数理解析研究所講究録 1222 巻 2001 年 90-97



# On Some Trices

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## 1 Motivation and Introduction

A semilattice $(S, *)$ is a set $S$ with a single binary, idempotent, commutative and associative operation $*$. Under the relation defined by

$$a \leq_* b \Longleftrightarrow a * b = b, \tag{1}$$

any semilattice $(S, *)$ is a partially ordered set $(S, \leq_*)$. Let $L$ be a lattice with two operations $\vee$ (join) and $\wedge$ (meet). Then, $(L, \vee)$ and $(L, \wedge)$ are semilattices. From (1), we can construct two ordered sets $(L, \leq_\vee)$ and $(L, \leq_\wedge)$, respectively ($a \leq_\vee b \Leftrightarrow a \vee b = b$ and $a \leq_\wedge b \Leftrightarrow a \wedge b = b$). The dual of $(L, \leq_\vee)$ is $(L, \leq_\wedge)$. That is, the order $\leq_\vee$ is nothing but the reverse of the order $\leq_\wedge$. The reason why $\vee$ and $\wedge$ introduce the reverse order is that lattices satisfy the absorption laws:

$$a \vee (a \wedge b) = a, \qquad a \wedge (a \vee b) = a \qquad (\textit{absorption}). \tag{2}$$

Suppose that there is an object with a rope on a line, and that we are able to pull the rope from the right (see Figure 1). Then, we can move the object to the right direction, but not to the left direction. This situation is considered to be irreversible.

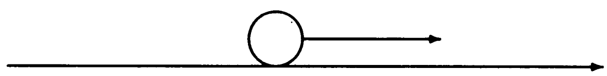


Figure 1: one dimensional irreversible

Next, see Figure 2. By pulling the rope from either right or left directions, we can move the object to any point on the line. This situation is considered to be reversible. A semilattice is a set having one order, i.e. one direction, like in Figure 1. A lattice

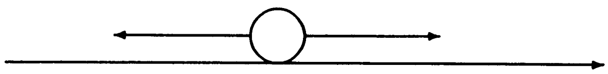


Figure 2: one dimensional reversible

is a set having two orders, i.e. two directions, like in Figure 2. Any object on the line can be moved to another arbitrary point on the line by pulling the object to the positive or negative directions.

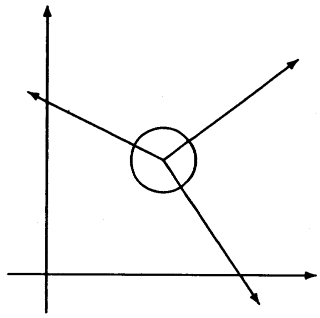


Figure 3: two dimensional reversible

Suppose that there is an object not on a line but in a plane (two dimensional Euclidian space). If we want to move the object to an arbitrary point in the plane, it is sufficient to have three "suitable" directions to pull the object, as shown in Figure 3.

If the three directions are not "suitable", we cannot move the original object to an arbitrarily chosen target point. Movements in three directions are not necessarily restricted to the two dimensional plane. Consider that a lattice corresponds to the one dimensional reversible case (Figure 2). What systems with three binary operations (semilattice) correspond to the two dimensional reversible cases (Figure 3)?

For $A$ a nonempty set and $n$ a positive integer, let $(A,\ *_1, *_2, ..., *_n)$ be an algebra with $n$ binary operations, and $(A,\ *_i)$ be a semilattice for every $i \in \{1, 2, ..., n\}$. Then, $(A,\ *_1, *_2, ..., *_n)$ is called a **n-semilattice**. We will deal mainly with triple-semilattice, (that is, $n = 3$). We denote each order on $A$ by $a \leq_i b \iff a *_i b = b$, respectively. Let $S_n$ be the symmetric group on $\{1, 2, ...n\}$. An algebra $(A, *_1, *_2, ..., *_n)$ has the **n-roundabout absorption laws** if it satisfies the following $n!$ identities:

$$((((a *_{\sigma(1)} b) *_{\sigma(2)} b) *_{\sigma(3)} b)... *_{\sigma(n)} b) = b. \tag{3}$$

for all $a, b \in A$ and for all $\sigma \in S_n$. Of cause, the 2-roundabout absorption laws is the absorption laws. An algebra $(A, *_1, *_2)$ which satisfies the 2-roundabout absorption laws is a lattice. The operations $*_1$ and $*_2$ are denoted by $\vee$ and $\wedge$.

An algebra $(A, *_1, *_2, *_3)$ which satisfies the 3-roundabout absorption laws is said to be a **trice**. To simplify explanation, we often omit "3" of "3-roundabout absorption laws." The operations $*_1$, $*_2$ and $*_3$ will be denoted by $\nearrow_1$, $\nwarrow_2$ and $\downarrow_3$ . Now, we can easily check that a lattice $L$ has following properties:

$$\forall a, b \in L\ \exists c \in L \text{ such that } a \leq_{\vee} c \leq_{\wedge} b$$
$$\forall a, b \in L\ \exists c \in L \text{ such that } a \leq_{\wedge} c \leq_{\vee} b.$$

**Proposition 1** Let an algebra $(A, *_1, *_2, ..., *_n)$ have the n-roundabout absorption laws. For every $a, b \in A$ and for evry $\sigma \in S_n$, there exists $c_1, ..., c_{n-1} \in A$ such that

$$a \leq_{\sigma(1)} c_1 \leq_{\sigma(2)} c_2 \leq_{\sigma(3)} \cdots \leq_{\sigma(n-1)} c_{n-1} \leq_{\sigma(n)} b. \tag{4}$$

We say that $(A, *_1, *_2, ..., *_n)$ is **attainable** if (4) is true for every $a, b \in A$. The "attainable" of lattice corresponds to the notion of reversiblity in the case of one dimension (Figure 2). We can consider that the "attainablity" of trice corresponds to the case of Figure 3, that is, the two dimensional reversible cases. The "trice" is a notion to correspond to "lattice."

# 2 Construction of trice

If we introduce three orders into a set under the condition that all two elements on it have a least upper bound for each order, then we can construct triple-semilattice. However, it is difficult to examine whether it is trice. Next question is naturally.

**Question 1** Let $(A, *_1, *_2, ..., *_n)$ be a n-semilattice. By adding any operations, is it possible to materialize the roundabout absorption laws?

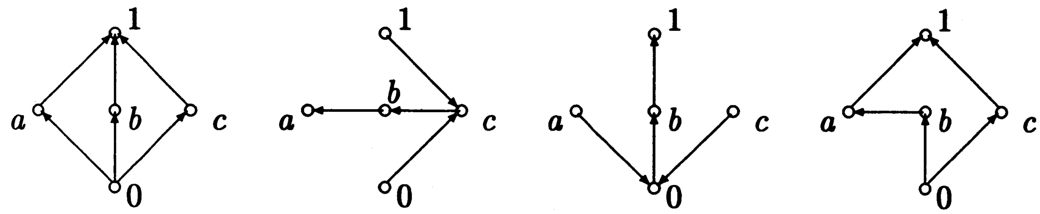
We consider about some fundamental cases of the above question.

## 2.1 From lattice to trice

**Question 2** Let $(L, \vee, \wedge)$ be a lattice. What kind of operation $*$, so that $(L, \vee, \wedge, *)$ is a trice, exists?

If the operation $*$ is equal to $\vee$ or $\wedge$, then the algebra $(L, \vee, \wedge, *)$ is a trice. This type n-semilattice is called stammered semilattice. It is being studied well (see [4]). Let $(L, *)$ be any linearly orderd set, then the algebra $(L, \vee, \wedge, *)$ is a trice.
And we can always make a trice by adding another type operation to a lattice.

**Example 1** The lattice $(M_3, \vee, \wedge)$ has five elements: $a$, $b$, $c$, 0, 1 and $a \wedge b = a \wedge c = b \wedge c = 0$, $a \vee b = a \vee c = b \vee c = 1$. Let $(M_3, *_1)$ be a semilattice which diagram is Figure 4-(a), let $(M_3, *_2)$ be a semilattice which diagram is Figure 4-(b). And let $(M_3, *_2)$ be a semilattice which diagram is Figure 4-(c). Then, the algebra $(M_3, \vee, \wedge, *_1)$, $(M_3, \vee, \wedge, *_2)$ and $(M_3, \vee, \wedge, *_3)$ are trices.




$(M_3, \vee, \wedge)$ (a) $(M_3, *_1)$ (b) $(M_3, *_2)$ (c) $(M_3, *_3)$

Figure 4: trice on $M_3$

## 2.2 From 2-semilattice to trice

**Definition 1** Let $(S, *_1, *_2)$ be a 2-semilattice. The $(S, *_1, *_2)$ is **tricable** when an algebra $(S, *_1, *_2, *_3)$ is a trice for some semilattice operation $*_3$ on $S$.

Every lattice $(L, \vee, \wedge)$ is tricable 2-semilattice. We will show another interested tricable 2-semilattice.

**The two delegates on linearly ordered set**
Suppose that $X$ is a linearly ordered set and $|X| \geq 2$. Let $X(2)$ be the set of all two points subset of $X$, that is,

$$X(2) = \{\{x_1, x_2\} |\ x_1, x_2 \in X\ s.t.\ x_1 < x_2\}.$$

Suppose that $\{a_1, a_2\}, \{b_1, b_2\} \in X(2)$. Let $c_2$ be $a_2 \vee b_2$. This $c_2$ is the largest member of $\{a_1, a_2, b_1, b_2\}$. And take

$$c_1 = \begin{cases} a_2 \vee b_1 & if\ a_2 < b_2 \\ a_1 \vee b_2 & if\ b_2 < a_2 \\ a_1 \vee b_1 & if\ a_2 = b_2 \end{cases}$$

Then $c_1$ is the second largest member of $\{a_1, a_2, b_1, b_2\}$. We define the binary operation $\sqcup$ on $X(2)$ by

$$\{a_1, a_2\} \sqcup \{b_1, b_2\} = \{c_1, c_2\}.$$

Clearly, $(X(2), \sqcup)$ is a semilattice. For example, let $X$ be a set of students, and we want to select two students as deledats to attend a convention. Some members of committee recommend $a_1$ and $a_2$. The other members of committee recommend $b_1$ and $b_2$. Then, we may select $c_1$ and $c_2$ from $a_1$, $a_2$, $b_1$ and $b_2$ according to this method. Hence, we called this operation **two delegates**.

Next, we define the dual operation of $\sqcup$. Let $d_1$ be the $a_1 \wedge b_1$. This $d_1$ be the smallest member of $\{a_1, a_2, b_1, b_2\}$. And take

$$d_2 = \begin{cases} a_2 \wedge b_1 & if\ a_1 < b_1 \\ a_1 \wedge b_2 & if\ b_1 < a_1 \\ a_2 \wedge b_2 & if\ a_1 = b_1 \end{cases}$$

Then, $d_2$ be the second smallest member of $\{a_1, a_2, b_1, b_2\}$. We define the binary operation $\sqcap$ on $X(2)$ by

$$\{a_1, a_2\} \sqcap \{b_1, b_2\} = \{d_1, d_2\}.$$

This $(X(2), \sqcap)$ is also a semilattice. The $(X(2), \sqcup, \sqcap)$ is not a lattice. But it is tricable 2-semilattice. Now, we define another operation $*$ on $X(2)$ by

$$\{a_1, a_2\} * \{b_1, b_2\} = \{d_1, c_2\}.$$

This $*$ is the operation which chose the smallest member and the largest member among $\{a_1, a_2, b_1, b_2\}$. Clearly, $(X(2), *)$ is a semilattice. The algebra $(X(2), \sqcup, \sqcap, *)$ is a trice.

**Example 2** Let $X$ be the four points linearly ordered set $\{1, 2, 3, 4\}$. We denote the two points subset $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$ and $\{3,4\}$ by $a$, $b$, $c$, $d$, $e$ and $f$. Then, the set $X(2)$ is $\{a, b, c, d, e, f\}$ (see Figure 5). This $(X(2), \sqcup, \sqcap, *)$ is a trice.

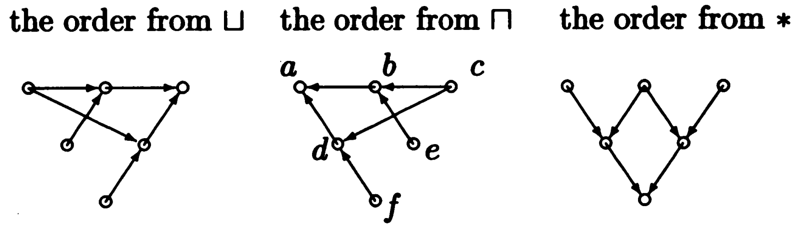



Figure 5: Example 2

**The three delegates on linearly ordered set**

We consider replacing "two delegates" with "three delegates". Suppose that $X$ is a linearly ordered set and $|X| \geq 3$. Define $X(3) = \{\{x_1, x_2, x_3\}|\ x_1, x_2, x_3 \in X$ *s.t.* $x_1 < x_2 < x_3\}$. For $\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\} \in X(3)$, suppose that $c_3$ is the largest member in $\{a_1, a_2, a_3, b_1, b_2, b_3\}$. Let $c_2$ be the second largest member and let $c_1$ be the third largest member. That is, we select three different points from top in descending order. We define the binary operations $\sqcup$ on $X(3)$ by

$$\{a_1, a_2, a_3\} \sqcup \{b_1, b_2, b_3\} = \{c_1, c_2, c_3\}.$$

Let $d_1$ be the smallest member of $\{a_1, a_2, a_3, b_1, b_2, b_3\}$, let $d_2$ be the second smallest member and let $d_3$ be the third smallest member. That is, we select three different points from bottom in ascending order. We define the binary operations $\sqcap$ on $X(3)$ by

$$\{a_1, a_2, a_3\} \sqcap \{b_1, b_2, b_3\} = \{d_1, d_2, d_3\}.$$

Then, the algebra $(X(3), \sqcup, \sqcap)$ is not tricable 2-semilattice (see Example 3). However, define the two operations $\bigtriangledown$ and $\bigtriangleup$ by

$$\{a_1, a_2, a_3\} \bigtriangledown \{b_1, b_2, b_3\} = \{d_1, c_2, c_3\}$$

$$\{a_1, a_2, a_3\} \bigtriangleup \{b_1, b_2, b_3\} = \{c_1, d_2, d_3\}.$$

Then, $\sqcup$, $\sqcap$, $\bigtriangledown$ and $\bigtriangleup$ have the 4-roundabout absorption laws.

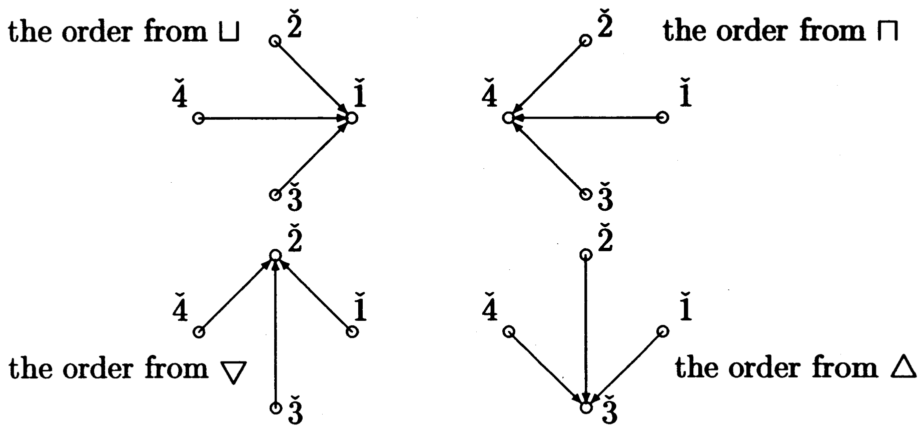



Figure 6: Example 3

**Example 3** Let $X$ be the four points linearly ordered set $\{1, 2, 3, 4\}$. We denote the three points subset $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$ and $\{2,3,4\}$ by $\check{4}$, $\check{3}$, $\check{2}$ and $\check{1}$. Then, the set $X(3)$ is $\{\check{1}, \check{2}, \check{3}, \check{4}\}$. (see Figure 6). Let $*$ be a semilattice on $X(3)$. If $\check{2} * \check{3} = \check{1}$, then $((\check{2} * \check{3}) \sqcup \check{3}) \sqcap \check{3} = \check{4}$. If $\check{2} * \check{3} = \check{2}$, then $((\check{2} * \check{3}) \sqcup \check{3}) \sqcap \check{3} = \check{4}$. If $\check{2} * \check{3} = \check{3}$, then $((\check{3} * \check{2}) \sqcup \check{2}) \sqcap \check{2} = \check{4}$. If $\check{2} * \check{3} = \check{4}$, then $((\check{2} * \check{3}) \sqcup \check{3}) \sqcap \check{3} = \check{4}$. In any case, the algebra $(X(3), \sqcup, \sqcap, *)$ is not a trice.

As the generalities, we have next question.

**Question 3** How many operations is it necessary to add to a n-semilattice to satisfy the roundabout absorption laws?

## 2.3 From semilattice to trice

When $(A, *_1)$ is a semilattice, the $*_1$ is not always one of the operations of a lattice. That is, it is not sufficient to add one operation to semilattice to satisfy the roundabout absorption laws.

**Question 4** By adding two operations, does any semilattice become a trice? Is there a concrete way of composing it? Then, can we make interested trice which it should pay attention to?

**Definition 2** Let $(A, *_1)$ be a semilattice. The semilattice $(A, *_1)$ is **tricable** when an algebra $(A, *_1, *_2, *_3)$ is a trice for some semilattice operations $*_2$ and $*_3$ on $A$.

Of course, if it knows that an algebra $(T, \nearrow_1, \nwarrow_2, \downarrow_3)$ is trice, then $(T, \nearrow_1)$, $(T, \nwarrow_2)$ and $(T, \downarrow_3)$ are tricable. If $(L, \vee, \wedge)$ is a lattice, then $(L, \vee)$ and $(L, \wedge)$ are tricable. If the 2-semilattice $(S, *_1, *_2)$ is tricable in the meaning of the definition 1, $(S, *_1)$ and $(S, *_2)$ are tricable. But, the 2-semilattice $(A, *_1, *_2)$ is not tricable even if $(A, *_1)$ and $(A, *_2)$ are tricable. For example, the $(X(3), \sqcup)$ and the $(X(3), \sqcap)$ are tricable, however the $(X(3), \sqcup, \sqcap)$ is not tricable.

**Example 4** The seven points semilattice which diagram is Figure 7-left is not tricable. But the eight points semilattice which diagram is Figure 7-right is tricable.

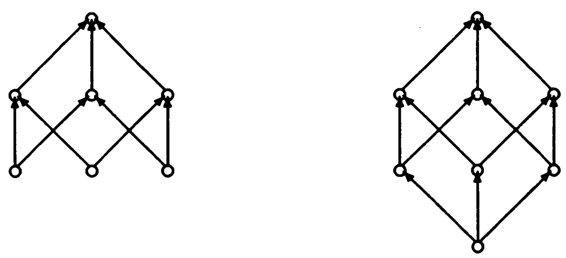


Figure 7: Example 4

Then, the sub-semilattice of $(A, *)$ is not tricable even if $(A, *)$ is tricable.

We gained the next propositions about the concrete way of composing a tricable semilattice.

Suppose we are given a family $\{(A_s, *_s)\}_{s \in S}$ of pairwise disjoint tricable semilattice, i.e., that $A_s \cap A_{s'}$ for $s \neq s'$.

**Proposition 2** Let $S$ be a linearly ordered set. Cosider the set $A = \bigcup_{s \in S} A_s$. And let the operation $\nearrow_1$ on $A$ defined by

$$x \nearrow_1 y = \begin{cases} x *_s y & if\ x, y \in A_s \\ x & if\ x \in A_s,\ y \in A_{s'} \text{ and } s > s' \\ y & if\ x \in A_s,\ y \in A_{s'} \text{ and } s < s'. \end{cases}$$

The algebra $(A, \nearrow_1)$ is a tricable semilattice.

**Proposition 3** Let $0 \in S$ and the semilattice $(A_0, *_0)$ have a minimum element $e$, that is, $e *_0 x = x$ for all $x \in A_0$. Cosider the set $A = \bigcup_{s \in S} A_s$. And let the operation

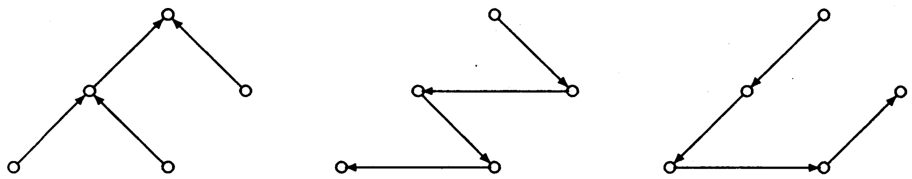
$\nearrow_1$ on $A$ defined by

$$x \nearrow_1 y = \begin{cases} x *_s y & if\ x, y \in A_s \\ x & if\ x \in A_0,\ y \in A_s \text{ and } s \neq 0 \\ y & if\ y \in A_0,\ x \in A_s \text{ and } s \neq 0 \\ e & if\ x \in A_s,\ y \in A_{s'} \text{ and } 0 \neq s \neq s' \neq 0. \end{cases}$$

The algebra $(A, \nearrow_1)$ is a tricable semilattice.

**Proposition 4** Suppose that the diagram of a semilattice is a tree structure, then it is made a trice by adding two operations which induced linearly ordered.

**Example 5** Suppose that $(S, \nearrow_1)$ is a semilattice which diagram is the tree of Figure 8-left. Let $(S, \nwarrow_2)$ and $(S, \downarrow_3)$ be next linearly ordered sets. Then, the algebra $(S, \nearrow_1, \nwarrow_2, \downarrow_3)$ is a trice.



the order from $\nearrow_1$ the order from $\nwarrow_2$ the order from $\downarrow_3$

Figure 8: Example 5

**Question 5** Will not there be another good method which composes a trice of a semilattice?

**Question 6** Will not there be a good way of judging that a semilattice is tricable?

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