数理解析研究所講究録 1222 巻 2001 年 24-33

Some type of commutative artin algebras*

山口大学理学部 菊政 勲 (Isao Kikumasa)
山口大学理学部 吉村 浩 (Hiroshi Yoshimura)

Department of Mathematics, Faculty of Science
Yamaguchi University
Yamaguchi 753-8512, Japan

Introduction

The class of quasi-Frobenius (QF) algebras, i.e., selfinjective artin algebras, is one of the most interesting classes of non-semisimple artin algebras. For example, any artin algebra A is a factor algebra of the trivial extension of A by its dual, which is a QF algebra. Our aim is to consider the possibility of QF test of factor algebras of a polynomial ring over a field by using computers and to determine the isomorphism classes of commutative QF algebras Λ over a field k satisfying the condition

$$\Lambda/J \cong k^{(m)} \quad \text{and} \quad J^3 = 0$$

where J is the radical of Λ and $k^{(m)}$ is a product of m copies of k for a positive integer m.

In §1 we show the existence of an algorithm for deciding whether a given factor algebra of a polynomial ring is QF, and demonstrate some examples by our implementation. In §2 we consider some type of commutative QF algebras satisfying the condition above, which will be said to be of type $(1,2,1)$, and determine their isomorphism classes.

Throughout this paper, k is a field with $k^* = k\backslash\{0\}$ and $\operatorname{ch} k$ the characteristic, all k-algebras mean finite dimensional *commutative* algebras over k and isomorphisms between k-algebras mean k-algebra isomorphisms. For a k-algebra Λ, we denote by $Rad(\Lambda)$ and $Soc(\Lambda)$ the radical and the socle of Λ, respectively.

*More general results of the last section in this paper will appear elsewhere.

1 QF test of polynomial algebras by computer

The following result is most essential for investigating local QF algebras.

Lemma 1.1 ([1]) *Let Λ be a local k-algebra. Then the following conditions are equivalent:*

(1) Λ *is QF;*

(2) $\dim_k \Lambda/Rad(\Lambda) = \dim_k Soc(\Lambda)$.

Using the lemma above and some results related to Gröbner basis theory, we have the following.

Theorem 1.2 *There exists an algorithm for deciding whether, for a given ideal I of the polynomial ring $R = k[x_1, \ldots, x_n]$ over a field k, R/I is QF.*

In fact, under some conditions on the base field k, the algorithm can be implemented in several computer algebra systems. The following are some examples computed by our program worked on SINGULAR [4], a computer algebra system.

Example 1.3 Let $R = \mathbb{Q}[x, y, z]$. By our program, we can easily check that the algebra $R/\langle z^3, yz - z^2, y^2, xz - z^2, xy, x^2 - z^2\rangle$ is a local QF algebra, while the algebra $R/\langle z^3, yz - z^2, y^2, xz - z^2, xy, x^2\rangle$ is local but not QF.

Example 1.4 (Tachikawa [6]) Let R be the polynomial ring $k[x_0, x_1, \ldots, x_n]$ in $n + 1$ variables $x_0, x_1, \ldots, x_n$ over a field k. For a positive integer t, set

$$I_t = \langle x_{[i]}x_{[i+1]} \cdots x_{[i+n-1]} - x^t_{[i+n]} \mid i = 0, 1, \ldots, n\rangle$$

where $[h]$ is the residue of h modulo $n + 1$. Then, Tachikawa proved that if $t \neq n$, then the k-algebra R/I_t is isomorphic to a direct sum of a group algebra and a local QF algebra (and hence, it is QF). Now, we shall particularly consider the algebra in case that $n = t - 1$ and denote it by T_t. The following is an execution result for the $\mathbb{Q}$-algebra $T_3 = \mathbb{Q}[x_0, x_1, x_2]/\langle x_0x_1 - x_2^3, x_1x_2 - x_0^3, x_2x_0 - x_1^3\rangle$ by our program, where the first three lines are our input and the rest are output.

```
> ring r=0, (x,y,z), lp;
> ideal I = (xy-z3, yz-x3, zx-y3);
> propideal(I);
<z9-z5,yz6-yz2,y2z5-y2z,y4-z4,xz-y3,xy-z3,x3-yz>
k-dimension = 27
number of components = 11
```

```
----- 1 ----- -----
Local QF ?              True
Groebner Basis of I = {z+1,y+1,x-1}
--     of (I:rad(I)) = {1}
--     of rad(I)     = {z+1,y+1,x-1}
k-basis of R/I       = {1}, 1
--  of R/(I:rad(I)) = {0}, 0
k-dim. of Loewy factors:
 upper(>): 1 (1)
----- 2 ----- -----
Local QF ?              True
Groebner Basis of I = {z-1,y-1,x-1}
--     of (I:rad(I)) = {1}
--     of rad(I)     = {z-1,y-1,x-1}
k-basis of R/I       = {1}, 1
--  of R/(I:rad(I)) = {0}, 0
k-dim. of Loewy factors:
 upper(>): 1 (1)
----- 3 ----- -----
Local QF ?              True
Groebner Basis of I = {z2+1,y+1,x-z}
--     of (I:rad(I)) = {1}
--     of rad(I)     = {z2+1,y+1,x-z}
k-basis of R/I       = {z,1}, 2
--  of R/(I:rad(I)) = {0}, 0
k-dim. of Loewy factors:
 upper(>): 2 (1)
----- 4 ----- -----
Local QF ?              True
Groebner Basis of I = {z5,yz2,y2z,y4-z4,xz-y3,xy-z3,x3-yz}
--     of (I:rad(I)) = {z4,yz2,y2z,y4,xz-y3,xy-z3,x3-yz}
--     of rad(I)     = {z,y,x}
k-basis of R/I       = {z4,z3,z2,yz,z,y3,y2,y,x2,x,1}, 11
--  of R/(I:rad(I)) = {z3,z2,yz,z,y3,y2,y,x2,x,1}, 10
k-dim. of Loewy factors:
 upper(>): 1 3 3 3 1 (5)
----- 5 ----- -----
Local QF ?              True
Groebner Basis of I = {z+1,y-1,x+1}
--     of (I:rad(I)) = {1}
--     of rad(I)     = {z+1,y-1,x+1}
k-basis of R/I       = {1}, 1
--  of R/(I:rad(I)) = {0}, 0
k-dim. of Loewy factors:
 upper(>): 1 (1)
----- 6 ----- -----
Local QF ?              True
Groebner Basis of I = {z+1,y2+1,x-y}
--     of (I:rad(I)) = {1}
--     of rad(I)     = {z+1,y2+1,x-y}
k-basis of R/I       = {y,1}, 2
--  of R/(I:rad(I)) = {0}, 0
k-dim. of Loewy factors:
 upper(>): 2 (1)
----- 7 ----- -----
Local QF ?              True
Groebner Basis of I = {z-1,y+1,x+1}
--     of (I:rad(I)) = {1}
--     of rad(I)     = {z-1,y+1,x+1}
k-basis of R/I       = {1}, 1
--  of R/(I:rad(I)) = {0}, 0
k-dim. of Loewy factors:
 upper(>): 1 (1)
----- 8 ----- -----
Local QF ?              True
Groebner Basis of I = {z-1,y2+1,x+y}
--     of (I:rad(I)) = {1}
--     of rad(I)     = {z-1,y2+1,x+y}
k-basis of R/I       = {y,1}, 2
--  of R/(I:rad(I)) = {0}, 0
k-dim. of Loewy factors:
 upper(>): 2 (1)
----- 9 ----- -----
Local QF ?              True
Groebner Basis of I = {z2+1,y-1,x+z}
--     of (I:rad(I)) = {1}
--     of rad(I)     = {z2+1,y-1,x+z}
k-basis of R/I       = {z,1}, 2
----- 10 ----- -----
Local QF ?              True
Groebner Basis of I = {z2+1,y-z,x+1}
--     of (I:rad(I)) = {1}
--     of rad(I)     = {z2+1,y-z,x+1}
k-basis of R/I       = {z,1}, 2
```

```
--  of R/(I:rad(I)) = {0}, 0
k-dim. of Loewy factors:
 upper(>): 2 (1)
----- 11 ----- -----
Local QF ?            True
Groebner Basis of I = {z2+1,y+z,x-1}
--    of (I:rad(I)) = {1}
--    of rad(I)     = {z2+1,y+z,x-1}
k-basis of R/I      = {z,1}, 2
--  of R/(I:rad(I)) = {0}, 0
k-dim. of Loewy factors:
 upper(>): 2 (1)
----- - ----- -----
QF : True
```

```
--  of R/(I:rad(I)) = {0}, 0
k-dim. of Loewy factors:
 upper(>): 2 (1)
```

This output, which is arranged for want of space, shows not only T_3 to be a (non-local) QF algebra but also further information about this algebra. For example, $\dim_{\mathbb{Q}} T_3 = 27$, and the number of local components of T_3 is 11; four components have dimension 1, six components have dimension 2 and one 'exceptional' component has dimension 11 and Loewy length 5.

We also run our program in case of the algebras T_4 and T_5 over $k = \mathbb{Q}$. It is then remarkable that T_4 and T_5 as well as T_3 have one exceptional local component, say C_4 and C_5, of dimension 131 and 1829, respectively. Omitting to show the output, we here give only a table showing the dimension and the number of components of the algebras and the series of dimensions of the factors in Loewy series of the components C_4 and C_5.

T_4		T_5	
dimension	the number of components	dimension	the number of components
1	1	1	16
4	31	2	640
131	1	1829	1
the number of components = 33		the number of components = 657	
$\dim T_4 = 256$		$\dim T_5 = 3125$	

the series of dimensions of the factors in Loewy series of C_4 and C_5:
C_4: 1 4 10 16 22 25 22 16 10 4 1
C_5: 1 5 15 35 65 105 151 195 225 235 225 195 151 105 65 35 15 5 1

Table 1: the dimension and the number of components of T_4 and T_5

2 Classification of QF algebras of type $(1, 2, 1)$

For the sake of completeness, we give a brief proof of the following known result.

Lemma 2.1 *For a commutative ring Λ, the following conditions are equivalent:*

(1) *Λ is a k-algebra with $\dim_k \Lambda/Rad(\Lambda) = 1$;*

(2) *$\Lambda \cong R/I$ where I is an ideal of the polynomial ring $R = k[x_1, \ldots . x_n]$ such that $\langle x_1, \ldots, x_n \rangle^m \subset I \subset \langle x_1, \ldots, x_n \rangle$ for some integers $m, n \geq 1$.*

In particular, if Λ is not a field, then the ideal I of (2) can be chosen to satisfy $I \subset \langle x_1, \ldots, x_n \rangle^2$.

Proof. (1) $\Rightarrow$ (2). Assume that Λ is a k-algebra with $J = Rad(\Lambda)$ such that $\dim_k \Lambda/J = 1$ and $J^m = 0$ for some integer $m \geq 1$. Since $\dim_k \Lambda/J = 1$, it follows that $\Lambda = k + J$. Let $\{u_i + J \mid i = 1, \ldots, n\}$ be a k-basis of J/J^2. Then it is shown that

$$\Lambda = k + \sum_{1 \leq i_1 \leq n} k u_{i_1} + \sum_{1 \leq i_1 \leq i_2 \leq n} k u_{i_1} u_{i_2} + \cdots + \sum_{1 \leq i_1 \leq \cdots \leq i_{m-1} \leq n} k u_{i_1} \cdots u_{i_{m-1}},$$

which implies the following epimorphism:

$$\varphi : R = k[x_1, \ldots, x_n] \to \Lambda, \ f(x_1, \ldots, x_n) \mapsto f(u_1, \ldots, u_n).$$

Now, setting $I = \operatorname{Ker} \varphi$, we will obtain the result.

(2) $\Rightarrow$ (1). This is easily shown. ∎

Let Λ be a k-algebra with $J = Rad(\Lambda)$ and n a positive integer. Then we say that Λ is *of type* $(1, n, 1)$ if

$$\dim_k \Lambda/J = 1, \quad \dim_k J/J^2 = n, \quad \dim_k J^2 = 1 \quad \text{and} \quad \dim_k J^3 = 0.$$

To determine the isomorphism classes of QF k-algebras Λ with $J = Rad(\Lambda)$ satisfying the condition

$$\Lambda/J \cong k^{(m)} \quad \text{and} \quad J^3 = 0$$

for some $m \geq 1$, we need to consider the k-algebras of type $(1, n, 1)$, for the local components of QF k-algebras satisfying this condition are the field k, the serial k-algebra of length 2, or the k-algebras of type $(1, n, 1)$.

The following lemma gives the "canonical forms" of k-algebras of type $(1, n, 1)$.

Lemma 2.2 *Let Λ be a k-algebra. Then the following conditions are equivalent:*

(1) *Λ is of type $(1, n, 1)$;*

(2) *There exist*
  (i) *p, q with $1 \le p \le q \le n$, and*
  (ii) *$a_{ij} \in k$ for $1 \le i \le j \le n$ with $(i, j) \ne (p, q)$*
  such that

$$\Lambda \cong k[x_1, \dots, x_n]/\langle x_i x_j - a_{ij} x_p x_q, x_p^2 x_q \mid 1 \le i \le j \le n \text{ with } (i,j) \ne (p,q)\rangle.$$

Proof. Set $R = k[x_1, \dots, x_n]$ and $J = \langle x_1, \dots, x_n\rangle$. For p, q and a_{ij}'s as in (i) and (ii) of (2), set

$$I = \langle x_i x_j - a_{ij} x_p x_q, x_p^2 x_q \mid 1 \le i \le j \le n \text{ with } (i,j) \ne (p,q)\rangle \subset R. \quad (\#)$$

(2) $\Rightarrow$ (1). Assume that $\Lambda = R/I$. For $f \in R$, set $\overline{f} = f + I \in \Lambda$. Since it is easy to see that $J^3 \subset I$, it follows from Lemma 2.1 that Λ is a k-algebra with $Rad(\Lambda) = J/I$. Furthermore, it can be easily seen that

$$\{\overline{x_1} + Rad(\Lambda)^2, \dots, \overline{x_n} + Rad(\Lambda)^2\} \quad \text{and} \quad \{\overline{x_p x_q}\}$$

are k-bases of $Rad(\Lambda)/Rad(\Lambda)^2$ and $Rad(\Lambda)^2$, respectively. Therefore, Λ is a k-algebra of type $(1, n, 1)$.

(1) $\Rightarrow$ (2). Let Λ be a k-algebra of type $(1, n, 1)$. By Lemma 2.1, we may assume that $\Lambda = R/H$ where H is an ideal of R such that $J^3 \subset H \subsetneq J^2$ and $\dim_k J^2/H = 1$. Then we can choose an $x_p x_q \notin H$ where $1 \le p \le q \le n$. Since $\dim_k J^2/H = 1$, it follows that for $1 \le i \le j \le n$ with $(i, j) \ne (p, q)$, there exist $a_{ij} \in k$ such that $x_i x_j - a_{ij} x_p x_q \in H$. For the (p, q) and the a_{ij}'s, define an ideal I of R as in (#). Since $I \subset H$ and $\dim_k R/I = \dim_k R/H = n + 2$, we have $H = I$, which shows (2). ■

It is easy to see that in case $n = 1$, any k-algebra of type $(1, 1, 1)$ is isomorphic to $k[x]/\langle x^3\rangle$, a serial k-algebra of length 3. In the rest of this paper, we shall consider a non-trivial case $n = 2$, i.e., k-algebras of type $(1, 2, 1)$. To this, we define k-algebras Γ and Σ_{ab} $(a, b \in k)$ by

$$\begin{aligned} \Gamma &= k[x, y]/\langle x^2, y^2\rangle \quad \text{and} \\ \Sigma_{ab} &= k[x, y]/\langle xy - ay^2, x^2 - by^2, y^3\rangle. \end{aligned}$$

Then we have the following.

Lemma 2.3 (1) *Any k-algebra Λ of type $(1,2,1)$ is isomorphic to Γ or Σ_{ab} for some $a, b \in k$.*

(2) *Any QF k-algebra of type $(1,2,1)$ is isomorphic to Γ or Σ_{ab} for some $a, b \in k$ with $a^2 - b \neq 0$.*

Proof. (1) It suffices to consider the factor algebras of $k[x,y]$ (where $x = x_1$ and $y = x_2$) in (2) of Lemma 2.2 for $(p,q) \in \{(1,2),(2,2)\}$. Then it can be checked that $\Lambda \cong \Gamma$ if $(p,q) = (1,2)$ and $a_{11} = a_{22} = 0$; $\Lambda \cong \Sigma_{ab}$ for some $a, b \in k$ otherwise.

(2) Since it is easily seen that Γ is QF, it suffices to show that Σ_{ab} is not QF if and only if $a^2 - b = 0$. Set $J = Rad(\Sigma_{ab})$ and $S = Soc(\Sigma_{ab})$. Then we see that $\{\overline{x} + J^2, \overline{y} + J^2\}$ and $\{\overline{y}^2\}$ are k-bases of J/J^2 and J^2, respectively, and that

$$J^2 = k\overline{y}^2 \subset S = \{f \in \Sigma_{ab} \mid f\overline{x} = f\overline{y} = 0\} \subset J.$$

It follows from Lemma 1.1 that Σ_{ab} is not QF if and only if $J^2 \subsetneq S$, which means that there exists $0 \neq (s,t) \in k^{(2)}$ such that $s\overline{x} + t\overline{y} \in S$. By the note above on S, we see that this is also equivalent to the condition $a^2 - b = 0$. ∎

Lemma 2.4 *Let $a, b, c, d \in k$ with $a^2 - b \neq 0$ and $c^2 - d \neq 0$. Then the following conditions are equivalent:*

(1) $\Sigma_{ab} \cong \Sigma_{cd}$;

(2) $\sqrt{(c^2 - d)(a^2 - b)^{-1}} \in k$.

Proof. (1) $\Rightarrow$ (2). Let φ be a k-isomorphism from Σ_{ab} to Σ_{cd} and

$$\begin{aligned} \varphi(\overline{x}) &= s\overline{x} + t\overline{y} + u\overline{y}^2 \quad (s,t,u \in k) \\ \varphi(\overline{y}) &= p\overline{x} + q\overline{y} + r\overline{y}^2 \quad (p,q,r \in k). \end{aligned}$$

Then,

$$0 \neq \varphi(\overline{y}^2) = \varphi(\overline{y})^2 = p^2\overline{x}^2 + 2pq\overline{x}\overline{y} + q^2\overline{y}^2 = (dp^2 + 2cpq + q^2)\overline{y}^2.$$

Set $w = dp^2 + 2cpq + q^2$. Since φ is an isomorphism, the determinant of the matrix of φ relative to k-bases $\{\overline{1}, \overline{x}, \overline{y}, \overline{y}^2\}$ of both Σ_{ab} and Σ_{cd} is

$$\det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & s & t & u \\ 0 & p & q & r \\ 0 & 0 & 0 & w \end{pmatrix} = w(qs - pt) \neq 0.$$

$$\begin{aligned} 0 &= \varphi(\overline{x}\overline{y} - a\overline{y}^2) = \varphi(\overline{x})\varphi(\overline{y}) - a\varphi(\overline{y})^2 \\ &= \{dsp + c(sq + tp) + tq - aw\}\overline{y}^2. \end{aligned}$$

Similarly,

$$0 = \varphi(\overline{x}^2 - b\overline{y}^2) = (ds^2 + 2cst + t^2 - bw)\overline{y}^2.$$

It follows that

$$\begin{aligned} a &= \{dsp + c(sq + tp) + tq\}w^{-1} \\ b &= (ds^2 + 2cst + t^2)w^{-1}. \end{aligned}$$

Now we substitute the a and b above for $(c^2 - d)(a^2 - b)^{-1}$ to obtain

$$(c^2 - d)(a^2 - b)^{-1} = w^2(qs - pt)^{-2},$$

which implies (2).

(2) $\Rightarrow$ (1). Assume that $u := \sqrt{(c^2 - d)(a^2 - b)^{-1}} \in k$. Then the following correspondence gives a k-isomorphism from Σ_{ab} to Σ_{cd} :

$$\overline{1} \mapsto \overline{1}, \quad \overline{x} \mapsto \overline{x} + (au - c)\overline{y}, \quad \overline{y} \mapsto u\overline{y}, \quad \overline{y}^2 \mapsto u^2\overline{y}^2,$$

which completes the proof. ∎

Now, consider the following algebra:

$$\Lambda_a := k[x, y]/\langle x^2 - ay^2, xy\rangle \quad (a \in k).$$

Then we can show the following.

Theorem 2.5 *Any QF k-algebra of type $(1, 2, 1)$ is isomorphic to*

$$\Gamma \quad or \quad \Lambda_a$$

for some $a \in k^*$. Moreover,

(1) *For $a, b \in k^*$, $\sqrt{ab^{-1}} \in k \;\Leftrightarrow\; \Lambda_a \cong \Lambda_b$.*

(2) (i) $\operatorname{ch} k \neq 2 \;\Leftrightarrow\;$ (ii) $\Gamma \cong \Lambda_{-1} \;\Leftrightarrow\;$ (iii) $\Gamma \cong \Lambda_c$ *for some $c \in k^*$.*

Proof. Note from Lemma 2.4 that for $a, b \in k$ with $a^2 - b \neq 0$, $\Sigma_{ab} \cong \Sigma_{0,b-a^2} = \Lambda_{b-a^2}$. Then the first assertion follows from Lemma 2.3.

(1) This follows immediately from $\Lambda_a = \Sigma_{0a}$ and Lemma 2.4.

(2) (i) $\Rightarrow$ (ii). If $\mathrm{ch}\, k \neq 2$, then the following correspondence gives a k-isomorphism from Γ to Λ_{-1}:

$$\overline{1} \mapsto \overline{1}, \quad \overline{x} \mapsto \overline{x} + \overline{y}, \quad \overline{y} \mapsto \overline{x} - \overline{y}, \quad \overline{xy} \mapsto -2\overline{y}^2.$$

(ii) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (i). Let $\varphi : \Gamma \to \Lambda_c$ ($c \in k^*$) be a k-isomorphism and

$$\begin{aligned} \varphi(\overline{x}) &= s\overline{x} + t\overline{y} + u\overline{y}^2 \quad (s, t, u \in k) \\ \varphi(\overline{y}) &= p\overline{x} + q\overline{y} + r\overline{y}^2 \quad (p, q, r \in k). \end{aligned}$$

Then,

$$0 = \varphi(\overline{x}^2) = \varphi(\overline{x})^2 = s^2\overline{x}^2 + t^2\overline{y}^2 = (cs^2 + t^2)\overline{y}^2,$$

from which $cs^2 + t^2 = 0$, i.e., $t^2 = -cs^2$. Similarly, $q^2 = -cp^2$. On the other hand,

$$0 \neq \varphi(\overline{xy}) = \varphi(\overline{x})\varphi(\overline{y}) = (csp + tq)\overline{y}^2.$$

Hence, $csp + tq \neq 0$. Therefore we have

$$0 \neq (csp + tq)^2 = c^2s^2p^2 + 2cstpq + t^2q^2 = 2(c^2s^2p^2 + cstpq),$$

which implies that $\mathrm{ch}\, k \neq 2$. ∎

Let $\{a_i \mid (a_i \in I)\}$ be a set of representative elements of the multiplicative group k^* modulo the subgroup $(k^*)^2$ of square elements, i.e., $k^*/(k^*)^2 = \{a_i(k^*)^2 \mid i \in I\}$. Then, Theorem 2.5 gives the isomorphism classes of QF k-algebras of type $(1, 2, 1)$ as follows.

Corollary 2.6 (1) *If* $\mathrm{ch}\, k \neq 2$, *then the isomorphism classes of QF k-algebras of type* $(1, 2, 1)$ *are* Λ_{a_i} $(i \in I)$. *In particular,* $\Gamma \cong \Lambda_{-1}$.

(2) *If* $\mathrm{ch}\, k = 2$, *then the isomorphism classes of QF k-algebras of type* $(1, 2, 1)$ *are* Γ *and* Λ_{a_i} $(i \in I)$. *In particular,* $\Gamma \not\cong \Lambda_a$ *for all* $a \in k^*$.

Remark 2.7 (1) For a given field k, the number N of the isomorphism classes of QF k-algebras of type $(1, 2, 1)$ is given as follows.

$$N = \begin{cases} 2^e \ (e \geq 0) \quad \text{or} \quad \infty \quad (\mathrm{ch}\, k \neq 2) \\ 2 \qquad\qquad\quad \text{or} \quad \infty \quad (\mathrm{ch}\, k = 2). \end{cases}$$

(2) For any $e \geq 0$, there exists a field k such that $N = 2^e$ (cf. [5, Chapter 4, §7]).

The following table shows the isomorphism classes of QF algebras of type $(1, 2, 1)$ over some typical fields k.

k		k-isomorphism classes
algebraically closed fields	$\mathrm{ch}\, k = 2$	$\Gamma,\ \Lambda_1$
	$\mathrm{ch}\, k \neq 2$	Γ
finite fields	$\mathrm{ch}\, k = 2$	$\Gamma,\ \Lambda_1$
	$\mathrm{ch}\, k \neq 2$	$\Lambda_1,\ \Lambda_a$ (a is a primitive element of k)
$\mathbb{R}$		$\Gamma,\ \Lambda_1$
$\mathbb{Q}$		Λ_{a_i} ($\{a_i \mid a_i \in \mathbb{N}\}$ is the set of square free integers)
$k_0(x_1, \ldots, x_m)$ (k_0 is an arbitrary field)		Λ_{a_i} $(i \in I,\ \lvert I \rvert = \infty)$
p-adic fields $\mathbb{Q}_p$	$p = 2$	Λ_{a_i} $(\mathbb{Q}_2/(\mathbb{Q}_2)^2 = \{a_i(\mathbb{Q}_2)^2 \mid i = 1, \ldots, 8\})$
	$p \neq 2$	Λ_{a_i} $(\mathbb{Q}_p/(\mathbb{Q}_p)^2 = \{a_i(\mathbb{Q}_p)^2 \mid i = 1, \ldots, 4\})$

Table 2: the isomorphism classes of QF k-algebras of type $(1, 2, 1)$

References

[1] H. Bass: On the ubiquity of Gorenstein rings, Math. Z. **82**(1963), 8–28.

[2] T. Becker and V. Weispfenning: Gröbner Bases, GTM **141**, Springer-Verlag, New York, 1993.

[3] D. Cox, J. Little and D. O'Shea: Ideals, Varieties, and Algorithms, 2nd ed., UTM, Springer, New York, 1997.

[4] G.-M. Greuel, G. Pfister, and H. Schönemann: Singular version 1.2 User Manual . In *Reports On Computer Algebra*, number 21. Centre for Computer Algebra, University of Kaiserslautern, June 1998. `http://www.mathematik.uni-kl.de/~zca/Singular`

[5] G. Karpilovsky: Field Theory, Monographs and Textbooks in Pure and Applied Math. **120**, Marcel Dekker, New York-Basel, 1988.

[6] H. Tachikawa: QF algebras and Gröbner bases of pure binomials, Preprint 00–014, SFB **343**, Bielefeld, 2000.