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Stokes Multipliers, Spectral Determinants and T-Q relations *

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Abstract

Recently, a remarkable correspondence has been unveiled between a certain class of ordinary linear differential equations (ODE) and integrable models. In the first part of the report, we survey the results concerning the 2nd order differential equations, the Schrödinger equation with a polynomial potential. We will observe that fundamental objects in the study of the solvable models, e.g., Baxter's $Q-$ operator, fusion transfer matrices come into play in the analyses on ODE. The second part of the talk is devoted to the generalization to higher order linear differential equations. The correspondence found in the case of the 2nd order ODE is naturally lifted up. We also mention a connection to the discrete soliton theory.

1 Introduction

Recent studies[1]-[7] reveal an unexpected connection between a certain category of ordinary differential equations (ODE) and integrable models (IM) with quantum group symmetry. We call this ODE/IM correspondence [1]. The success inherits the fruitful results from the exact WKB analysis[8]-[16] and progress in the study of integrable structure [17]-[20].

The aim of the present talk is two-fold, the survey on the 2nd order ODE case, and the brief preview on the generalization to higher order cases.

Firstly, we will review the results on the 2nd order differential equations, the Schrödinger equation with a polynomial potential,

$$-\frac{d^2y}{dx^2} + x^\ell y = Ey.$$

We will mainly follow the argument in [2] but employ some simplifications and add some materials. Motivated by the success of the exact WKB method, we regard the coordinate x as a complex variable. The complex x plane is conveniently divided into sectors. See section 2. Each sector possesses two (= the order of the equation) linear independent solutions. We call them a fundamental set of solutions (FSS). The relations among FSS of different sectors are of our interest. To be precise, we would like to evaluate the Stokes multiplier which characterizes the connection rule. In this view point, it is natural to regard that the problem consists of two coupled equations, the original differential equation and the difference equation for the Stokes multiplier. It will be then shown that fundamental objects in the study of the solvable models,

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[1]Following R. Tateo.

e.g., Baxter's $Q-$ operator, fusion hierarchy of transfer matrices based on $U_q(A_1^{(1)})$ and their functional relations naturally come into play in the analyses on ODE. Especially, (unfused) transfer matrix is identified with the Stokes multiplier. Reflecting the ODE/IM correspondence, the Stokes multiplier has two representations, the Wronskian representation, which arises from ODE, and the DVF representation, originated from IM. They both play a role in generalizing the results in the second part of the talk.

Physically, the Stokes multiplier may be less interesting. Rather, the quantity of importance is the spectral determinant, $D(E) = \det(\mathcal{H} - E)$ or eigenvalues themselves. Remarkably, $D(E)$ also belongs to the fusion hierarchy. Thus the result provides a unified view of Stokes multipliers and spectral determinant.

In the second part, a generalization to higher order ODE will be addressed,

$$-\frac{d^{n+1}y}{dx^{n+1}} + x^{\ell}y = Ey.$$

In [6], functional relations were derived among Stokes multipliers and their generalizations. These are identical to functional relations among transfer matrices of solvable models with $U_q(A_n^{(1)})$ symmetry, which generalizes the observation for $n = 1$. The relations were evolved by use of the machinery in the solvable models, a quantum analogue of the Jacobi-Trudi formula. Here we will give an alternative, much simpler derivation, resulting from the Wronskian representation of the Stokes multipliers. We also note that the possible connection of the relations to discrete soliton equations (the Hirota-Miwa equation)[21, 22].

The parallelism to IM will be further exploited. The eigenvalues of transfer matrices possess a universal structure called the dressed vacuum form (DVF). The universality has a deep origin in analyticity of their expressions under Bethe ansatz equations and the Yang-Baxter integrability. We will show that Stokes multipliers also assume the same DVF.

The paper is organized as follows. The asymptotic form for $n+1-$th ODE will be discussed in section 2. Several notations and symbols, such as sectors, Stokes matrices, are introduced for n general. In the next three sections, we restrict ourselves to the $n = 1$ case. The recursion relations and functional relations for the Stokes multiplier and its generalizations are derived in section 3. Under certain assumptions, one transforms the algebraic relations to a set of integral equations modulo one unknown parameter. Remarkably, the integral equations take identical forms to thermodynamic Bethe ansatz equations. We shall discuss the DVF representation for the Stokes multiplier in section 4. The spectral problem is addressed in section 5. Utilizing the previous results, spectral determinants are identified and the one missing parameter in section 3 will be determined. The extension to arbitrary n is the topics of sections 6 and 7. We conclude the paper with a brief summary and discussion in section 8.

While preparing the manuscript, I find the preprint [23] appearing on e-print. The content of the paper largely overlaps with the second part of the present manuscript. They actually treated a more general set of ODE but without the argument of the functional relations.

2 Asymptotic Expansion, FFS and Stokes multipliers

The details of the present section can be found in [24, 25, 26].

We first discuss the asymptotic behavior of a slightly generalized differential equation,

$$\partial^{n+1}y + (-1)^n P(x)y = 0 \tag{1}$$
$$P(x) = \sum_{j=0}^{\ell} a_j x^{\ell-j}$$

where a_j are complex numbers and $a_1 = 1$. Note that the factor $(-1)^n$ is not essential. It can be adsorbed into re-definition of the angle of x. For later convenience, we will include this factor throughout out this report.

Now that $x = \infty$ is an irregular singular point of the equation, analytic properties of the solutions are different for different angle regions in complex x plane. Let $\mathcal{S}_k$ be a region in the plane satisfying

$$|\arg x + k\theta| \leq \frac{\pi}{\ell + n + 1}$$

for $x \in \mathcal{S}_k$, where $\theta = \frac{2\pi}{\ell+n+1}$. We first analyze the asymptotic behavior of a subdominant solution in $\mathcal{S}_0$. Following [25, 24], we define $b_h (h = 1, 2, \cdots)$ by the relation,

$$(1 + \sum_{k=1}^{\ell} a_k x^{-k})^{1/(n+1)} = 1 + \sum_{h=1}^{\infty} b_h x^{-h}.$$

A key function $E(x, \mathbf{a})$ is defined by b_h,

$$\begin{aligned} E(x, \mathbf{a}) &:= \int (1 + \sum_{h=1}^{h_\ell} b_h x^{-h}) x^{\ell/(n+1)} dx \\ &= \frac{n+1}{\ell+n+1} x^{(\ell+n+1)/(n+1)} + \sum_{h=1}^{h_\ell} \frac{b_h}{\frac{\ell}{n+1} - h + 1} x^{\ell/(n+1)+1-h} \end{aligned}$$

where $h_\ell = N$ for $\ell = N(n+1) - j, (j = 1, \cdots, n)$. $\mathbf{a}$ stands for $(a_1, a_2, \cdots, a_\ell)$

In addition, we introduce an exponent ν_ℓ by

$$\nu_\ell = \begin{array}{ll} \frac{n\ell}{2}, & \text{for } \ell \neq 0 \bmod n+1 \\ \frac{n\ell}{2} + (n+1) b_{h_\ell + 1}, & \text{for } \ell = 0 \bmod n+1 \,. \end{array} \tag{2}$$

Theorem 1. *In $\mathcal{S}_0$, there exists a subdominant solution to (1) $y(x, \mathbf{a})$ which has the asymptotic behavior,*

$$y(x, \mathbf{a}) \sim C^{-1} x^{-\nu_\ell/(n+1)} e^{-E(x, \mathbf{a})}. \tag{3}$$

A normalization factor C is introduced for convenience in the later discussion,

$$C^{n+1} := \exp(-\frac{\pi n}{2} i) \prod_{0 \leq i < j \leq n} (w^j - w^i), \quad w := \exp(-\frac{2\pi}{n+1} i).$$

As argued in [6], the range of the validity of the asymptotic form is wider if one forgets the subdominance. Explicitly, it is valid for $|\arg x| < \frac{n+2}{\ell+n+1}\pi$.

The intriguing feature in the differential equation (1) is a certain symmetry in rotating x plane.

Theorem 2. *if $y(x, \mathbf{a})$ is the prescribed solution, then*

$$y_k(x, \mathbf{a}) := y(xq^{-k}, G^{(k)}(\mathbf{a})) q^{nk/2}$$

is also a solution to (1).

The parameter q signifies $\exp(i\theta) = \exp(i\frac{2\pi}{\ell+n+1})$. The operation $G^{(k)}(\mathbf{a})$ is defined by $G^{(k)}(\mathbf{a}) = G(G^{(k-1)}(\mathbf{a})), k \geq 2$ and $G(\mathbf{a}) = (a_1/q, a_2/q^2, \cdots a_\ell/q^\ell)$.

From now on, we restrict our discussion to a single potential term case ,

$$P(x) = x^\ell + a_\ell, \quad a_\ell = \lambda^{n+1}.$$

One immediately verifies that $b_{h_\ell+1} = 0$ and thus $\nu_\ell = n\ell/2$ for $\ell > n + 1$. Under the operation of G, $G(a_\ell) = a_\ell q^{-\ell} = a_\ell q^{n+1}$. In term of λ, the action of G is simply given by $G^{(k)}(\lambda) = \lambda q^k$. Consequently, $y_k = q^{nk/2} y(xq^{-k}, \lambda q^k)$.

A set of fundamental solutions (FSS) in $\mathcal{S}_k$ is formed by by $(y_k, y_{k+1}, \cdots, y_{k+n})$. We introduce a $(n+1)\times(n+1)$ matrix $\Phi_k(x)$

$$\Phi_k(x) := \begin{pmatrix} y_k, & y_{k+1}, & \cdots, & y_{k+n} \\ \partial y_k, & \partial y_{k+1}, & \cdots, & \partial y_{k+n} \\ \vdots & & & \vdots \\ \partial^n y_k, & \partial^n y_{k+1}, & \cdots, & \partial y_{k+n} \end{pmatrix}. \tag{4}$$

We denote the Wronskian, the determinant of $\Phi_k(x)$, by W_k. Note that the above asymptotic expansion is valid for $y_{k+j}, (j = 0, \cdots, n)$ in the common sector $\mathcal{S}_{k+1/2} \cup \mathcal{S}_{k-1/2}$. As W_k is constant in x, one easily checks the linear independence of these solutions by using the asymptotic expansion (3) at the sector. Due to the present normalization of y_k, we have $W_k = 1$.

A Stokes matrix S_k connects FFS of $\mathcal{S}_k$ and $\mathcal{S}_{k+1}$

$$\Phi_{k+1}(x) = \Phi_k(x) S_k. \tag{5}$$

The linear independence of solutions demands S_k in the following form,

$$S_k = \begin{pmatrix} \tau_1^{(1)}(\lambda q^k), & 1, & 0, & 0, & \cdots, & 0 \\ \tau_1^{(2)}(\lambda q^k), & 0, & 1, & 0, & \cdots, & 0 \\ \vdots & & & & & \vdots \\ \tau_1^{(n)}(\lambda q^k), & 0, & 0, & 0, & \cdots, & 1 \\ \tau_1^{(n+1)}(\lambda q^k), & 0, & 0, & 0, & \cdots, & 0 \end{pmatrix}. \tag{6}$$

We call elements τ Stokes multipliers.

By the Cramer's formula, one represents $\tau_1^{(j)}(\lambda q^k)$ as

$$\tau_1^{(j)}(\lambda q^k) = \det \begin{pmatrix} y_{k+1}, & y_{k+2}, & \cdots, & y_k, & \cdots, & y_{k+n+1} \\ \vdots & & & & & \vdots \\ \partial^n y_{k+1}, & \partial^n y_{k+2}, & \cdots, & \partial^n y_k, & \cdots, & \partial^n y_{k+n+1} \end{pmatrix} \tag{7}$$

that is, $(y_k, \partial y_k, \cdots, \partial^n y_k)$ is inserted in the j−th column in the denominator. Evidently $\tau_1^{(n+1)}(\lambda q^k) = (-1)^n W_{k+1}/W_k = (-1)^n$.

The above representation (7) of Stokes multipliers will be referred to as the Wronskian representation.

Determinants of such structure will be hereafter abbreviated to, by specifying only the first row, $[y_{k+1}, y_{k+2}, \cdots, y_k, \cdots, y_{k+n+1}]$. Generally,

$$[y_{i_1}, y_{i_2}, \cdots, y_{i_n}] := \begin{pmatrix} y_{i_1}, & y_{i_2}, & \cdots, & y_{i_n} \\ \vdots & & & \vdots \\ \partial^{n-1} y_{i_1}, & \partial^{n-1} y_{i_2}, & \cdots, & \partial^{n-1} y_{i_n} \end{pmatrix}.$$

We have prepared materials needed for study on general n. In the next few sections, however, we confine ourselves to the $n = 1$ case. There are two reasons for the separated argument. First, only for $n = 1$ case, we have a clear bridge between the connection problem and the spectral problem. Second, the second order ODE may be the most relevant to physics.

3 Fusion Stokes matrices for the 2nd order ODE

The ingenious idea in [2] lies in the introduction of the generalized (or fusion) Stokes matrices connecting the second neighboring sectors, the third neighboring sectors, and so on. We denote by $S_k^{(j)}$ the fusion Stokes matrices connecting two FSS, Φ_k and Φ_{k+j}

$$\Phi_k = \Phi_{k+j} S_k^{(j)}.$$

Obviously, the recursion relation holds,

$$S_k^{(j)} = S_{k+1}^{(j-1)} S_k^{(1)}. \tag{8}$$

Theorem 3. $S_k^{(j)}$ *has an expression*

$$S_k^{(j)} = \begin{pmatrix} \tau_j^{(1)}(\lambda q^k), & \tau_{j-1}^{(1)}(\lambda q^{k+1}) \\ -\tau_{j-1}^{(1)}(\lambda q^k), & -\tau_{j-2}^{(1)}(\lambda q^{k+1}) \end{pmatrix}$$

where we adopt $\tau_0^{(1)}(\lambda) = 1, \tau_{-1}^{(1)}(\lambda) = 0$. *Thanks to the condition* $y_{\ell+2+k} = -y_k$, $\tau_\ell^{(1)}(\lambda) = -\tau_{\ell+2}^{(1)}(\lambda) = 1$ *and* $\tau_{\ell+1}^{(1)}(\lambda) = 0$. *Naturally,* $\tau_j^{(1)}, (j \geq 2)$ *are referred to as the generalized Stokes multipliers. Due to (8) they satisfy relations,*

$$\tau_j^{(1)}(q\lambda)\tau_1^{(1)}(\lambda) = \tau_{j+1}^{(1)}(\lambda) + \tau_{j-1}^{(1)}(q^2\lambda). \tag{9}$$

example

For $\ell = 1$,

$$-\frac{d^2}{dx}y + xy = \lambda^2 y,$$

it is well known that the eigenfunction is given by Airy function $y = \mathrm{Ai}(x)$. The above connection rule then fixes the Stokes multiplier for Airy function $\tau_1^{(1)} = 1$.

Let $\tau_j^{(1)}(\lambda) = T_j^{(1)}(\lambda q^{(j+1)/2})$. One can then prove

$$T_j^{(1)}(\lambda q^{1/2}) T_j^{(1)}(\lambda q^{-1/2}) = 1 + T_{j+1}^{(1)}(\lambda) T_{j-1}^{(1)}(\lambda). \tag{10}$$

using (9) and the mathematical induction. *These functional relations exactly coincide with those among fusion transfer matrices of* $U_q(A_1^{(1)})$. In the latter context, the suffix j specifies the spin $j/2$ assigned to the auxiliary space. They are the closed set of equations among finitely many unknown functions $T_j^{(1)}, (j = 0, 1, \cdots, \ell)$. Thus they may be of significance in the estimation of the quantity of our original interest, $\tau_1^{(1)}(\lambda)$. Actually, with additional assumptions on the analyticity and asymptotic behavior of $\tau_j^{(1)}(\lambda)$, one can fix $\tau_1^{(1)}(\lambda)$ via coupled nonlinear integral equations resulting from (10). To see this, we conveniently put $Y_j(\lambda) = T_{j+1}(\lambda)T_{j-1}(\lambda)$ and $\lambda = \mathrm{e}^{\ell/(\ell+2)\theta}$. Now the functional relations read

$$Y_j(\theta + i\frac{\pi}{h}) Y_j(\theta - i\frac{\pi}{h}) = (1 + Y_{j-1}(\theta))(1 + Y_{j+1}(\theta)), \quad j = 1, 2, \cdots, \ell - 1 \tag{11}$$

where $h = \ell$. Note that $Y_0 = Y_\ell = 0$.

Assumption

$(\log Y_j(\theta))'$ $((\log(1 + Y_j(\theta))')$ are analytic, nonzero and have constant asymptotic behavior (ANZC) in the strips $\mathrm{Im}\theta \in [-\frac{\pi}{h}, \frac{\pi}{h}]$, $\mathrm{Im}\theta \in [-0^+, 0^+]$ respectively.

The validity of this assumption will be discussed in section 5.

Once this is granted, one immediately derives from (11) [2, 29, 30],

$$\epsilon_j(\theta) = m_j r \exp\theta - \frac{1}{2\pi} \sum_{k=1}^{h-1} \phi_{j,k} * L_k(\theta) \tag{12}$$

where $m_j = \sin(\pi j/h)/\sin(\pi/h)$, $Y_j(\theta) = \exp(\epsilon_j(\theta))$ and $L_j(\theta) = \log(1 + 1/Y_j(\theta))$. The asterisk denotes the convolution, $A * B(\theta) = \int A(\theta - \theta')B(\theta')d\theta'$.

This type of coupled integral equations is known as thermodynamic Bethe ansatz equation(TBA). One finds them in various branches of IM, e.g., the thermodynamics of 1D spin chains or the perturbation theory of CFT. The kernel, $\phi_{j,k}$, is related to the two particle S-matrix $S_{j,k}$ of quantum field theory based on A_{h-1} by $\phi_{a,b}(\theta) = -i\partial_\theta \log S_{j,k}(\theta)$ and

$$\begin{aligned} S_{j,k}(\theta) &= \prod_{j=0}^{\min(j,k)-2} \{|j-k| + 2j + 1\}, \\ & \{p\} := (p-1)(p+1), \quad (p) = \frac{\sinh(\theta/2 + i\pi/2h)}{\sinh(\theta/2 - i\pi/2h)}. \end{aligned}$$

Theorem 4. *The set of equations (12) fixes $Y_j(\theta)$ for a given r.*

To determine the factor r, it needs an independent ingredient from the spectral theory. We will come back to this point in section 5.

We have a remark. For the later use in the spectral problem, we have introduced generalized Stokes matrices and derived functional relations (10) from the obvious recursion relation (8). They can be also easily extracted from the following Wronskian representation of $\tau_j^{(1)}(E)$,

$$\tau_j^{(1)}(E) = \det \begin{pmatrix} y_0, & y_{j+1} \\ y_0', & y_{j+1}' \end{pmatrix}. \tag{13}$$

For $n > 1$, the situation is different. The generalized Stokes matrices can be defined similarly. Their elements, however, do not contain nice generalization of $\tau_1^{(a)}$'s. The formal definition of the Wronskian type like (13) still works efficiently. See the discussion in section 6.

4 Dressed Vacuum Forms of Stokes multipliers

As shown in the previous section, the Stokes multipliers share same functional relations with the transfer matrices of IM. Below we will discuss if this correspondence carries forward.

The eigenvalues of the transfer matrices in solvable models exhibit a universal structure often referred to as the dressed vacuum form (DVF). We shall explain DVF for the simplest the $A_1^{(1)}$ case with the dimension of the auxiliary space being 2.

Obviously, the highest weight state (= vacuum) is the trivial eigenstate of the transfer matrix. Its eigenvalue consists of two terms, reflecting the dimensionality of the auxiliary space. Each of them is given by the simple product of the local weights which is termed as the vacuum expectation value,

$$T_{\text{vacuum}}(\lambda) = f_1(\lambda) + f_2(\lambda).$$

This expression must be modified for general eigenvalues. The quantum inverse scattering method yields the exact expression. The result tells that $T_{\text{vacuum}}(\lambda)$ must be modified by "dressing " the vacuum expectation values with ratios of Baxter's Q operator (or its eigenvalue) which commutes with T, $[T, Q] = 0$,

$$T_1^{(1)}(\lambda) = f_1(\lambda)\frac{Q(\lambda q)}{Q(\lambda)} + f_2(\lambda)\frac{Q(\lambda q^{-1})}{Q(\lambda)}. \tag{14}$$

The fact that the eigenvalue must be pole free results the famous Bethe ansatz equation (BAE),

$$\frac{f_1(\lambda_j)}{f_2(\lambda_j)} = -\frac{Q(\lambda_j q^{-1})}{Q(\lambda_j q)}$$

where $Q(\lambda_j) = 0$. This kind of representation is called DVF.

Clearly, eq(14) has an interpretation as the second order difference equation (Baxter's T-Q relation),

$$T_1^{(1)}(\lambda)Q(\lambda) = f_1(\lambda)Q(\lambda q) + f_2(\lambda)Q(\lambda q^{-1}).$$

Thus two independent solutions exist which we call $Q_{\pm}$.

In [27, 28], it is shown that DVF is universal for models based on general $U_q(\hat{g})$ under certain assumptions. The key ingredient in the argument is the analyticity under BAE. Thus we may conclude that DVF embodies the BAE or Yang-Baxter integrable structure.

Now we turn to the $n = 1, k = 0$ of (5). The Stokes multiplier $\tau_1^{(1)}(\lambda)$ is given by $y's$ in two manners,

$$\tau_1^{(1)}(\lambda) = \frac{y_0 + y_2}{y_1}|_{x=0} \quad = \frac{y_0' + y_2'}{y_1'}|_{x=0}. \tag{15}$$

Originally, the rhs can be evaluated at any x yielding the same $\tau_1^{(1)}(\lambda)$. We adopt a convention to enumerate them at the origin for the later convenience. See section 5.

By comparison of (14) and (15) and the identification, $\tau_1^{(1)}(\lambda) = T_1^{(1)}(\lambda q)$, made after (9) , we deduce $y_j \propto Q_-(\lambda q^j)$ and $y_j' \propto Q_+(\lambda q^j)$. Precisely, the argument in the next section concludes $y_j = q^{j/2}Q_-(\lambda q^j)$, and $y_j' = q^{-j/2}Q_+(\lambda q^j)$.

The linear independence of FSS implies that $\tau_1^{(1)}(\lambda)$ is pole-free. On the other hand, $y(0, \lambda)$ can generally be zero for some $\lambda = \lambda_j$. Thus we have BAE for Stokes multipliers.

It is interesting that dy/dx, which is by no means a solution to the original ODE, now appears as the second "solution" to the difference equation. This issue will be further pursued in a later section.

The coincidence is not only for the spin 1/2 case, but also for cases of arbitrary spins. This can be easily seen as they share the same initial condition and the functional relations. One can also verify this directly using the Wronskian representation. For this we rewrite the condition $W_k = 1$ in the form,

$$y_k y_{k+1}' - y_k' y_{k+1} = 1 \Longrightarrow \frac{y_{k+1}'}{y_{k+1}} - \frac{y_k'}{y_k} = \frac{1}{y_k y_{k+1}}.$$

With use of this, one obtains

$$\begin{aligned}\tau_j^{(1)}(\lambda) &= y_0 y_{j+1}\left(\frac{y_{j+1}'}{y_{j+1}} - \frac{y_0'}{y_0}\right) = y_0 y_{j+1} \sum_{k=1}^{j}\left(\frac{y_{j+1-k}'}{y_{j+1-k}} - \frac{y_{j-k}'}{y_{j-k}}\right) \\ &= y_0 y_{j+1} \sum_{k=1}^{j} \frac{1}{y_{j-k}y_{j-k+1}},\end{aligned}$$

which coincides with the known expression for the transfer matrix. Actually the discussion like above has been firstly found as the operator identity under the name of the quantum Wronskian form. We follow the discussion in [19, 20] for reproducing the DVF for the spin $j/2$ case.

Before closing the section, we present simplest examples ($\ell = 1, 2$) where explicit solutions are available by elementary functions [1, 2, 4, 12, 15]. We shall use E instead of λ ($E = \lambda^2$) and adopt same symbols, y, τ etc as the function of E.

The case $\ell = 1$

This is a well known example in quantum mechanics. The wave function y is given by the Airy function.

On the other hand, for $\ell = 1$, we have $\tau_1^{(1)} = 1$ (theorem 3). Thus $T - Q$ relation simplifies,

$$\begin{aligned}Q_-(E) &= q^{-1/2}Q_-(Eq^{-2}) + q^{1/2}Q_-(Eq^2) \\ Q_+(E) &= q^{1/2}Q_+(Eq^{-2}) + q^{-1/2}Q_+(Eq^2).\end{aligned} \tag{16}$$

These relations coincide with the 3-solution dependence relation for the Airy function[12],

$$\begin{aligned} q^{-1}\mathrm{Ai}(q^{-1}E) + \mathrm{Ai}(E) + q\mathrm{Ai}(qE) &= 0 \\ q^{-2}\mathrm{Bi}(q^{-1}E) + \mathrm{Bi}(E) + q^2\mathrm{Bi}(qE) &= 0 \end{aligned} \tag{17}$$

where $\mathrm{Bi}(x) := \frac{d\mathrm{Ai}(x)}{dx}$.

To check this, we use $q^3 = 1$ in the arguments in the first of (16),

$$Q_-(E) = q^{-1/2}Q_-(Eq) + q^{1/2}Q_-(Eq^{-1}),$$

and substitute $q = -q^{-1/2}$ in the coefficients of the first relation in (17),

$$-q^{1/2}\mathrm{Ai}(q^{-1}E) + \mathrm{Ai}(E) - q^{-1/2}\mathrm{Ai}(qE) = 0, \Longrightarrow \mathrm{Ai}(E) = q^{-1/2}\mathrm{Ai}(qE) + q^{1/2}\mathrm{Ai}(q^{-1}E).$$

The second relations can be checked similarly.

<u>The case $\ell = 2$</u>

The case with the harmonic oscillator is slightly complicated as the asymptotic formula must be modified. We utilize known facts on the Weber's function $D_\eta(z)$,

$$\frac{d^2D_\eta(z)}{dz^2} + (\eta + \frac{1}{2} - \frac{z^2}{4})D_\eta(z) = 0,$$

which has an asymptotic behavior for $\eta \neq 0$, integer,

$$D_\eta(z) \sim z^\eta \exp(-z^2/4).$$

It has the 2nd order irregular singularity at ∞ and regular elsewhere.

The FSS consists of $\{D_\eta(z), D_{-\eta-1}(iz)\}$ or $\{D_\eta(-z), D_{-\eta-1}(-iz)\}$. The connection rule reads,

$$\frac{\sqrt{2\pi}}{\Gamma(\eta+1)}D_\eta(z) = i^\eta D_{-\eta-1}(iz) + i^{-\eta}D_{-\eta-1}(-iz). \tag{18}$$

There exist recurrence relations,

$$D'_\eta(z) = z/2D_\eta(z) - D_{\eta+1}(z) = -z/2D_\eta(z) + \eta D_{\eta-1}(z). \tag{19}$$

Obviously, $y(x, E)$ is given in terms of $D_\eta(z)$. In order to cancel the phase factor arising from the asymptotic behavior, we define precisely

$$y_k := q^{k/2+k/2E_k}\frac{D_{\eta_k}(\sqrt{2}xq^{-k})}{2^{\eta_k/2}\sqrt{2i}}$$

where $E_k = Eq^{2k}$, $2\eta_k + 1 = E_k$ and $\eta = \eta_0$. They constitute our FSS.

By definition, $T_1^{(1)}(E)y_0 = y_1 + y_{-1}$. Remembering $q = i$ so that $\eta_1 = \eta_{-1} = -\eta - 1$, we rewrite this into the form,

$$\frac{T_1^{(1)}(E)}{2^{\eta+1/2}}D_\eta(z) = i^\eta D_{-1-\eta}(iz) + i^{-\eta}D_{-1-\eta}(-iz),$$

with $z = \sqrt{2}x$. By comparing this with (18) we conclude

$$T_1^{(1)}(E) = 2^{\eta+1}\frac{\sqrt{\pi}}{\Gamma(\eta+1)} = 2^{E/2+1/2}\frac{\sqrt{\pi}}{\Gamma(E/2+1/2)}$$

which coincides with the result from CFT[2, 4]. The expectation values of $Q_\pm(E)$ are proportional to Weber's function and its derivative at the origin. Thanks to the recursion relations (19), we can replace the latter by again Weber's function with the unit shift in η,

$$Q_-(E) \propto D_\eta(0), \qquad Q_+(E) \propto D_{\eta+1}(0).$$

We shall utilize the following integral representation for $D_\eta(z)$,

$$D_\eta(z) = -\frac{\Gamma(\eta+1)}{2\pi i} e^{-z^2/4} \int_C e^{-t^2/2-zt}(-t)^{-(\eta+1)}dt,$$

where C surrounds the positive real axis counterclockwise. The evaluation at $z = 0$ is then straightforward,

$$D_\eta(0) = \frac{2^{\eta/2}\sqrt{\pi}}{\Gamma((1-\eta)/2)} = \frac{2^{(E-1)/4}\sqrt{\pi}}{\Gamma((3-E)/2)}.$$

Hence,

$$Q_-(E) \propto \frac{1}{\Gamma((3-E)/4)}, \qquad Q_+(E) \propto \frac{1}{\Gamma((1-E)/4)}. \tag{20}$$

For general values of ℓ, the representations of $T_1^{(1)}$ or $Q_\pm$ by elementary functions are not known. Still, we can evaluate them, e.g., from solutions to TBA (12). To fix one missing parameter r there, we next consider the spectral problem.

5 Spectral Determinants and Stokes multipliers

The final section for the $n = 1$ case is devoted to the spectral problem for $\ell = 2M$ and M being an integer. We will still use E instead of λ.

We first put some remarks on elementary facts. Let $H(x)$ be a our Hamiltonian operator, $H(x) = -\frac{d^2}{dx^2} + x^{2M}$.

Definition 1. *We call $\psi(x)$ the eigen-function and E, the eigenvalue of $H(x)$ if $H(x)\psi_E(x) = E\psi_E(x)$ and $\psi_E(x)$ is a vector in the Hilbert space satisfying, e.g., $||\psi(x)|| < \infty$.*

Definition 2. *Let P be a spatial inversion operator such that $Pf(x) = f(-x)$ for any operators or vectors.*

Obviously, $[H(x), P] = 0$. Thus if $H(x)\psi_E(x) = E\psi_E(x)$ then $P\psi_E(x) = p\psi_E(x)$. Since P^2 is an identity operator, $P\psi_E(x) = \psi_E(-x) = \pm\psi_E(x)$. Consequentially, we have

Lemma 1. *If $H(x)\psi_E(x) = E\psi_E(x)$ then $\psi_E(x=0) = 0$ or $\frac{d\psi_E(x)}{dx}|_{x=0} = 0$.*

The above lemma does not require the boundary condition $\lim_{x\to\pm\infty}|\psi_E(x)| = 0$ imposed by our potential.

Two conditions can not be satisfied simultaneously. Or otherwise, $\psi_E^{(n)}(x = 0) = 0$ for arbitrary n, resulting a trivial ψ. Thus a lemma follows.

Lemma 2. *Eigenvalues are classified by the parities of the associated eigenfunctions. We denote E_j^+ if $\frac{d\psi_{E_j^+}(x)}{dx}|_{x=0} = 0$ and E_j^- if $\psi_{E_j^-}(x=0) = 0$*

On the positive real axis, $\psi_E(x) = y_0 + a(E)y_1$ up to normalizations. We have $[\psi_E(x), y_0] = a(E)$ from the obvious asymptotic behavior, $\lim_{x\to\infty}\psi_E(x)/y_1 = a(E)$. As the eigenfunction must be bounded, $a(E) = 0$ if $E \in \{E_j^+\} \cup \{E_j^-\}$. Conversely, if $a(E') = 0$ for some E' then $\psi_{E'}(x)$ is proportional to y_0. Thus $\psi_{E'}(x)$ is bounded as $x \to +\infty$ *and* it is recessive as $x \to -\infty$ due to the parity argument. In addition, it is a solution to the eigenvalue equation. Then, by definition, E' belongs to the set of eigenvalues. We conclude,

Lemma 3. *If $\psi_E(x) = y_0 + a(E)y_1$ on the positive real axis, then $a(E) \propto D(E) := D_+(E)D_-(E)$ where $D_\pm(E) := \prod_j(1 - E/E_j^\pm)$.*

Finally we quote results from the WKB analysis,

Lemma 4. *For the potential x^{2M}, the energy levels E_k and the spectral determinant $D(E)$ behave asymptotically as*

$$b_0(E_k)^\mu \sim 2\pi(k+1/2), \quad k \to \infty \tag{21}$$

$$\ln D(E) \sim \frac{b_0}{2\sin(\mu\pi)} E^\mu \tag{22}$$

$$\mu = \frac{M+1}{2M}, \quad b_0 = \frac{\pi^{1/2}\Gamma(\frac{1}{2M})}{M\Gamma(\frac{1}{2M}+\frac{3}{2})}. \tag{23}$$

We shall apply the above general observation to results obtained in the preceding sections. The connection rule enables a representation of $D(E)$ in terms of y_0. To check this, we consider the Stokes matrix $S_0^{(M+1)}$. It connects FSS on the positive and the negative real axes. We start from the negative real axis. If E takes an eigenvalue, then $\psi_E(x) = y_{M+1}$ apart from a normalization. The connection rule demands it behave on the positive axis,

$$\psi(x, E_\alpha) \sim -\tau_{M-1}^{(1)}(q^2 E_\alpha) y_0 + \tau_M^{(1)}(E_\alpha) y_1.$$

Lemma 3 tells $\tau_M^{(1)} \propto D(E)$.

On the other hand, we consider the $j = M$ case of (13). Note that $q^{M+1} = -1$, and $y_{M+1} = iy(-x, E)$. Then $\tau_M^{(1)} = i(y(x,E)y'(-x,E) - y(-x,E)y'(x,E))$. Since the lhs is independent of x and the rhs is not singular at $x = 0$, we conveniently put $x = 0$ in the rhs and find $\tau_M^{(1)} \propto y(0,E)y'(0,E)|_{x=0}$. Thus, as a function of E, $y_0 y_0'|_{x=0}$ has only zeros at eigenvalues. Then the above lemma leads to their identification with $D^{\pm}(E)$. The choice of the evaluation at $x = 0$ here and in the previous section is now clear.

Summarizing, we have a theorem.

Theorem 5. *The fusion hierarchy contains $D(E)$ as its $M-$ th member,*

$$\tau_M^{(1)} \propto D(E), \quad \text{equivalently} \quad T_M^{(1)}(E) \propto D(-E).$$

A base of FSS and its derivative at the origin are proportional to spectral determinants depending on parities,

$$y(0,E) \propto D_-(E), \quad y'(0,E) \propto D_+(E).$$

The previous explicit result (20) for $M = 1(\ell = 2)$ is quite consistent with this. $Q_\pm(E)$'s are nothing but $D_\pm(E)$ here. They are vanishing at known spectra of the harmonic oscillator, $E = 2n+1$, where n =(even/odd) corresponds to the parity = (even/odd).

These identifications lead to the expression for $T_1^{(1)}$ via spectral determinants,

$$T_1^{(1)}(E) = q^{1/2}\frac{D_+(Eq^2)}{D_+(E)} + q^{-1/2}\frac{D_+(Eq^{-2})}{D_+(E)} = q^{-1/2}\frac{D_-(Eq^2)}{D_-(E)} + q^{1/2}\frac{D_-(Eq^{-2})}{D_-(E)}.$$

More significantly, we have BAEs,

$$\frac{D_\epsilon(E_j q^2)}{D_\epsilon(E_j q^{-2})} = -q^\epsilon, \qquad \epsilon = \pm.$$

These equations, combined with the WKB result, are efficient enough to determine the spectral determinants, being transformed into coupled nonlinear integral equations. We, however, take a different route here and utilize them as a tool to investigate TBA (12).

Let us revisit to the assumption 3 raised in section 3. Suppose all energy levels are enumerated exactly so that $D_\pm(E)$ are constructed. Then $T_1^{(1)}(E)$ is estimated. By the use of the analogue of the relation(9), we can successively generate $T_j^{(1)}(E)$ and check the validity of the assumption. Strictly speaking, as we have infinitely many levels, this procedure can not be accomplished. One

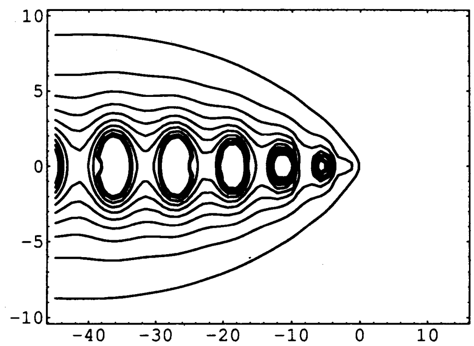

Figure 1: the contourplot for $M = 3$, $|\mathrm{e}^{E/7} T_2^{(1)}(E)|$

however knows that the WKB approximation is fairy accurate for higher energy levels. Thus we input first 100 exact energy levels and approximate rests by the WKB results, to evaluate $D_\pm(E)$.

Our numerical results indicate the remarkable patterns,

Conjecture 1. *Zeros of $T_j^{(1)}(E)$ are of the first order and always distribute on the negative real E axis.*

This supports the assumption, although by no means a proof. As an example, the contourplot for $M = 3$, $|\mathrm{e}^{E/7} T_2^{(1)}(E)|$ is depicted in Fig.1. These patterns imply that the state corresponds to "the vacuum" (the ground state) in IM [1, 4].

There is also an independent support to this conjecture [2, 31, 32]. As shown in [2], zeros of $T_j^{(1)}(E)$ coincide with negative of eigenvalues associated to $\mathcal{PT}$-symmetric Hamiltonian $p^2 + x^{2j}(ix)^\epsilon$ with $\epsilon = 2M - 2j$. The numerical and analytical studies on the $\mathcal{PT}$-symmetric Hamiltonian in [31, 32] conclude positive and real eigenvalues for $M \geq 1$, which is consistent with the conjecture. The studies reveal, at the same time, the breakdown of the conjecture, when M being continued to a real number less than 1 [2, 31, 32].

We assume that the validity of integral equations (12). Then all we have to do is fix r in evaluating Stokes multipliers and spectral determinants. The theorem 5 tells $T_M^{(1)}(-E_k) = 0$, which results $Y_M(-E_k q^2) = -1$ or $\log Y_M(\theta_k + i\frac{\pi}{2}) = (2k+1)\pi i$. Remember $E = \exp(\theta/\mu)$. For large values of θ, numerical data concludes that the contribution form the integral is negligible so that we have an approximation, $\log Y_M(\theta) \sim m_M \exp(\theta)$. Thus $T_M^{(1)}(-E_j) = 0$ means $m_M \exp(\theta_k) = m_M E_k^\mu = (2k+1)\pi$ for large enough j. Comparing this with the WKB result, we conclude $m_M r = b_0$, which derives the desired quantity.

Summarizing the results for 2nd order ODE, we have the following correspondence,

energy	$\Longleftrightarrow$	spectral parameter
Stokes multipliers, $D(E)$	$\Longleftrightarrow$	fusion transfer matrices
$y\vert_{x=0}, y'\vert_{x=0}$	$\Longleftrightarrow$	(vacuum) expectation values of $Q_\pm$

In the next two sections, the higher order ODE will be briefly discussed. Our results indicate a natural extension of the ODE/IM correspondence examined for $n = 1$ above.

6 Functional relations in Stokes multipliers

As in the case of $n = 1$, we can introduce generalized Stokes matrices connecting disjoint sectors for arbitrary n. The obvious recursion relation leads to functional relations, however, among complex objects corresponding to Young tableaux of the hook shape. Then the restriction of relations among Young tableaux of the rectangular shape results the desired relation [6]. This procedure requires some technique in integrable models e.g., quantum analogue of the Jacobi-Trudi formula.

We derive the same relation in a simpler way using the Wronskian representation of Stokes multipliers. Let auxiliary functions $\tau_m^{(a)}(\lambda)$ be

$$\tau_m^{(a)}(\lambda) = [y_1, y_2, \cdots y_{a-1}, y_0, y_{a+m}, y_{a+m+1} \cdots y_{n+m}]. \tag{24}$$

Note that we adopt the abbreviation defined in section 2.

Due to $y_{n+1+\ell} = (-)^n y_0$, $\tau_m^{(a)}(\lambda) = 0$ for $m \geq \ell+1, \ell+2, \cdots$. This is an analogue to the quantum group reduction. Remark that the set contains the original Stokes multipliers as $m = 1$ cases.

Then the claim is the following functional relations ,

Theorem 6.

$$\tau_m^{(a)}(\lambda)\tau_m^{(a)}(\lambda q) = \tau_m^{(a+1)}(\lambda)\tau_m^{(a-1)}(\lambda q) + \tau_{m+1}^{(a)}(\lambda)\tau_{m-1}^{(a)}(\lambda q) \tag{25}$$

where $\tau_1^{(0)} = 1, \tau_0^{(a)} = (-1)^{a-1}$.

We utilize a lemma in [33] for the proof of the above theorem.

Lemma 5.

$$[f_1, f_2, \cdots f_N, a_0, a_1][f_1, f_2, \cdots f_N, a_2, a_3] - \tag{26}$$
$$[f_1, f_2, \cdots f_N, a_0, a_2][f_1, f_2, \cdots f_N, a_1, a_3] + \tag{27}$$
$$[f_1, f_2, \cdots f_N, a_0, a_3][f_1, f_2, \cdots f_N, a_1, a_2] = 0. \tag{28}$$

When $N = 1$, this relation follows from the Laplace expansion of the trivial relation,

$$0 = \det \begin{pmatrix} f & a_0 & a_1 & 0 & a_2 & a_3 \\ f' & a_0' & a_1' & 0 & a_2' & a_3' \\ f" & a"_0 & a"_1 & 0 & a"_2 & a"_3 \\ 0 & 0 & a_1 & f & a_2 & a_3 \\ 0 & 0 & a_1' & f' & a_2' & a_3' \\ 0 & 0 & a"_1 & f" & a"_2 & a"_3 \end{pmatrix}.$$

With the similar argument, the validity of the above lemma is verified for arbitrary N.

Proof. of theorem 6 :

We shall adopt identifications,

$$\begin{aligned} (f_1, f_2, \cdots, f_N) &\leftrightarrow (y_1, \cdots, y_{a-1}, y_{a+m+1}, \cdots, y_{n+m}) \\ (a_0, a_1, a_2, a_3) &\leftrightarrow (y_0, y_a, y_{a+m}, y_{n+m+1}). \end{aligned}$$

For $a = 1$, the left hand side of the first relation should read as $(y_2, \cdots, y_{n+1})$. The six elements in eq.(28) are interpreted as

$$\begin{array}{cc} -\tau_m^{(a+1)}(\lambda), & (-1)^{n-2}\tau_m^{(a-1)}(\lambda q), \\ \tau_m^{(a)}(\lambda), & (-1)^{n-1}\tau_m^{(a)}(\lambda q), \\ (-1)^{n-a}\tau_{m+1}^{(a)}(\lambda), & (-1)^{a-1}\tau_{m-1}^{(a)}(\lambda q) \end{array}$$

respectively. This immediately leads to the theorem. □

We have a note. The substitutions $\tau_m^{(a)}(\lambda) \to (-1)^{a-1} T_m^{(a)}(\lambda q^{(a+m)/2})$ brings the equation to the form known as the $T-$ system for $A_n^{(1)}$ in solvable models,

$$T_m^{(a)}(\lambda q^{1/2}) T_m^{(a)}(\lambda q^{-1/2}) = T_m^{(a+1)}(\lambda) T_m^{(a-1)}(\lambda) + T_{m+1}^{(a)}(\lambda) T_{m-1}^{(a)}(\lambda). \tag{29}$$

This observation supports the ODE/IM correspondence for n arbitrary. In view of the solvable models, $T_m^{(a)}(\lambda)$ should be understood as the (eigenvalues of) transfer matrix associated to the auxiliary space $W_m^{(a)}(\lambda)$. (As the module in classical Lie algebra, $W_m^{(a)}(\lambda)$ is isomorphic to $m\Lambda_a$, of which Young diagram takes a rectangular shape.)

The relation also finds a connection to a discrete soliton system. We parameterize $\lambda = q^{p/2}$ and denote $f(p, a, m) = T_m^{(a)}(\lambda)$. Let $D_i, i = 1, 2, 3$ be Hirota operators acting on i th variable. Then the eq (29) reads

$$(\exp D_1 - \exp D_2 - \exp D_3) f \cdot f = 0.$$

This equation is known as the Hirota-Miwa equation with $Z_1 = -Z_2 = -Z_3 = 1$ [21, 22]. The present construction imposes the periodicity and boundary conditions,

$$\begin{aligned} f(\ell+n+1, a, m) &= f(0, a, m) \\ f(p, -1, m) &= f(p, n+2, m) = f(p, n+1, \ell+1) = 0. \end{aligned}$$

7 DVF for arbitrary n

The DVF in the Stokes multipliers are also found for arbitrary n.

Let us check this for $\tau_1^{(1)}(\lambda)$. The following lemma is useful for this purpose.

Lemma 6. *For $m \geq 2$ we have a recursion relation among ratios of determinants,*

$$\frac{[y_0, y_2, \cdots y_m]}{[y_1, \cdots, y_m]} = \frac{[y_0, y_2, \cdots y_{m-1}]}{[y_1, \cdots, y_{m-1}]} + \frac{[y_0, y_1, \cdots y_{m-1}][y_2, \cdots, y_m]}{[y_1, \cdots, y_m][y_1, \cdots, y_{m-1}]}. \tag{30}$$

We should interpret $[y_1] \to y_1$ and $[y_0, y_1, \cdots y_{m-1}] \to y_0$ for $m = 2$.

Proof. The lemma is equivalent to

$$[y_0, y_2, \cdots y_m][y_1, \cdots, y_{m-1}] = [y_0, y_2, \cdots y_{m-1}][y_1, \cdots, y_m] + [y_0, y_1, \cdots y_{m-1}][y_2, \cdots, y_m]. \tag{31}$$

Since (31) is linear in $y_0^{(j)}$, it suffices to show the equality of the coefficients of them in the both sides. First consider the coefficient of y_0, We need to show the equality,

$$[y_1, \cdots, y_{m-1}][y_2', \cdots y_m'] = [y_2, \cdots, y_m][y_1', \cdots y_{m-1}'] + [y_1, \cdots, y_m][y_2', \cdots y_{m-1}']. \tag{32}$$

To verify this, we prepare a matrix $\mathcal{M}$

$$\mathcal{M} := \begin{pmatrix} y_1, & y_2 & \cdots & y_m \\ y_1', & y_2' & \cdots & y_m' \\ \vdots & & \cdots & \vdots \\ y_1^{(m-1)} & y_2^{(m-1)} & \cdots & y_m^{(m-1)} \end{pmatrix}.$$

Denote by

$$D\begin{bmatrix} i_1, & i_2, & \cdots \\ j_1, & j_2, & \cdots \end{bmatrix}$$

the minor, the determinant of a matrix obtained by deleting $i_1, i_2 \cdots$ rows and $j_1, j_2, \cdots$ columns from $\mathcal{M}$. Then eq.(32) is represented as,

$$D\begin{bmatrix} m \\ m \end{bmatrix} D\begin{bmatrix} 1 \\ 1 \end{bmatrix} = D\begin{bmatrix} 1 \\ m \end{bmatrix} D\begin{bmatrix} m \\ 1 \end{bmatrix} + DD\begin{bmatrix} 1, & m \\ 1, & m \end{bmatrix},$$

where $D = \det\mathcal{M}$. Obviously this is the Jacobi identity. Thus the equality of coefficients of y_0 in both sides is established. The equalities are similarly proven up to those of $y_0^{(m-2)}$. For $y_0^{(m-1)}$ case, the first term of the rhs in (31) does not contribute. We can check , however, the equality of the reminding terms. Thus the lemma is proved. □

Next we will show

Theorem 7. *$\tau_1^{(1)}(\lambda)$ can be represented in the following DVF*

$$\begin{aligned}\tau_1^{(1)}(\lambda) &= \frac{[y_2, y_3, \cdots, y_{n+1}]}{[y_1, y_2, \cdots, y_n]} + \frac{[y_2, y_3, \cdots, y_n][y_0, y_1, \cdots, y_{n-1}]}{[y_1, y_2, \cdots, y_{n-1}][y_1, y_2, \cdots, y_n]} + \\ &\cdots + \frac{[y_2, y_3][y_0, y_1, y_2]}{[y_1, y_2][y_1, y_2, y_3]} + \frac{[y_0, y_1]y_2}{[y_1, y_2]y_1} + \frac{y_0}{y_1}.\end{aligned} \tag{33}$$

Proof. Firstly we substitute $\tau_1^{(n+1)} = (-1)^n$ to eq.(5) and obtain

$$\begin{pmatrix} y_1, & y_2, & \cdots, & y_n \\ y_1' & y_2', & \cdots, & y_n' \\ \vdots & & \cdots, & \vdots \\ y_1^{(n)} & y_2^{(n)}, & \cdots, & y_n^{(n)} \end{pmatrix} \begin{pmatrix} \tau_1^{(1)}(\lambda) \\ \tau_1^{(2)}(\lambda) \\ \vdots \\ \tau_1^{(n)}(\lambda) \end{pmatrix} = \begin{pmatrix} y_0 - (-1)^n y_{n+1} \\ y_0' - (-1)^n y_{n+1}' \\ \vdots \\ y_0^{(n)} - (-1)^n y_{n+1}^{(n)} \end{pmatrix}.$$

The application of Cramer's formula yields $\tau_1^{(1)}$ in the form,

$$\tau_1^{(1)} = \frac{[y_2, y_3, \cdots, y_{n+1}]}{[y_1, y_2, \cdots, y_n]} + \frac{[y_0, y_2, \cdots, y_n]}{[y_1, y_2, \cdots, y_n]}.$$

We use Lemma 6 to the second term in the rhs to obtain,

$$\tau_1^{(1)} = \frac{[y_2, y_3, \cdots, y_{n+1}]}{[y_1, y_2, \cdots, y_n]} + \frac{[y_0, y_1, \cdots, y_{n-1}][y_2, y_3, \cdots, y_{n-1}]}{[y_1, y_2, \cdots, y_n][y_1, y_2, \cdots, y_{n-1}]} + \frac{[y_0, y_2, \cdots, y_{n-1}]}{[y_1, y_2, \cdots, y_{n-1}]}.$$

It is now obvious that repeated applications of Lemma 6 to the last term results the expression (33). □

We mimic the case of $n = 1$ and introduce D functions

$$[y_j, \cdots, y_{k+j}]|_{x=0} = q^{(n-k)(k+1)(j+k/2)/2} D^{(k+1)}(\lambda q^{j+k/2}).$$

Then the $a+1-$th term in (33) reads,

$$\frac{[y_2, y_3, \cdots, y_{a+1}][y_0, y_1, \cdots, y_a]}{[y_1, y_2, \cdots, y_a][y_1, y_2, \cdots, y_{a+1}]} = q^{a-n/2} \frac{D^{(a+1)}(\lambda q^{a/2}) D^{(a)}(\lambda q^{(a+3)/2})}{D^{(a+1)}(\lambda q^{(a+2)/2}) D^{(a)}(\lambda q^{(a+1)/2})}.$$

The DVF consists of $n+1$ terms for the solvable $U_q(A_1^{(1)})$ model of which auxiliary space is Λ_1 as a classical module. It is characterized by Baxter's Q operators of n species,

$$\begin{aligned}T_1^{(1)}(\lambda) &= f_n \frac{Q^{(n)}(\lambda q^{(n+1)/2})}{Q^{(n)}(\lambda q^{(n-1)/2})} + f_{n-1} \frac{Q^{(n)}(\lambda q^{(n-3)/2}) Q^{(n-1)}(\lambda q^{n/2})}{Q^{(n)}(\lambda q^{(n-1)/2}) Q^{(n-1)}(\lambda q^{(n-2)/2})} + \cdots \\ &+ f_a \frac{Q^{(a+1)}(\lambda q^{(a-2)/2}) Q^{(a)}(\lambda q^{(a+1)/2})}{Q^{(a+1)}(\lambda q^{a/2}) Q^{(a)}(\lambda q^{(a-1)/2})} + \cdots \\ &+ f_1 \frac{Q^{(2)}(\lambda q^{-1/2}) Q^{(1)}(\lambda q)}{Q^{(2)}(\lambda q^{1/2}) Q^{(1)}(\lambda q)} + f_0 \frac{Q^{(1)}(\lambda q^{-1})}{Q^{(1)}(\lambda)}.\end{aligned}$$

Clearly, we have

Theorem 8. *Under the identification,* $\tau_1^{(1)}(\lambda) \leftrightarrow T_1^{(1)}(\lambda q)$,

$$Q^{(a)}(\lambda) \leftrightarrow D^{(a)}(\lambda), \quad f_a \leftrightarrow q^{-n/2+a} \tag{34}$$

two DVFs coincide.

The pole-free property of $\tau_1^{(1)}(\lambda)$, required from the linear independence of FSS, results BAE,

$$-q^{-1} = \frac{Q^{(a-1)}(\lambda_j^{(a)} q^{-1/2}) Q^{(a)}(\lambda_j^{(a)} q) Q^{(a+1)}(\lambda_j^{(a)} q^{-1/2})}{Q^{(a-1)}(\lambda_j^{(a)} q^{1/2}) Q^{(a)}(\lambda_j^{(a)} q^{-1}) Q^{(a+1)}(\lambda_j^{(a)} q^{1/2})}, \qquad (a = 1, 2, \cdots, n)$$

where $Q^{(a)}(\lambda_j^{(a)}) = 0$ and $Q^{(n+1)} = Q^{(0)} = 1$.

Thus we have verified the common algebraic structure for arbitrary n.

The representation (33) or the identification (34) is , however, not unique. One easily recognizes this by remembering the simplest case ($n = 1$) where two different expressions are available for $\tau_1^{(1)}$. This originates from the simple fact that both y and its derivative are solutions to Baxter's $T - Q$ relation. The situation is also true for $n > 1$. One can show that the identification

$$Q^{(a)}(\lambda q^{a+k}) \leftrightarrow \frac{d^j}{dx^j}[y_{k+1}, y_{k+2}, \cdots, y_{k+a}]|_{x=0}, j = 0, 1, \cdots \tag{35}$$

works and we have a variety of representations for the same $\tau_1^{(1)}$. This is shown by using formulas analogous to Lemma 6, and the detail will be published elsewhere.

Before closing the section, we comment on $\tau_m^{(a)}$, $(a > 1)$. The corresponding DVFs are known in the integrable models, but explicit forms are quite involved. Still, one can parameterize them by analogue of Young tableaux[28]. We utilize the "tableaux" representation in proving the equivalence of the DVF in integrable models and Stokes multipliers $\tau_m^{(a)}$ for a, n and m general. This point may be further discussed in a separate publication.

8 Summary and discussions

In the present report, we discuss a curious connection between $n+1$ the order ODE and integrable models. When $n = 1$, the connection is efficient enough to derive analytic equations which yields estimations of eigenvalues and Stokes multipliers.

For higher n, the correspondence is still at the algebraic relation level. Unfortunately, the definition of the eigenvalue problem is not necessary clear for higher order differential equations. The characterization of the eigenspace (it is the Hilbert space for $n = 1$) is not obvious. More technically, there are several subdominant solutions in each sector. This obscures the identification of $D(E)$ in the general Stokes matrices. The lack of the connection prevents us from writing down the integral equations and evaluating parameters like "r" for $n = 1$. We however comment some progress made in [3, 23]

The observation made in the last few sections may be interesting. Suppose that the ODE/IM correspondence even occurs at the construction of models. Then one may find the variable x also in IM. Once if one of Baxter's $Q^{(1)}$ is constructed, the other independent $Q^{(1)}$ functions are found in derivatives of the $Q^{(1)}$ with respect to the hidden variable x. Moreover, one can generate higher Q functions, i.e., $Q^{(a)}, a \geq 2$ mere by taking determinants of fundamental $Q^{(1)}$. The other $Q^{(a)}$'s are again obtainable via taking derivatives. On the other hand, construction of Q functions via the standard "pair-propagation" argument [17] seems to be far more complicated for $n > 1$. To the authors' knowledge, the explicit construction of Q is done only for cases corresponding $n = 1$ [17, 34, 35], and the procedure is already involved. The systematic construction found in ODE is not obvious in IM. The present results for ODE may be a clear guide for analyses in the analogous issue in IM, but it needs further research.

Finally, we comment that the ODE/IM correspondence is still at the "phenomenological" stage. The fundamental question as to the origin of the correspondence is still open. The complete classification of the ODE tractable with the IM approach may need the answer to this fundamental question.

I hope to clarify these issues in future publications.

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