数理解析研究所講究録 1222 巻 2001 年 34-39

On structures of weak interlaced bilattices

近藤通朗
島根大学総合理工学部
Michiro KONDO
Department of Mathematics and Computer Science, Shimane University

Abstract

We sutdy fundamental properties of weak interlaced bilattices and show that

1. For any bounded lattice L, there exists an interlaced bilattice $\mathcal{B}$ such that $\mathcal{K}(L) \cong Cons(\mathcal{B})$;
2. For any interlaced bilattice $\mathcal{B}$ with negation, there exists a lattice L such that $\mathcal{K}(L) \cong Cons(\mathcal{B})$.

1 Introduction

It is well-known the Kleene's 3-valued logic in the field of multiple-valued logics. The logic has three values *false, true*, and $\perp$ (*unknown*) as truth values. These values have two informal orderings concerning "amount of knowledge" and "degree of truth". For example, if we think of a certain proposition such as *Riemann's conjecture* assigned $\perp$ as truth value, then it is possible that we can conclude the truth value of the proposition as *true* or *false* with increasing knowledge. Thus in the ordering of knowledge, $\perp$ is smaller than *true* and *false*. A sentence with $\perp$ is between *false* and *true* in the ordering of degree of truth. In this way it can be considered that the three valued logic has two orderings. Belnap ([2]), Ginsberg([5]), and others proposed concept of a *bilattice* which has two orderings and proved some fundamental results ([1, 3, 4]). It is shown by Fitting ([3]) that bilattices can give a uniform semantics for many lanuages of logic programming. Since then the theory of bilattices is a hot reserach field.

On the other hand, as in *Fuzzy logics*, a truth value can be taken as a closed interval $[a, b]$. Let L be a lattice and $\mathcal{K}(L)$ be the set of all closed intervals of L. In this case we also define two orderings. For $[a, b], [c, d] \in \mathcal{K}(L)$, if $[a, b] \subseteq [c, d]$ then the knowledge in $[a, b]$ is greater than that in $[c, d]$. Thus we set $[a, b] \sqsubseteq_k [c, d]$ if $[a, b] \subseteq [c, d]$. Likewise we also define $[a, b] \sqsubseteq_t [c, d]$ if $a \leq c$ and $b \leq d$, because $[c, d]$ is greater than $[a, b]$ in the ordering degree of truth. The structure $\mathcal{K}(L) =< \mathcal{K}(L), \sqsubseteq_t, \sqsubseteq_k>$ which precise definition is given below has the property of *weak interlaced bilattice.*

In [3, 4], Fitting, Font and Moussavi have investigated the strucutre of $\mathcal{K}(L)$ and proved some results :

1. If L is a bounded lattice, then $\mathcal{K}(L)$ is a weak interlaced bilattice ([4]) ;
2. If L is a complete lattice with an *involution*, then $\mathcal{K}(L) \cong Cons(\mathcal{B})$, where $Cons(\mathcal{B})$ is the set of all *consistent* elements of an *interlaced bilattice* $\mathcal{B}$ with *negation* and *conflation* ([3]) ;

3. If $\mathcal{B}$ is a distributive bilattice with commutative negation and conflation, then $Cons(\mathcal{B}) \cong \mathcal{K}(L)$ for some complete distributive lattice L ([3]).

Now it is natural to ask the following questions :

Q1 Is there a lattice L such that $\mathcal{W} \cong \mathcal{K}(L)$ for every weak interlaced bilattice $\mathcal{W}$?

Q2 Is there an interlaced bilattice $\mathcal{B}$ such that $\mathcal{K}(L) \cong Cons(\mathcal{B})$ for every bounded lattice L ?

Q3 Is there a lattice L such that $\mathcal{K}(L) \cong Cons(\mathcal{B})$ for every interlaced bilattice with negation $\mathcal{B}$?

In the following, we study properties of $\mathcal{K}(L)$ and answer the questions above.

2 Definition of $\mathcal{K}(L)$

We define a structure $\mathcal{K}(L)$ for any lattice L. Let $L = (L, \leq)$ be a lattice and $K(L)$ be the set of all closed intervals of L, that is,

$$K(L) = \{[a,b] | a \leq b, a, b \in L\}$$
$$[a,b] = \{x | a \leq x \leq b\}.$$

For any $[a,b],[c,d] \in K(L)$, we define two orderings $\sqsubseteq_t, \sqsubseteq_k$ on $K(L)$ as follows :

$$[a,b] \sqsubseteq_t [c,d] \Longleftrightarrow a \leq c, b \leq d$$
$$[a,b] \sqsubseteq_k [c,d] \Longleftrightarrow a \leq c, b \geq d$$

We set $\mathcal{K}(L) =< K(L), \sqsubseteq_t, \sqsubseteq_k>$. It is obvious from definition that $[0,0]$ ($[1,1]$) is the minimum (maximum) element with respect to $\sqsubseteq_t$. On the other hand, while $[0,1]$ is the minimum element, there is no maximum element with respect to the ordering $\sqsubseteq_k$. This means that $\mathcal{K}(L)$ is a lattice with respect to $\sqsubseteq_t$ and is a semi-lattice concering $\sqsubseteq_k$. Four operators $\sqcap_t, \sqcup_t, \sqcap_k, \sqcup_k$ are defined by

$$inf_{\sqsubseteq_t}\{a,b\} = a \sqcap_t b$$
$$sup_{\sqsubseteq_t}\{a,b\} = a \sqcup_t b$$
$$inf_{\sqsubseteq_k}\{a,b\} = a \sqcap_k b$$
$$sup_{\sqsubseteq_k}\{a,b\} = a \sqcap_k b \quad (if\ it\ is\ defined)$$

A relational system $< B, \leq_t, \leq_k>$ is called an *interlaced bilattice* if it satisfies

1. B is a non-empty set
2. $< B, \leq_t>$, $< B, \leq_k>$ are bounded lattices and satisfy
   (a) $x \leq_t y \Longrightarrow x \otimes z \leq_t y \otimes z,\ x \oplus z \leq_t y \oplus z$
   (b) $x \leq_k y \Longrightarrow x \wedge z \leq_k y \wedge z,\ x \vee z \leq_k y \vee z$

By $0(1)$, we mean the minimum (maximum) element with respect to the ordering $\leq_t$. We also denote by $\bot(\top)$ the minimum (maximum) element concering to $\leq_k$.

Any interlaced bilattice is called *distributive* when it satisfies

$$x \circ (y \bullet z) = (x \circ z) \bullet (y \circ z)$$

for $\circ, \bullet \in \{\wedge, \vee, \otimes, \oplus\}$. This means twelve equations such as

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$
$$x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$$
$$\vdots$$

A map $\neg$ from B into itself is called a *negation* if

$$x \leq_t y \Longrightarrow \neg y \leq_t \neg x$$
$$x \leq_k y \Longrightarrow \neg x \leq_k \neg y$$
$$\neg\neg x = x.$$

For lattices $L_1 =< L_1, \wedge_1, \vee_1 >$ and $L_2 =< L_2, \wedge_2, \vee_2 >$, we define operations $\wedge, \vee, \otimes, \oplus$ on the product $L_1 \times L_2$: For $(a, b), (c, d) \in L_1 \times L_2$,

$$(a, b) \wedge (c, d) = (a \wedge_1 c, b \vee_2 d)$$
$$(a, b) \vee (c, d) = (a \vee_1 c, b \wedge_2 d)$$
$$(a, b) \otimes (c, d) = (a \wedge_1 c, b \wedge_2 d)$$
$$(a, b) \oplus (c, d) = (a \vee_1 c, b \vee_2 d).$$

The structure $L_1 \odot L_2 =< L_1 \times L_2, \wedge, \vee, \otimes, \oplus >$ is called a *Ginsberg product.* There are some fundamental results about the structure :

Proposition 1 (Fitting). *If L_1, L_2 are bounded lattices then the Ginsberg product $L_1 \odot L_2 =< L_1 \times L_2, \wedge, \vee, \otimes, \oplus >$ is an interlaced bilattice. Espectially, $L \odot L$ is an interlaced bilattice with negation $\neg$, where $\neg$ is defined by $\neg(a, b) = (b, a)$.*

It is proved that the converse holds by Avron ([1]).

Proposition 2 (Avron). *For any interlaced bilattice $\mathcal{B}$, there are bounded lattices L_1, L_2 such that $\mathcal{B} \cong L_1 \odot L_2$. In particular, for any interlaced bilattice $\mathcal{B}$ with negation, there is a bounded lattice L such that $\mathcal{B} \cong L \odot L$.*

It is clear from definition that orderings $\sqsubseteq_t, \sqsubseteq_k$ on $\mathcal{K}(L)$ are the same as $\leq_t, \leq_k$ on Ginsberg product $L \odot L$, respectively :

$$\sqsubseteq_t \text{ in } \mathcal{K}(L) \Longleftrightarrow \leq_t \text{ in } L \odot L$$
$$\sqsubseteq_k \text{ in } \mathcal{K}(L) \Longleftrightarrow \leq_k \text{ in } L \odot L$$

Next we give a definition of a *weak interlaced bilattice* according to Font ([4]). A structure $\mathcal{W} =< W, \leq_t, \leq_k >$ is called a *weak interlaced bilattice* if

1. $< W, \leq_t >$: lattice
2. $< W, \leq_k >$: meet semilattice
3. $a \leq_k b, c \leq_k d \Longrightarrow a \wedge c \leq_k b \wedge d, a \vee c \leq_k b \vee d$
4. $a \leq_t b, c \leq_t d \Longrightarrow a \otimes c \leq_t b \otimes d,$
5. $a \leq_t b, c \leq_t d \Longrightarrow a \oplus c \leq_t b \oplus d$ if $a \oplus c$ and $b \oplus d$ exist.

3 Properties of weak interlaced bilattices

For any weak interlaced bilattice $\mathcal{W}$, if we define

$$L_1 = \{x \in \mathcal{W} \mid x \leq_k 0\} = [\bot, 0]_k$$
$$L_2 = \{x \in \mathcal{W} \mid x \leq_k 1\} = [\bot, 1]_k,$$

then we have

Proposition 3.

$$L_1 = [\bot, 0]_k = [0, \bot]_t$$
$$L_2 = [\bot, 1]_k = [\bot, 1]_t$$

Proof. Let $x \in [\bot, 0]_k$. Since $\bot \leq_k x \leq_k 0$, we have $\bot \vee \bot \leq_k x \vee \bot \leq_k 0 \vee \bot$ by definition of weak interlaced bilattice. From $\bot \vee \bot = 0 \vee \bot = \bot$, it follows that $x \vee \bot = \bot$ and hence that $x \leq_t \bot$. This means $[\bot, 0]_k \subseteq [0, \bot]_t$.

Conversely, suppose $x \in [0, \bot]_t$. If we put $u = 0 \otimes x$, then it is clear that $u \leq_k 0$ and $u \leq_k x$. Since $0 \leq_t x$, we have $0 \otimes x \leq_t x \otimes x = x$ and hence $u \leq_t x$. It follows from $\bot \leq_k u$ that $x \wedge \bot \leq_k x \wedge u$. Since $x \leq_t \bot$, we also have $x \wedge \bot = x$. On the other hand, since $u \leq_t x$, we get $u \wedge x = u$. Theses imply that $x \leq_k u$ and hence that $x = u$. Thus we have $x \leq_k 0$. Namely, we have $[0, \bot]_t \subseteq [\bot, 0]_k$.

The second equation can be proved similarly.

□

The result implies that L_1 and L_2 are lattices with ordering $\leq_1$ and $\leq_2$ in $\mathcal{B}$, respectively, where $\leq_1$ and $\leq_2$ are defined by

$$\leq_1 = \leq_t = \geq_k$$
$$\leq_2 = \leq_t = \leq_k$$

Thus we can consider the Ginsberg product $L_1 \odot L_2$, which becomes an *interlaced bilattice*. Moreover we can prove

Proposition 4. *Let $\mathcal{W}$ be any weak interlaced bilattice. For any $x \in \mathcal{W}$, we have* $x = (x \otimes 0) \oplus (x \otimes 1) = (x \wedge \bot) \vee (x \vee \bot)$

Proof. See Avron [1] Cor.3.8 □

Now we investigate a realtion between a weak interlaced bilattice $\mathcal{W}$ and an interlaced bilattice $L_1 \odot L_2$ constructed by $\mathcal{W}$.

Lemma 1. *A map $\xi : \mathcal{W} \to L_1 \times L_2$ defined by $\xi(x) = (x \otimes 1, x \otimes 0) = (x \vee \bot, x \wedge \bot)$ is an embedding.*

This means that

Theorem 1. *Any weak interlaced bilattice can be embedded into an interlaced bilattice.*

As to the question **Q1**, we can give a negative answer by presentiong a counter example. Let $\mathcal{W}$ be the set $\{0, 1, a, b, \bot, 1\}$ such that $0 \leq_t a \leq_t \bot \leq_t b \leq_t 1$ and $\bot \leq_k a \leq_k 0, \bot \leq_k b \leq_k 1$. It is obvious that $\mathcal{W}$ is a weak interlaced bilattice. Suppose that there is a lattice L such that $\mathcal{W} \cong \mathcal{K}(L)$. If $|L| \geq 3$, then there exists an element $a \in L$ such that $0 < a < 1$. For that element we have $[0,0], [0,a], [0,1], [a,1], [a,a], [1,1] \in \mathcal{K}(L)$ and $|\mathcal{K}(L)| \geq 6$. Since $|\mathcal{W}| = 5$, it must be $|L| \leq 2$. But, in this case, we have $|\mathcal{K}(L)| \leq 3$. This means that there is no lattice L such that $\mathcal{W} \cong \mathcal{K}(L)$.

4 Characterization of $\mathcal{K}(L)$

In this section we consider the properties of $\mathcal{K}(L)$ for a bounded lattice L. Let $\mathcal{I}(L) = \{(a,b) \mid a \leq b\} \subseteq L \times L$. It is clear that $\mathcal{I}(L)$ is closed under the operations $\wedge, \otimes$ and $\oplus$ but not closed under $\vee$. If we define a map $\eta : \mathcal{K}(L) \to \mathcal{I}(L)$ by $\eta([a,b]) = (a,b)$, then we can prove that

Lemma 2. $\eta : \mathcal{K}(L) \to \mathcal{I}(L)$ *: bijection and*

$$\begin{aligned} \eta([a,b] \sqcap_t [c,d]) &= \eta([a,b]) \otimes \eta([c,d]) \\ \eta([a,b] \sqcup_t [c,d]) &= \eta([a,b]) \oplus \eta([c,d]) \\ \eta([a,b] \sqcap_k [c,d]) &= \eta([a,b]) \wedge \eta([c,d]) \\ \textit{if } [a,b] \oplus [c,d] \textit{ exists} \quad \eta([a,b] \sqcup_k [c,d]) &= \eta([a,b]) \vee \eta([c,d]) \end{aligned}$$

We call the map η a *t-k dual isomorphism* and identify the isomorphism with the t-k dual isomorphism, that is,

$$\mathcal{K}(L) \cong \mathcal{I}(L) \subseteq L \odot L$$

In any interlaced bilattice $L \odot L$, the negation $\neg$ is defined by

$$\neg(a,b) = (b,a).$$

An element (a,b) in $L \odot L$ is called *consistent* when it satisfies $(a,b) \leq_t \neg(a,b)$, that is,

$$(a,b) : \text{consistent} \iff a \leq b$$

If we denote by $Cons(\mathcal{B})$ the set of all consistent elements of an interlaced bilattice $\mathcal{B}$, since $Cons(L \odot L) = \mathcal{I}(L)$, then we have

Theorem 2. *For any bounded lattice L,*
$\mathcal{K}(L) \cong \mathcal{I}(L) = Cons(L \odot L)$

This means that we can answer the question **Q2** as **Yes**. Moreover, for the structure $\mathcal{I}(L)$, we can show

Theorem 3. *$\mathcal{I}(L)$ is the weak interlaced bilattice generated by $\Delta = \{(a,a) \mid a \in L\}$.*

Proof. Let $\mathcal{W}$ be any weak interlaced bilattice such that $\Delta \subseteq \mathcal{W}$. For every element $(a,b) \in \mathcal{I}(L)$ $(a \leq b)$, since $(a,a),(b,b) \in \Delta \subseteq \mathcal{W}$, we have $(a,a) \wedge (b,b) = (a \wedge b, a \vee b) = (a,b) \in \mathcal{W}$. Thus $\mathcal{I}(L) \subseteq \mathcal{W}$. □

As to the question **Q3**, let $\mathcal{B}$ be any interlaced bilattice with negation. Since there is a lattice L such that $\mathcal{B} \cong L \odot L$, by identifying $\mathcal{B}$ with $L \odot L$, we have $Cons(\mathcal{B}) = Cons(L \odot L) = \mathcal{I}(L) \cong \mathcal{K}(L)$. This means that

Theorem 4. *For any interlaced bilattice $\mathcal{B}$ with negation, there is a lattice L such that*

$$Cons(\mathcal{B}) \cong \mathcal{K}(L)$$

Therefore we can answer the question **Q3** as **Yes**.

References

[1] A.Avron, *The structure of interlaced bilattices*, Math. Struct. in Comp. Science, vol.**6** (1996), 287-299.

[2] N.D.Belnap, Jr., *A useful four-valued logic*, J.M.Dunn and G.Epstein (eds) Modern Uses of Multiple-Valued Logic, D.Reidel (1977).

[3] M. Fitting, *Kleene's Logic, generalized*, Jour.Logic Comput., vol.**1** (1991), 797-810.

[4] J.M.Font and M.Moussavi, *Notes on a six-valued extension of three-valued logic*, Jour. of Applied Non-Classical Logic, vol.**3** (1993), 173-187.

[5] M.Ginsberg, *Multivalued logic : a uniform approach to reasoning in artificial intelligence*, Computational Intelligence, vol.**4** (1988), 265-316.

Michiro Kondo
e-mail:kondo@cis.shimane-u.ac.jp
Department of Mathematics and Computer Science
Shimane University, Matsue, 690-8504
JAPAN