by Lusin's theorem there exists a closed set F_i ($i \geq 1$) with $\mu(F_i) \geq 1 - \varepsilon/2^i$ satisfying

$$F_i \ni \tilde{x} \mapsto \int \varphi_i d\mu_{\tilde{x}} : \text{continuous}.$$

Then $F_\varepsilon = \bigcap_{i=1}^{\infty} F_i$ has the desired property.

□

For $\nu \in \mathcal{M}(M_f)$ and $E \in \mathcal{B}$ let $\nu|_E$ denote the restriction of ν to E, i.e. $\nu|_E(A) = \nu(A \cap E)$ for $A \in \mathcal{B}$. Clearly $\nu|_E$ is a finite measure. We denote as $B(\tilde{x}, r)$ and $U(\tilde{x}, r)$ the closed and open balls in M_f with center $\tilde{x} \in M_f$ and radius $r > 0$ respectively. Let $\{\varphi_1, \varphi_2, \cdots\}$ be as above and let $\nu \in \mathcal{M}(M_f)$. For $\tilde{x} \in \text{supp}(\nu)$ and $\varepsilon > 0$ we can find i such that

$$\int_{U(\tilde{x},\varepsilon)} \varphi_i d\nu > \int \varphi_i d\nu - \varepsilon.$$

Since the inequality holds for ν' sufficiently close to ν, we can easily prove that

$$\mathcal{M}(M_f) \ni \nu \mapsto \text{supp}(\nu) \in \mathcal{C}$$

is lower semi-continuous and so the map is measurable ([3] Corollary III.3). Since $\nu \mapsto \text{diam}(\text{supp}(\nu))$ is lower semi-continuous,

$$\begin{aligned} \mathcal{P}(M_f) = &\quad \{\nu \in \mathcal{M}(M_f) : \nu \text{ is a point measure}\} \\ = &\quad \{\nu \in \mathcal{M}(M_f) : \text{diam}(\text{supp}(\nu)) = 0\} \end{aligned}$$

is a closed set of $\mathcal{M}(M_f)$. Since $(\tilde{f}^n\xi)(\tilde{x}) \subset \eta(\tilde{x})$, we have

$$\text{supp}(\mu_{\tilde{x}}^n) \subset \text{supp}(\mu_{\tilde{x}}) \quad (n \in \mathbf{Z})$$

for μ-almost all $\tilde{x} \in M_f$.

Lemma 5 *Let f, μ and $\{\mu_{\tilde{x}} | \tilde{x} \in M_f\}$ be as above. Then for μ-almost all $\tilde{x} \in M$,* supp$(\mu_{\tilde{x}})$ *has no isolated points.*

Proof. Let ξ and $\mu_{\tilde{x}}^n$ be as above. Then it is easily checked that for $n \in \mathbb{Z}$

$$P_n = \{\tilde{x} \in M_f : \mu_{\tilde{x}}^n \in \mathcal{P}(M_f)\} \supset \{\tilde{x} \in M_f : \mu_{\tilde{x}}|_{(\tilde{f}^n\xi)(\tilde{x})} \text{ is a point measure}\}.$$

If this lemma is false, then there exists a measurable set with positive measure such that for any $\tilde{x}$ belonging to the set, supp$(\mu_{\tilde{x}})$ has an isolated point. Since diam$((\tilde{f}^{-k}\xi)(\tilde{x})) \to 0$ $(k \to \infty)$ by (1), we have $\mu(P_{-k}) > 0$ for k large enough. Put $P = \bigcap_{j\geq 1}\bigcup_{n\geq j} \tilde{f}^n P_{-k}$ and then $\mu(P) = 1$ because μ is ergodic.

By (3) we have

$$\begin{aligned} \tilde{f}^n(P_{-k}) &= \{\tilde{f}^n(\tilde{x}) \in M_f : \mu_{\tilde{x}}^{-k} \in \mathcal{P}(M)\} \\ &= \{\tilde{x} \in M_f : \tilde{f}_*^n \mu_{\tilde{f}^{-n}\tilde{x}}^{-k} \in \mathcal{P}(M)\} \\ &= \{\tilde{x} \in M_f : \mu_{\tilde{x}}^{n-k} \in \mathcal{P}(M_f)\} \\ &= P_{n-k} \qquad (n \in \mathbb{Z}), \end{aligned}$$

and so $P = \bigcap_{j\geq 1}\bigcup_{n\geq j} P_{n-k}$. Thus, for $\tilde{x} \in P$ there exists an increasing sequence $\{n_i\}_{i\geq 0}$ such that $\tilde{x} \in P_{n_i}$ for $i \geq 0$. Since $\mu_{\tilde{x}} = \lim_{i\to\infty} \mu_{\tilde{x}}^{n_i}$ (by (4)) and $\mu_{\tilde{x}}^{n_i} \in \mathcal{P}(M_f)$ for i, we have $\mu_{\tilde{x}} \in \mathcal{P}(M_f)$ for $\tilde{x} \in P$.

Since $\xi \geq \eta$ and μ_x is a point measure for μ-almost all $\tilde{x} \in M_f$, so is $\mu_{\tilde{x}}^\xi$. Thus $\mu_{\tilde{x}}^\xi((\tilde{f}^{-1}\xi)(\tilde{x})) = 1$ for μ-almost all $\tilde{x}$. Therefore

$$h_\mu(\tilde{f}) = H_\mu(\tilde{f}^{-1}\xi|\xi) = \int -\log \mu_{\tilde{x}}^\xi((\tilde{f}^{-1}\xi)(\tilde{x}))d\mu(\tilde{x}) = 0$$

by Lemma 3. This is a contradiction.

□

3 Proof of Theorem C

In this section we will prove Theorem C. Let f, μ, η and $\{\mu_{\tilde{x}}|\tilde{x} \in M_f\}$ be as in §2. By Lemma 5, supp$(\mu_{\tilde{x}})$ is perfect for μ-almost all $\tilde{x} \in M_f$. Therefore, to obtain the conclusion it suffices to show the following.

Proposition 1 *If $\mu_{\tilde{x}}$ is not a point measure for μ-almost all $\tilde{x} \in M_f$, then supp$(\mu_{\tilde{x}})$ is a $*$-chaotic set for μ-almost all $\tilde{x} \in M_f$.*

Proof. The proof of this proposition is similar to that of [31] Proposition 2. However, for completeness we give the proof.

Fix $0 < \varepsilon < 1$ and let F_ε be as in Lemma 4. By assumption we can take and fix $\tilde{x}_0 \in$ supp$(\mu|F_\varepsilon)$ such that $\mu_{\tilde{x}_0}$ is not a point measure. Choose two distinct points $\tilde{y}_1, \tilde{y}_2 \in$ supp$(\mu_{\tilde{x}_0})$ and put $\tau = d(\tilde{y}_1, \tilde{y}_2)/2 (> 0)$. For $0 < r < \tau/2$ we can take $\delta = \delta(r) > 0$ with

$$\mu_{\tilde{x}_0}(U(\tilde{y}_i, r)) > \delta \quad (i = 1, 2).$$

Since $U(\tilde{y}_i, r)$ are open, there exists a large integer $m' = m'(r) > 0$ such that if $\rho(\nu, \mu_{\tilde{x}_0}) < 1/m'$ $(\nu \in \mathcal{M}(M_f))$, then

$$\nu(U(\tilde{y}_i, r)) > \delta = \delta(r) \quad (i = 1, 2). \tag{5}$$

By Lemma 4 we can find $\epsilon' = \epsilon'(r) > 0$ such that for $\tilde{x} \in U(\tilde{x}_0, \varepsilon') \cap F_\varepsilon$

$$\rho(\mu_{\tilde{x}}, \mu_{\tilde{x}_0}) < 1/2m' = 1/2m'(r). \tag{6}$$

Remark that

$$d(U(\tilde{y}_1, r), U(\tilde{y}_2, r)) = \inf\{d(\tilde{x}, \tilde{y}) : d(\tilde{x}, \tilde{y}_1) < r,\ d(\tilde{y}, \tilde{y}_2) < r\} > \tau.$$

Let ξ be as in Lemma 2 and put

$$B_m(n) = \left\{ \tilde{x} \in M_f \;\middle|\; \begin{array}{l} \rho(\mu_{\tilde{x}}^{[k/2]}, \mu_{\tilde{x}}) < 1/m, \\ \operatorname{diam}((\tilde{f}^{-k+[k/2]}\xi)(\tilde{f}^{-k}\tilde{x})) < 1/m \quad (k \geq n) \end{array} \right\}$$

for $n, m \geq 1$. Then $B_m(n) \subset B_m(n+1)$ and $\mu(\bigcup_{n=0}^{\infty} B_m(n)) = 1$ by (1) and (4), and so there exists an increasing sequence $\{n_m\}$ such that $\mu(B_m(n_m)) \geq 1 - 1/2^{m+1}$ $(m \geq 1)$. Since $\mu(\cap_{k=m}^{\infty} B_k(n_k)) \geq 1 - 1/2^m$ for $m \geq 1$, we can find $D_m \in \mathcal{B}$ with $\mu(D_m) \geq 1 - 2^{-m/2}$ satisfying

$$\mu_{\tilde{x}}(\cap_{k=m}^{\infty} B_k(n_k)) \geq 1 - 2^{-m/2} \quad (\tilde{x} \in D_m). \tag{7}$$

For $0 < r < \tau/2$ we put

$$K_r = \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} \left(\bigcap_{n=0}^{\infty} \bigcup_{\ell=n}^{\infty} \tilde{f}^{-\ell}(U(\tilde{x}_0, \varepsilon'(r)) \cap F_\varepsilon \cap D_m) \right).$$

Since $\mu(U(\tilde{x}_0, \varepsilon'(r)) \cap F_\varepsilon \cap D_m) \geq \mu(U(\tilde{x}_0, \varepsilon'(r)) \cap F_\varepsilon) - 2^{-m/2} > 0$ for m sufficiently large, we have $\mu(K_r) = 1$ $(0 < r < \tau/2)$ by the ergodicity of μ. Therefore, to obtain the conclusion it suffices to show that $\operatorname{supp}(\mu_{\tilde{x}})$ is a $*$-chaotic set for $\tilde{x} \in K = \cap_{n \geq 1} K_{1/n}$.

To do this fix $\tilde{x} \in K_r$ $(r = 1/n, n \geq 1)$ and suppose that nonempty open sets U_1 and U_2 satisfy

$$U_1 \cap U_2 \neq \emptyset, \quad U_j \cap \operatorname{supp}(\mu_{\tilde{x}}) \neq \emptyset \quad (j = 1, 2).$$

Choose $m_0 > 0$ with

$$0 < 2^{-m_0/2} < \min\{\mu_{\tilde{x}}(U_j) : j = 1, 2\} \quad \text{and} \quad m_0 \geq 2m'.$$

Since $\tilde{x} \in K_r$, by the definition of K_r, there exist $m_1 > m_0$ and a sequence of positive integers $\{\ell_k\}_k$ with $\ell_k > n_k$ such that

$$\tilde{f}^{\ell_k}(\tilde{x}) \in U(\tilde{x}_0, \varepsilon'(r)) \cap F_\varepsilon \cap D_{m_1} \quad (k \geq 1). \tag{8}$$

Thus, by (3) and (7) we have

$$\begin{aligned} \mu_{\tilde{x}}(\tilde{f}^{-\ell_k}(B_k(n_k))) &\geq \mu_{\tilde{x}}(\tilde{f}^{-\ell_k}(\cap_{k=m_1}^{\infty} B_k(n_k))) \\ &= \mu_{\tilde{f}^{\ell_k}(\tilde{x})}(\cap_{k=m_1}^{\infty} B_k(n_k)) \\ &\geq 1 - 2^{-m_1/2} \geq 1 - 2^{-m_0/2} \quad (k \geq m_1), \end{aligned}$$

and so $\mu_{\tilde{x}}(U_j \cap \tilde{f}^{-\ell_k}(B_k(n_k))) \geq \mu_{\tilde{x}}^{u}(U_j) - 2^{-m_0/2} > 0$. Therefore we can choose

$$\tilde{z}_j = \tilde{z}_j(k) \in U_j \cap \tilde{f}^{-\ell_k}(B_k(n_k)) \cap \eta(\tilde{x})$$

for $j = 1, 2$ and $k \geq m_1$.

Since $\tilde{f}^{\ell_k}(\tilde{z}_j) \in B_k(n_k) \cap \tilde{f}^{\ell_k}(\eta(\tilde{x})) \subset B_k(\ell_k) \cap \eta(\tilde{f}^{\ell_k}(\tilde{x}))$, we have

$$\begin{array}{l} \rho(\mu_{\tilde{f}^{\ell_k}(\tilde{z}_j)}^{[\ell_k/2]}, \mu_{\tilde{f}^{\ell_k}(\tilde{x})}) = \rho(\mu_{\tilde{f}^{\ell_k}(\tilde{z}_j)}^{[\ell_k/2]}, \mu_{\tilde{f}^{\ell_k}(\tilde{z}_j)}) < 1/k \leq 1/m_0 \leq 1/2m', \\ \operatorname{diam}((\tilde{f}^{-\ell_k+[\ell_k/2]}\xi)(\tilde{z}_j)) < 1/k \end{array} \tag{9}$$

for $j = 1, 2$ and $k \geq m_1$. By use of (6) and (8)

$$\begin{aligned} \rho(\mu_{\tilde{f}^{\ell_k}(\tilde{z}_j)}^{[\ell_k/2]}, \mu_{\tilde{x}_0}) &\leq \rho(\mu_{\tilde{f}^{\ell_k}(\tilde{z}_j)}^{[\ell_k/2]}, \mu_{\tilde{f}^{\ell_k}(\tilde{x})}) + \rho(\mu_{\tilde{f}^{\ell_k}(\tilde{x})}, \mu_{\tilde{x}_0}) \\ &< 1/2m' + 1/2m' = 1/m', \end{aligned}$$

and so $\mu_{\tilde{z}_j}^{-\ell_k+[\ell_k/2]}(\tilde{f}^{-\ell_k}U(\tilde{y}_i, r)) = \mu_{\tilde{f}^{\ell_k}(\tilde{z}_j)}^{[\ell_k/2]}(U(\tilde{y}_i, r)) > \delta$ by (5). Thus we have

$$(\tilde{f}^{-\ell_k+[\ell_k/2]}\xi)(\tilde{z}_j) \cap \tilde{f}^{-\ell_k}U(\tilde{y}_i, r) \neq \emptyset$$

for $1 \le i, j \le 2$ and $k \ge m_1$. Since $\tilde{z}_j \in U_j$, by (9) we may assume

$$\tilde{z}_j \in (\tilde{f}^{-\ell_k + [\ell_k/2]}\xi)(\tilde{z}_j) \subset U_j$$

for k large enough. Therefore

$$U_j \cap \tilde{f}^{-\ell_k} U(\tilde{y}_i, r) \supset (\tilde{f}^{-\ell_k + [\ell_k/2]}\xi)(\tilde{z}_j) \cap \tilde{f}^{-\ell_k} U(\tilde{y}_i, r) \neq \emptyset$$

for $1 \le i, j \le 2$ and k large enough.

Now we take $b_{i,j} = b_{i,j}(k) \in U_j \cap \tilde{f}^{-\ell_k} U(\tilde{y}_i, r)$ for $1 \le i, j \le 2$ and then

$$\begin{aligned} &b_{i,j} \in U_j \quad (1 \le i, j \le 2), \\ &d(\tilde{f}^{\ell_k}(b_{1,1}), \tilde{f}^{\ell_k}(b_{2,2})) > d(U(\tilde{y}_1, r), U(\tilde{y}_2, r)) > \tau \quad \text{and} \\ &d(\tilde{f}^{\ell_k}(b_{1,1}), \tilde{f}^{\ell_k}(b_{1,2})) \le \operatorname{diam}(U(\tilde{y}_1, r)) = 2r = 2/n. \end{aligned}$$

This implies that $\operatorname{supp}(\mu_{\tilde{x}})$ is a $*$-chaotic set for $\tilde{x} \in K = \cap_{n \ge 1} K_{1/n}$.

□

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