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ON ORDERED MONOID RINGS

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A monoid M is said to be *ordered* if the elements of M are linearly oredered with respect to the relation $<$ and that, for all $x, y, z \in G$, $x < y$ implies $zx < zy$ and $xz < yz$. It is well known that torsion-free nilpotent groups and free groups are ordered groups (see [8, Lemma 13.1.6 and Corollary 13.2.8]). Hence any submonoid of a torsion-free nilpotent group or a free group is an ordered monoid.

Let R be a ring. A left (resp. right) annihilator of a subset U of R is defined by $l_R(U) = \{a \in R \mid aU = 0\}$(*resp.* $r_R(U) = \{a \in R \mid Ua = 0\}$). Let G be an ordered monoid. Put $rAnn_R(2^R) = \{r_R(U) \mid U \subseteq R\}$ and $lAnn_R(2^R) = \{l_R(U) \mid U \subseteq R\}$. If U is a subset of R, then $r_{RG}(U) = r_R(U)RG$. Hence we have a map $\Phi : rAnn_R(2^R) \longrightarrow rAnn_{RG}(2^{RG})$ defined by $\Phi(I) = I(RG)$ for every $I \in rAnn(R)$. For an element $f \in RG$, C_f denotes the set of coefficients of f and for a subset V of RG, C_V denotes the set $\cup_{f \in V} C_f$. Then $r_{RG}(V) \cap R = r_R(V) = r_R(C_V)$. Hence we also have a map $\Psi : rAnn_{RG}(2^{RG}) \longrightarrow rAnn_R(2^R)$ defined by $\Psi(I) = I \cap R$ for every $I \in \Delta$. Obviously Φ is injective and Ψ is surjective. Clearly Φ is surjective if and only if Ψ is injective, and in this case Φ and Ψ are the inverses of each other. We consider when Φ is surjective.

Following Rege and Chhawchharia [10] a ring R is called an *Armendariz ring* if whenever polynomials $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$, satisfy $f(x)g(x) = 0$ we have $a_i b_j = 0$ for every i and j. This name is conneted with the work of Armendariz [2]. Some results of Armendariz rings can be found in [1], [5], [7] and [10]. Let G be an ordered monoid. A ring R is called a *G-Armendariz ring* if whenever $p = \sum_{g \in G} a_g g, q = \sum_{h \in G} b_h h \in RG$ satisfy $pq = 0$ we have $a_g b_h = 0$ for every g and h in G.

The following proposition shows that Φ is bijective if and only if R is Armendariz.

Proposition 1. Let R be a ring and let G be an ordered monoid. The following statements are equivalent:

1) R is G-Armendariz.
2) $rAnn_R(2^R) \longrightarrow rAnn_{RG}(2^{RG}); A \rightarrow A(RG)$ is bijective.
3) $lAnn_R(2^R) \longrightarrow lAnn_{RG}(2^{RG}); B \rightarrow (RG)B$ is bijective.

Following Kaplansky [6], a ring R is called a *Baer ring* if the left annihilator of each subset is generated by an idempotent. We note that the definition of Baer rings is left-right symmetric. A ring R is called a *left (resp. right) p.p.-ring* if the left (resp. right) annihilator of each element of R is generated by an idempotent. A left and right p.p. ring is celled a p.p. ring. Using Proposition 1, we can generalize [7, Theorems 9 and 10]) as follows:

Corollary 2. Let G be an ordered monoid and let R be a G-Armendariz ring. Then R is a Baer ring (resp. p.p. ring) if and only if RG is a Baer ring (resp. p.p. ring).

A ring R is called a *G-quasi-Armendariz ring* if whenever $p = \sum_{i=0}^{m} a_i g_i$, $q = \sum_{j=0}^{n} b_j h_j \in RG$ satisfy $pRGq = 0$, then we have $a_i R b_j = 0$ for every i and j. In case $G = \{x^i \mid i = 0, 1, 2, \cdots\}$, a G-quasi-Armendariz ring is simply called a *quasi-Armendariz ring*. In [5], we studied quasi-Armendariz rings. Let $rAnn_R(id(R))$ (resp. $lAnn_R(id(R))$) denote the set $\{r_R(U) \mid U$ is an ideal of $R\}$ (resp. $\{l_R(U) \mid U$ is an ideal of $R\}$).

Proposition 3. Let R be a ring and let G be an ordered monoid. Then the following statements are equivalent:

1) R is G-quasi-Armendariz.

2) $rAnn_R(id(R)) \longrightarrow rAnn_{RG}(id(RG)); A \to A(RG)$ is bijective.

3) $lAnn_R(id(R)) \longrightarrow lAnn_{RG}(id(RG)); B \to (RG)B$ is bijective.

A submodule N of a left R-module M is called a *pure* submodule if $L \otimes_R N \to L \otimes_R M$ is a monomorphism for every right R-module L.

Theorem 4. Let G be an ordered monoid. Then the following are equivalent;

(1) $l_R(Ra)$ is pure as a left ideal in R for any element $a \in R$;

(1) $l_{RG}(RGz)$ is pure as a left ideal in RG for any element $z \in RG$;

In this case, R is a G-quasi-Armendariz ring.

Corollary 5. Let R be a commutative ring and let G be an abelian ordered monoid. Then each principal ideal of R is flat if and only if each principal ideal of RG is flat. In this case R is a G-Armendariz ring.

A ring R is called *quasi-Baer* if the left annihilator of every left ideal of R is generated by an idempotent. Note that this definition is left-right symmetric. Some results of a quasi-Baer ring can be found in [3] and [9].

Let R be a quasi-Baer ring and let $a \in R$. Then $l_R(aR) = Re$ for some idempotent $e \in R$, and so $l_R(aR)$ is pure as a left ideal in R. Therefore a quasi-Baer ring satisfies the hypothesis of Theorem 4. Hence we obtain the following.

Corollary 6 ([4, Theorem 1]). Let G be an ordered monoid. A ring R is a quasi-Baer ring if and only if RG is a quasi-Baer ring. In this case, R is a quasi-Armendariz ring.

Now we consider some extensions of G-quasi-Armendariz rings. Let R be a ring and let n be a positive integer. Let $\mathrm{M}_n(R)$ denote the ring of $n \times n$ matrices over R and e_{ij} denote the (i, j)-matrix unit. $T_n(R)$ denotes the ring of all $n \times n$ upper triangular matrices over R.

Theorem 7. Let G be an ordered monoid. If R a G-quasi-Armendariz ring and let S be a subring of $M_n(R)$ such that $e_{ii}Se_{jj} \subseteq S$ for all i, $j \in \{1, \cdots, n\}$. Then S is also a G-quasi-Armendariz ring.

Corollary 8. Let G be an ordered monoid. If R a G-quasi-Armendariz ring, then, for any positive integer n, $T_n(R)$ is also a G-quasi-Armendariz ring.

For $f \in RG$, the content A_f of f is the ideal of R generated by the coefficients of f. For any subset S of RG, A_S denotes the ideal $\sum_{f \in S} A_f$. In case $G = \{ x^i \mid i = 0, 1, 2, \cdots\}$, a commutative ring R is *Gaussian* if $A_{fg} = A_f A_g$ for all $f, g \in R[x]$ (See [1]). We extend this notion to noncommutative rings as follows. A ring R is said to be *G-quasi-Gaussian* if $A_{fRg} = A_f A_g$ for all $f, g \in RG$.

Theorem 9. A ring R is G-quasi-Gaussian if and only if every homomorphic image of R is quasi-Armendariz.

Example 10. A ring R is *fully idempotent* if $I^2 = I$ for every twosided ideal I of R. Obviously a ring R is fully idempotent if and only if every homomorphic image of R is semiprime. Von Neumann regular rings are fully idempotent. We can easily see that a semiprime ring is G-quasi-Armendariz. Therefore by Theorem 9, a fully idempotent ring is a G-quasi-Gaussian ring.

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