数理解析研究所講究録 1222 巻 2001 年 53-57

Characterization of finite Automata by the Images and the Kernels of their Transition Functions [1]

T. Saito (斎藤 立彦)

Mukunoura 374, Innoshima,
Hiroshima, Japan, 722-23221
e-mail: tatsusaito@mx4.tiki.ne.jp

1. Introduction

By an automaton $\mathcal{A}$, we mean here a 3-tuple (X, A, δ), where X is a finite set (the set of states), A is a finite alphabet (the set of inputs) and δ is a mapping of $X \times A$ into X (the transition function).

As usual, A^* and A^+ denotes the free monoid and free semigroup generated by A, respectively, and δ is extended from $X \times A$ to $X \times A^*$. In this case, $\delta(x, s)$ is denoted simply by xs for $x \in X, s \in A^*$

Let $\rho = \{(s,t) \in A^* \times A^* : xs = xt$ for every $x \in X\}$. Then ρ is a congruence on A^* and A^*/ρ is a finite transformation semigroup on X by defining the action of $s\rho \in A^*/\rho$ on X as $x(s\rho) = xs$. The semigroup A^*/ρ is called the *characteristic semigroup* of $\mathcal{A}$. Let $\mathcal{V}$ be a class of semigroups not necessarily a variety. Then an automaton $\mathcal{A}$ is called a *$\mathcal{V}$-type* if $A^+/\rho \in \mathcal{V}$.

For $s \in A^*$, let $\operatorname{im} s = \{xs : x \in X\} = Xs$ and $\ker s = \{(x,y) \in X \times X : xs = ys\}$, which are called the image and the kernel of s, respectively. Then $\ker s$ is an equivalence on X.

Let $\mathcal{V}$ and $\mathcal{U}$ be two classes of semigroups. Then the direct product of $\mathcal{V}$ and $\mathcal{U}$ is defined by $\mathcal{V} \times \mathcal{U} = \{V \times U : V \in \mathcal{V}, U \in \mathcal{U}\}$. Let $U \in \mathcal{U}$ and let S be a semigroup. If for each $s \in V$, there exists $U_s \in \mathcal{U}$ such that $S = \sqcup\{U_s : s \in U\}$ and $U_s \cdot U_t = \{u_s u_t : v_s \in U_s, v_t \in U_t\} \subseteq U_{st}$, then we say that S belongs to $\mathcal{V}(\mathcal{U})$, where $\sqcup$ denotes a disjoint union..

As our start, we consider the following classes of semigroups: $\mathcal{G}$={groups}, $\mathcal{LZ}$={left zero semigroups $[st = s]$}, $\mathcal{RZ}$={right zero semigroups $[st = t]$} and $\mathcal{SL}$ ={semilattices $[st = ts, s^2 = s]$}. We first characterize, for the classes $\mathcal{G}, \mathcal{LZ} \times \mathcal{G}, \mathcal{G} \times \mathcal{RZ}$ and $\mathcal{LZ} \times \mathcal{G} \times \mathcal{RZ}$, their types automata by the images and the kernels of their transition functions. By using the results, we characterize, for $\mathcal{V} \in \{\mathcal{SL}, \mathcal{LZ}, \mathcal{RZ}\}$ and $\mathcal{U} \in \{\mathcal{G}, \mathcal{LZ} \times \mathcal{G}, \mathcal{G} \times \mathcal{RZ}, \mathcal{LZ} \times \mathcal{G} \times \mathcal{RZ}\}$, $\mathcal{V}(\mathcal{U})$-type automaton by the same way.

---

[1] This is an abstract and the details will be published eleswhere

For a example, we show later that an automaton $\mathcal{A}$ is a $\mathcal{SL}(\mathcal{LZ} \times \mathcal{G})$-type if and only if im $st =$ im $s \cap$ im t for every $s, t \in A^+$.

2. Preliminaries

For a set Y, $|Y|$ denotes the cardinality of Y, for an equivalence λ, $x\lambda$ denotes the λ-class containing x and $|\lambda|$ denotes the number of λ-classes. i.e., $|\lambda| = |\{x\lambda : x \in X\}|$, and for $s \in A^+$, $|s|$ denotes the length of s. Then clearly $|\text{im } s| = |\text{ker } s|$ for every $s \in A^+$.

For $s \in A^*$, let fix $s = \{x \in X : xs = x\}$. Let $E(A^+) = \{e \in A^+ : (e, e^2) \in \rho\}$. Then it is easy to see that $e \in E(A^+)$ if and only if im $e =$ fix e. and that $(e, f) \in \rho$ for $e, f \in E(A^+)$ if and only if im $e =$ im f and ker $e =$ ker f. Since A^+/ρ is finite, for every $s \in A^+$, there exists a positive integer m such that $s^m \in E(A^+)$.

Lemma 1. *Let $s, t \in A^+$. Then*

(1) *If* $|\text{ker } st| = |\text{ker } t|$, *then* im $st =$ im t.

(2) *If* $|\text{im } st| = |\text{im } s|$, *then* ker $st =$ ker s.

Lemma 2. *Let $s \in A^+$. Then the following are equivalent :*

(1) im $s \cap x$ker $s \neq \emptyset$ *for every* $x \in X$.

(2) im $s^m =$ im s *and* ker $s^m =$ ker s *for every* $m \in \mathbf{N}^+$.

(3) *There exists* $e \in E(A^+)$ *such that* im $s =$ im e, ker $s =$ ker e, $(s, se) \in \rho$ *and* $(s, es) \in \rho$.

Lemma 3. *The following are equivalent :*

For every $s, t \in A^+$,

(1) im $s \cap x$ker $t \neq \emptyset$ *for every* $x \in X$,

(2) im $st =$ im t.

(3) ker $st =$ ker s.

Let $\mathcal{A} = (X, A, \delta)$ be an automaton, and let $Y = \cup\{\text{im } a : a \in A\}$ and $\kappa = \cap\{\text{ker } a : a \in A\}$. Then we have $Y = \cup\{\text{im } s; s \in A^+\}$ and $\kappa = \cap\{\text{ker } s : s \in A^+\}$. In fact, if $s \in A^+$, then $s = s'a = bs"$ for some $a, b \in A, s's" \in A^*$, so that im $s'a \subseteq$ im a and ker $b \subseteq$ ker $bs"$.

Since $Ys \subseteq Y$ for every $s \in A^+$, the restriction s_Y of s to Y can be defined. Let $A_Y = \{a_Y : a \in A\}$. Then the automaton $\mathcal{A}_Y = (Y, A_Y, \delta)$ is called the *subautomaton of $\mathcal{A}$ with respect to Y.*

Let $s, t \in A^+$ and $x \in X$. Since $(xs)t_Y = (xs)t$, the action of st_Y on X is defined by $x(st_Y) = x(st)$.

Let κ be as above. Define the action of $s \in A^+$ on X/κ by $(x\kappa)s = (xs)\kappa$. Then the action is well-defined. In fact, if $x\kappa = y\kappa$, then $(x, y) \in \kappa \subseteq \ker s$, so that $xs = ys$. When the action of s is on X/κ, s is denoted by s_κ. Let $A_\kappa = \{a_\kappa : a \in a\}$. Then the automaton $\mathcal{A}_\kappa = (X/\kappa, A_\kappa, \delta)$ is called the *automaton induced from $\mathcal{A}$ by κ.*

Let $s, t \in A^+$ and $x \in X$. Then clearly $(x\ker s)s = xs$. Since $\kappa \subseteq \ker s$, we have $(x\kappa)s = xs$, so that $((x\kappa)s_\kappa)t = ((xs)\kappa)t = (xs)t$, Thus the action of $s_\kappa t$ on X is defined by $x(s_\kappa t) = x(st)$.

For an automaton $\mathcal{A} = (X, A, \delta)$, let $Im(A^+) = \{\text{im } s : s \in A^+\} = \{Y_i : i \in I\}$, i.e., for each $i \in I$, $Y_i = \text{im } s$ for some $s \in A^+$ and $\text{im } s \in Im(A^+)$ for every $s \in A^+$, and let $Ker(A^+) = \{\ker s : s \in A^+\} = \{\kappa_\mu : \mu \in M\}$, $Im(A_\kappa^+) = \{\text{im } s_\kappa : s \in A^+\} = \{Z_i : i \in I'\}$ and $Ker(A_Y^+) = \{\ker s_Y : s \in A^+\} = \{\kappa_\mu : \mu \in M'\}$. In this case, if $\text{im } s \cap x\ker s \neq \emptyset$ holds for every $s \in A^+$ and $x \in X$, then $Im(A^+) = Im(E(A^+))$ and $Ker(A^+) = Ker(E(A^+))$.

3. Main Results

A semigroup in $\mathcal{LZ} \times \mathcal{G}$ is called a left group whose class is denoted simply by $\mathcal{LG}$, i.e., $\mathcal{LG} = \mathcal{LZ} \times \mathcal{G}$.

Theorem 1. *Let $\mathcal{A} = (X, A, \delta)$ be an automaton. Then the following are equivalent :*

(1) *There exists a subset Y of X such that* $\text{im } a = Y$ *and* $Y \cap x\ker a \neq \emptyset$ *for every $a \in A$ and $x \in X$.*

(2) *There exists a subset Y of X such that* $\text{im } s = Y$ for every $s \in S$.

(3) *$\mathcal{A}$ is a left group type.*

From Theorem 1 we obtain the following results

Corollary 1.1. *An automaton $\mathcal{A} = (X, A, \delta)$ is a $\mathcal{SL}(\mathcal{LG})$-type if and only if* $\text{im } st = \text{im } s \cap \text{im } t$ *for every $s, t \in A^+$.*

Corollary 1.2. *An automaton $\mathcal{A} = (X, A, \delta)$ is a $\mathcal{RZ}(\mathcal{LG})$-type if and only if* $\text{im } st = \text{im } t$ *for every $s, t \in A^+$.*

A semigroup in $\mathcal{G} \times \mathcal{RZ}$ is called a right group whose class is denoted by $\mathcal{RG}$, i.e., $\mathcal{RG} = \mathcal{G} \times \mathcal{RZ}$.

Theorem 2. *Let $\mathcal{A} = (X, A, \delta)$ be an automaton. Then the following are equivalent :*

(1) *There exists an equivalence* κ *on* X *such that* $\ker a = \kappa$ *and* $\mathrm{im}\ a \cap x\kappa \neq \emptyset$ *for every* $a \in A$.

(2) *There exists an equivalence* κ *on* X *such that* $\ker s = \kappa$ *for every* $s \in A^+$.

(3) $\mathcal{A}$ *is a right group type.*

Corollary 2.1. *An automaton* $\mathcal{A} = (X, A, \delta)$ *is a* $\mathcal{SL}(\mathcal{RG})$*-type if and only if* $\ker st = \ker s \vee \ker t$ *for every* $s, t \in A^+$.

Corollary 2.2. *An automaton* $\mathcal{A} = (X, A, \delta)$ *is a* $\mathcal{LZ}(\mathcal{RG})$*-type if and only if* $\mathrm{im}\ st = \mathrm{im}\ t$ *for every* $s, t \in A^+$.

From Corollaries 1.2 and 2.2 we obtain :

Corollary 2.3. *An automaton* $\mathcal{RZ}(\mathcal{LG})$*-type if and only if it is* $\mathcal{LZ}(\mathcal{RG})$*-type.*

Remark. It can be easily show that $\mathcal{LZ}(\mathcal{LG}) = \mathcal{LZ}(\mathcal{G}) = \mathcal{LG}$ and $\mathcal{RZ}(\mathcal{RG}) = \mathcal{RZ}(\mathcal{G}) = \mathcal{RG}$.

Theorem 3. *Let* $\mathcal{A} = (X, A, \delta)$ *be an automaton. Then the following are equivalent* :

(1) *There exist a subset* Y *of* X *and an equivalence* κ *on* X *such that* $\mathrm{im}\ a = Y$ *and* $\ker a = \kappa$ *for every* $a \in A$ *and* $Y \cap x\kappa \neq \emptyset$ *for every* $x \in X$.

(2) *There exist a subset* Y *of* X *and an equivalence* κ *on* X *such that* $\mathrm{im}\ s = Y$ *and* $\ker s = \kappa$ *for every* $s \in A^+$,

(3) $\mathcal{A}$ *is a group-type.*

A semigroup in $\mathcal{SL}(\mathcal{G})$ is called a *Cliford semigroup,*

Corollary 3.1. *An automaton* $\mathcal{A} = (X, A, \delta)$ *is a Cliford smigroup type if and only if* $\mathrm{im}\ st = \mathrm{im}\ s \cap \mathrm{im}\ t$ *and* $\ker st = \ker s \vee \ker t$ *for every* $s, t \in A^+$.

Theorem 4. *Let* $\mathcal{A} = (X, A, \delta)$ *be an automaton, and Let* $Y = \cup\{\mathrm{im}\ a : a \in A\}, \kappa = \cap\{\ker a : a \in A\}$. *Suppose that* $\mathrm{im}\ s \cap x\ker s \neq \emptyset$ *for every* $s \in A^+, x \in X$. *Then the following are equivalent:*

(1) $\mathcal{A}$ *is a* $\mathcal{LZ} \times \mathcal{G} \times \mathcal{RZ}$*-type.*

(2) $\ker s_Y = \ker t_Y$ *for every* $s, t \in A^+$.

(3) $\mathrm{im}\ s_\kappa = \mathrm{im}\ t_\kappa$ *for every* $s, t \in A^+$.

Corollary 4.1. *With the assumption of Theorem 4, the following are equivalent* :

(1) $\mathcal{A}$ *is a* $\mathcal{SL}(\mathcal{LZ} \times \mathcal{G} \times \mathcal{RZ})$*-type.*

(2) ker $s_Y t_Y =$ ker $s_Y \vee$ ker t_Y *for every* $s, t \in A^+$.
(3) im $s_\kappa t_\kappa =$ im $s_\kappa \cap$ im t_κ *for every* $s, t \in A^+$.

Suppose that an automaton $\mathcal{A}$ is a $\mathcal{LZ} \times \mathcal{G} \times \mathcal{RZ}$-type. As is seen in the proof of Theorem 4, $A^+/\rho = \{(i, g, \mu) : i \in I, g \in G, \mu \in M\}$. For $i \in I$ and $\mu \in M$, let $A_i/\rho = \{(i, g, \mu) : g \in G, \mu \in M\}$ and $A_\mu = \{(i, g, \mu) : i \in I, g \in G\}$, respectively. Then $A_i/\rho \in \mathcal{RG}$ and $A_\mu/\rho \in \mathcal{LG}$. For $s\rho = (i, g.\nu), t\rho = (j, h, \mu)$, since $(st)\rho = (i, gh, \mu)$, by Theorems 1 and 2, we have ker $st =$ ker s and im $st =$ im t. Thus we obtain ;

Corollary 4.2. *If an automaton* $\mathcal{A}$ *is a* $\mathcal{LZ} \times \mathcal{G} \times \mathcal{RZ}$*-type, then it is a* $\mathcal{RZ}(\mathcal{LG})$*-type. The converse is not true,*

There is a simple example that a $\mathcal{RZ}(\mathcal{LG})$-type automaton which is not a $\mathcal{LZ} \times \mathcal{G} \times \mathcal{RZ}$-type.

References

1. Howie, J. M., "Fundamentals of Semigroup Theory", Oxford Science Publications, Oxford, 1995.

2. Howie, J. M., "Automata and Languages", Oxford Science Publications, Oxford, 1991.

3. Petrich, M., "Lectures in Semigroups", John Wiley and Sons, London, 1977.

4. Saito, T. *Band-type acts over free monoids* (preprint).