数理解析研究所講究録 1222 巻 2001 年 152-159

No trade theorem in an S-4 logic model

慶応義塾大学大学院経済学研究科
(Graduate School of Economics, Keio University)*
平瀬　和基 (Kazuki Hirase) †

Abstract

This paper introduces an application of the S-4 logic. There are two aims in this paper. Aim 1 is to check the relation between our model and the S-4 logic. We'll see the soundness and completeness of the S-4 logic with respect to the model by using the concept of structure. Aim 2 is to prove a variation of no trade theorem in the model.

1 Introduction

The word "knowledge" and especially "common knowledge" plays a very important role in game theory. Intuitively, an event is common knowledge if everyone knows it, everyone knows that everyone knows it, everyone knows that everyone knows that everyone knows it and so on. Then how can we treat (common) knowledge formally?

Aumann (1976) tried to solve this problem. He introduced the formal notion of common knowledge using set based and partitional information structure and showed, so called, agreeing to disagree theorem[1]

After Aumann, many papers have studied knowledge. Milgrom (1981), and Monderer and Samet (1989) treated knowledge by different approaches. Milgrom (1981) applied axiomatic approach [2] to modeling knowledge. Monderer and Samet (1989) used probability approach[3]. They managed to approximate knowledge with belief. We note that these approaches also use partiotional information structure.

Samet (1990) have studied non-partitional information structure. He showed agreeing to disagree theorem based on non-partitional information structure.

This paper also studies non-partitional information structure like Samet. We would like to prove a kind of no trade theorem which is introduced by Milgrom and Stokey(1982)

*Mita 2-15-45, Minatoku, Tokyo, 108-8345, Japan

†E-mail address: hirase@pc.highway.ne.jp

[1]This theorem insists that if players' posteriors for a given event are common knowledge, then these must be equal, even though they are based on different information.

[2]This is the approach which defines the set of all states in which a player knows a given event. After Milgrom's paper, many papers have been written by this approach.

[3]Probability approach defines the event in which player n believes E with probability at least p.

after showing the relation between the model and the logic. In section 5, we will see the model is one of the S-4 logic. a variation of no trade theorem is proved in section 8.

2 S-4 logic

S-4 logic is denoted as $\langle Sy, S, AR, Prov\rangle$.

Sy means symbols. Symbols consists of N, PV, logical connectives, and players' modal operators. N means a set of players i and j. Now, we restrict the number of the players to 2 persons for simplicity. But we can extend the results to n persons case easily. PV is a set of propositional variables, or atomic sentences. Logical connectives are $\wedge, \vee, \rightarrow, (,)$, and $\neg$. Players' modal operators are $\Box_i$ and $\Box_j$.

The second element of S-4 logic is S. S means a set of sentences, or a set of formulae. S is inductively constructed from L inductively.

$$(S1) : PV \subseteq S$$

$$(S2) : \phi, \psi \in S \Rightarrow \neg\phi, \phi \rightarrow \psi, \phi \wedge \psi, \phi \vee \psi, \Box_n \phi \in S \ (n = i, j)$$

$(S3)$: Every sentence is constructed by a finite number of applications of (S1) and (S2).

The third element of the S-4 logic is AR. AR means axioms and rules. AR consists of PL, modal axioms and rules.

PL is propositional logic, or a set of all tautologies, that is, for all $\phi, \psi, \chi \in S$.

$$(PL1) : \phi \rightarrow (\psi \rightarrow \phi)$$

$$(PL2) : (\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi)$$

$$(PL3) : (\neg\phi \rightarrow \neg\psi) \rightarrow ((\neg\phi \rightarrow \psi) \rightarrow \phi)$$

$$(PL4) : \phi \wedge \psi \rightarrow \phi$$

$$(PL5) : \phi \rightarrow \phi \vee \psi$$

$$(MP) : \frac{\phi \rightarrow \psi \quad \phi}{\psi}$$

$$(\wedge - rule) : \frac{\phi \rightarrow \psi \quad \phi \rightarrow \chi}{\phi \rightarrow \psi \wedge \chi}$$

$$(\vee - rule) : \frac{\psi \rightarrow \phi \quad \chi \rightarrow \phi}{\psi \vee \chi \rightarrow \phi}$$

where $\phi, \psi, \chi \in S$.

For modal part, we assume axioms $K, T, 4$ and N. K is, in other words, the Axiom of Distribution. T is the Axiom of Knowledge. 4 is the Positive Introspection. And N is the Necessitation rule.

$$(K):(\Box_n(\phi \to \psi) \to (\Box_n\phi \to \Box_n\psi))$$

$$(T):\Box_n\phi \to \phi$$

$$(4):\Box_n\phi \to \Box_n\Box_n\phi$$

$$(N):\frac{\phi}{\Box_n\phi}$$

where $\phi, \psi \in S$ and $n = i, j$

With these axioms and rules, we can define the provability *Prov.* of a sentence in the logic.

Definition 1. A *proof* is a finite tree satisfying (PR1) and (PR2).

$(PR1)$: A sentence is associated with each node, and the sentence associated with every leaf node is an instance of $(PL1)-(PL5), K$, T, or 4.

$(PR2)$: Each adjoining node forms an instance of (MP), $(\wedge - rule)$, $(\vee - rule)$, or (N).

We say that $\phi(\in S)$ is *provable* in the S-4 logic if and only if there exist a proof which root is associated with ϕ.

3 Structure

Structure is $\langle \Omega, P_i, P_j \rangle$. Ω is a nonempty finite state space. So 2^Ω is called a set of events. Players' *information functions* P_i and P_j is a function from the state space Ω to the event set 2^Ω. The set $P_n(\omega)$ means the event which player n recognize when the real state is ω. The set $P_n(\omega)$ is called player n's information set or possibility set at ω.

We assume that each players' information function satisfy the following.

$$(P-1):\omega \in P_n(\omega)$$

$$(P-2):\omega' \in P_n(\omega) \Rightarrow P_n(\omega') \subseteq P_n(\omega)$$

for $\forall \omega, \omega' \in \Omega$ and $n = i, j$.

P-1 means the condition that a player never excludes the real state. When the real state is ω, the playern thinks that ω may have occurred. From P-2, we have that if there is a state ξ, so that $\xi \in P_n(\omega')$ and $\xi \notin P_n(\omega)$ then $\omega' \notin P_n(\omega)$. So, P-2 says that player n at ω can make consideration as follows: "The state ξ is excluded. If it were the state ω', I would not exclude ξ. Thus it must be that the state is not ω'." P-1 and P-2 play very important roles in the relation to the S-4 logic. We call these three tuples an *information* *structure.*

Partiotional information structure need to assume P-3: $\omega' \in P_n(\omega) \Rightarrow P_n(\omega') \supseteq P_n(\omega)$ $for \omega, \omega' \in \Omega$.

To interpret P-3, consider the case that $\omega' \in P(\omega)$ and there is a state $\xi \in P(\omega)$ that is not in $P(\omega')$. Then, P-3 says that a player at ω can conclude, from the fact that he (she) can not exclude ξ, that the state is not ω', a state at which he (she) would be able to exclude ξ.

Note the following proposition holds.

Proposition 1. Player n's information function P_n satisfies P-1, 2, and 3 if and only if there is a partition of Ω such that for any $\omega \in \Omega$ the set $P_n(\omega)$ is the element of the partition that contains ω.

Proof. Suppose that P_n satisfies P-1, 2, and 3. If $P_n(\omega)$ and $P_n(\omega')$ intersect and $\xi \in P_n(\omega) \cap P_n(\omega')$ then by P-2 and 3, we have $P_n(\omega) = P_n(\omega') = P_n(\xi)$. By P-1 we have $\bigcup_{\omega\in\Omega} P_n(\omega) = \Omega$. The other direction is obvious. □

We don't assume P-3. Thus we treat a non-partitional information structure.

4 Model

The model $\mathcal{M}$ consists of Sy, S, an information structure, a truth assignment π, and a validity $\models$, i.e., $\mathcal{M} = \langle Sy, S, \Omega, P_i, P_j, \pi, \models \rangle$. A truth assignment π is a function from $PV \times \Omega$ to the set $\{\top, \bot\}$. From this truth assignment, $\models$ decides the validity of the sentences. For any sentence ϕ and ψ, we define the validity as follows.

(V1): For any $v \in PV$, $\models_\omega v \iff \pi(v, \omega) = \top$

(V2): $\models_\omega \neg\phi$ if and only if $\models_\omega \phi$ does not hold.

(V3): $\models_\omega \phi \to \psi \iff \models_\omega \neg\phi$ or $\models_\omega \psi$

(V4): $\models_\omega \phi \wedge \psi \iff \models_\omega \phi$ and $\models_\omega \neg\phi$

(V5): $\models_\omega \phi \vee \psi \iff \models_\omega \phi$ or $\models_\omega \psi$

(V6): $\models_\omega \Box_n\phi \iff P_n(\omega) \subseteq \{\xi \in \Omega : \models_\xi \phi\}$ for $n = i, j$

5 Soundness and Completeness

With these preparations of logic, structure, and model, we can prove the following theorem, This theorem is well known by logicians as *soundness and completeness* (of the S-4 logic). The theorem insists that a sentence is provable in the logic if and only if the sentence is valid at every state in any S-4 model.

Theorem 1. *A sentence ϕ is provable in the S-4 logic* $\iff \models_\omega \phi$ *for* $\forall\omega \in \Omega$ *in the model* $\mathcal{M}$.

sketch of the proof. For soundness ($\Rightarrow$), we can verify that each sentence [4] in AR is valid at $\forall\omega \in \Omega$ in the model using the properties P-1 and P-2. For completeness ($\Leftarrow$), first, we can construct the canonical model.[5] Second, we can see that a sentence ϕ is valid in the canonical model if and only if the sentence ϕ is provable in the S-4 logic. Third, we can easily show that if a sentence ϕ is valid in any S-4 model then the sentence ϕ is valid in the canonical model, because the canonical model is also a model. These arguments show the completeness, that is, a sentence is provable if it is valid. □

6 Knowledge

From this section, we analyze the model, S-4 logic model. Since no trade Theorem treats an epistemic condition for no trade, we have to define the concept of knowledge formally. This section defines the knowledge, common knowledge and mutual knowledge.

Definition 2. Player n *knows* an event $E(\in 2^\Omega)$ at ω if and only if $P_n(\omega) \subseteq E$. $(n = i, j)$.

From the meaning of the information function, the player n knows that some state in $P_n(\omega)$ has occurred. Hence if $P_n(\omega) \subseteq e$, (of course) the player n know the state in E has occurred. With this interpretation, we have defined the player's knowledge.

Before defining common knowledge, we define the self-evident event.

Definition 3. An event $F(\in 2^\Omega)$ is a *self evident* between i and j if and only if $\omega \in F \Rightarrow P_n(\omega) \subseteq F$ for $n = i, j$.

F is a self-evident event among S, if whenever it occurs players i and j know that it occurs. Now, we define common knowledge.

Definition 4. An event E is *common knowledge* at ω between i and j if and only if there exist a self evident event F between i and j such that $\omega \in F \subseteq E$.

E is common knowledge between i and j, if there is a self-evident event between i and j containing ω whose occurrence implies E.

Here, we define *mutual knowledge*, concept between knowledge and common knowledge. Intuitively, an event is mutual knowledge when all players know the event. The formal definition is as follows.

Definition 5. Let $R := P_i(\omega) \cup P_j(\omega)$. An event E is *mutual knowledge* at ω between i and j if and only if $R \subseteq E$.

It is obvious that the following proposition holds.

Proposition 2. If an event E is mutual knowledge at ω between i and j, player i knows E at ω. And if an event E is common knowledge at ω between i and j, E is mutual knowledge at ω between i and j.

---

[4]A rule $\frac{\phi}{\psi}$ must be modified by a sentence $\phi \to \psi$.

[5]See Chellas (1980), Hughes and Cresswell (1996).

7 Economy

In this section, we consider a pure exchange economy where the players interact. Pure exchange economy is a well known concept in economics literature. It is denoted as $\langle N, \Omega, P_i, P_j, C, e, U_i, U_j, \mu \rangle$. N is a set of players. $\langle \Omega, P_i, P_j \rangle$ is an information structure. Here, there are l kind of commodities, and $C := \Re^l_+$ is a *commodity space*. $e = (e_i, e_j) : \Omega \to C \times C$ is an *initial endowment* of commodities for players. $U_i, U_j : \Omega \times C \to \Re$ is a player's *utility function*. We assume that each player's utility function is strictly increasing with respect to C. We suppose the existence of a prior and it is common for both players. (So, a prior is called common prior.) Let the *common prior* be a probability measure on Ω, μ. We denote the common prior to E as $\mu(E)$. And we assume $\mu(E) > 0$ for any event E.

In this economy, given $x : \Omega \to C \times C$, players compute expected utility *ex ante*, *interim*, and *ex post*.

- player n's *ex ante* expected utility is $\sum_{\omega \in \Omega} \mu(\{\omega\}) U_n(\omega, x_n(\Omega))$.
- player n's *interim* expected utility at ω is $\sum_{\omega' \in P_n(\omega)} \frac{\mu(\{\omega'\}) U_n(\omega', x_n(\omega'))}{\mu(\{P_n(\omega)\})}$
- player n's *ex post* expected utility at ω is $U_n(\omega, x_n(\omega))$

8 No Trade Theorem

To show no trade theorem, we have to define trade, feasible trade, and Pareto optimality.

Definition 6 (trade and feasible trade). A *trade* $t = (t_i, t_j)$ is a function from Ω to $\Re^l \times \Re^l$. A trade t is *feasible* if and only if for all $\omega \in \Omega$,

$$e_n(\omega) + t_n(\omega) \geq \mathbf{0} \quad \forall n \in N$$

and

$$t_i(\omega) + t_j(\omega) \leq \mathbf{0}.$$

Feasibility of a trade means that each player's trade have to be within the budget of his (her) own initial endowment and that trade does not make any commodity.

Definition 7 (Pareto optimality). $x : \Omega \to C \times C$ is *(ex ante) Pareto optimal* if and only if there does not exist $y : \Omega \to C \times C$ such that

$$\forall n \in N, E_n[U_n(y_n)] \geq E_n[U_n(x_n)]$$

and

$$\exists n \in N, E_n[U_n(y_n)] > E_n[U_n(x_n)].$$

Pareto optimality is one of the most famous normative concept in economics.
With these preparations, we can prove the following theorem.

Theorem 2 (No Trade Theorem). *Suppose that an initial endowment e is ex ante Pareto optimal and that t is a feasible trade. Then, if it is mutual knowledge at ω that each player ex post weakly prefers t to e, every player at ω is interim indifferent between t and the zero trade, that is,*

$$\sum_{\omega' \in P_n(\omega)} \frac{\mu(\{\omega'\})U_n(\omega', e_n(\omega') + t_n^*(\omega'))}{\mu(\{P_n(\omega)\})} = \sum_{\omega' \in P_n(\omega)} \frac{\mu(\{\omega'\})U_n(\omega, e_n(\omega'))}{\mu(\{P_n(\omega)\})}, n = i, j.$$

The result says that the trade t is meaningless for every player at ω with his (her) information. So I would like to think this as some stability, or an equilibrium. Hence, no trade theorem shows an epistemic condition for an equilibrium.

Proof. Denote $R := P_i(\omega) \cup P_j(\omega)$. From the condition of mutual knowledge, $\forall \omega' \in R, \forall n \in N$,

$$U_n(\omega', e_n(\omega') + t_n(\omega')) \geq U_n(\omega', e_n(\omega')). \tag{1}$$

Now, define $t_n^* := t_n \mathbf{1}_R$ for $\forall n \in N$.[6] ($\mathbf{1}_R(\omega') = 1$ if $\omega' \in R$, and $\mathbf{1}_R(\omega') = 0$ otherwise.) Viewing t^* *ex ante*,

$$\begin{aligned}
&\sum_{\omega' \in \Omega} \mu(\{\omega'\})U_n(\omega', e_n(\omega') + t_n^*(\omega')) \\
&= \sum_{\omega' \in \Omega} \mu(\{\omega'\})U_n(\omega', e_n(\omega') + t_n(\omega')\mathbf{1}_R(\omega')) \\
&= \sum_{\omega' \in \Omega} \mu(\{\omega'\})\mathbf{1}_R(\omega')U_n(\omega', e_n(\omega') + t_n(\omega')) + \sum_{\omega' \in \Omega} \mu(\{\omega'\})\mathbf{1}_{R^C}(\omega')U_n(\omega', e_n(\omega')) \\
&= \sum_{\omega' \in R} \mu(\{\omega'\})U_n(\omega', e_n(\omega') + t_n(\omega')) + \sum_{\omega' \in R^C} \mu(\{\omega'\})U_n(\omega', e_n(\omega')) \\
&\geq \sum_{\omega' \in R} \mu(\{\omega'\})U_n(\omega', e_n(\omega')) + \sum_{\omega' \in R^C} \mu(\{\omega'\})U_n(\omega', e_n(\omega')) \\
&= \sum_{\omega' \in \Omega} \mu(\{\omega'\})U_n(\omega', e_n(\omega') + t_n^*(\omega')) \qquad (2)
\end{aligned}$$

where R^C denotes the complement of R, and the inequality follows from (1).

Suppose that the inequality of (1) is strict for $\exists n \in N$ and $\exists \omega' \in R$. Then the inequality of (2) must be strict for $\exists n \in N$. This contradicts our assumption that the initial endowment e is Pareto optimal. Hence,

$$U_n(\omega', e_n(\omega') + t_n(\omega')) = U_n(\omega', e_n(\omega')), \quad \forall \omega' \in R, n = i, j.^7 \tag{3}$$

---

[6] t^* is also feasible.

[7] Each player at ω is *ex post* indifferent between t and the zero trade.

From $P_i(\omega) \subseteq R$ and $P_j(\omega) \subseteq R$, (3) implies;

$$\sum_{\omega' \in P_n(\omega)} \frac{\mu(\{\omega'\})U_n(\omega', e_n(\omega') + t_n^*(\omega'))}{\mu(\{P_n(\omega)\})} = \sum_{\omega' \in P_n(\omega)} \frac{\mu(\{\omega'\})U_n(\omega, e_n(\omega'))}{\mu(\{P_n(\omega)\})}, n = i, j.$$

Therefore, each player at ω is *interim* indifferent between t and the zero trade. □

Note that common knowledge is not necessary for our theorem, while the original no trade theorem in Milgrom and Stokey need common knowledge. For this point, our assumption is weaker than the original, since common knowledge means mutual knowledge. Also note the original needs the word " each player *interim* weakly prefers t to e " but does not need the word " each player *ex post* weakly prefers t to e ." For this point, our assumption is more strong because if a player *ex post* weakly prefers t to e he *interim* weakly prefers t to e in this case.

References

Aumann, R.: "Agreeing to Disagree," *The Annals of Statistics*, 4 (1976), 1236-1239.

Chellas, B.: *Modal Logic: An Introduction.*: Cambridge University Press, 1980.

Hughes, G., and Cresswell, M.: *A New Introduction to Modal Logic.* : Routledge, 1996.

Milgrom, P.: "An Axiomatic Characterization of Common Knowledge," *Econometrica*, 49 (1981), 219-222.

Milgrom, P., and Stokey, N.: "Information, Trade and Common Knowledge," *Journal of Economic Theory*, 26 (1982), 17-27.

Monderer, D., and Samet, D.: "Approximating Common Knowledge with Common Belief," *Games and Economic Behavior*, 1 (1989), 170-190.

Samet, D.: "Ignoring Ignorance and Agreeing to Disagree," *Journal of Economic Theory*, 52 (1990), 190-207.