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Exceptional surgeries and genera of knots

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1 Introduction

A 3-manifold is said to be *hyperbolic* if it is homeomorphic to the quotient of the 3-dimensional hyperbolic space via a torsion free kleinian group acting as isometries. We also say that a knot in a 3-manifold is *hyperbolic* if it has the hyperbolic complement.

A *Dehn surgery* is one of the well-known operations producing a new 3-manifold from a prescribed one. When a 3-manifold and a knot in it are given, one can yield a lot of new 3-manifolds by performing Dehn surgeries The well-known Hyperbolic Dehn Surgery Theorem due to Thurston [22] says that all but finitely many Dehn surgeries on a hyperbolic knot give hyperbolic 3-manifolds. Also see [19] for a detailed proof.

In view of this result, a Dehn surgery along a hyperbolic knot is called *exceptional* if it yields a non-hyperbolic manifold. A lot of study to know which surgeries are exceptional. A well arranged survey was given in [7]. In particular, it was shown that on the number of exceptional surgeries, there exists a *universal* upper bound [9, 15].

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One of the main subject of the study of exceptional surgeries is those on knots in the 3-sphere S^3. It is well-known that a Dehn surgery on a knot in S^3 is characterized by the *surgery slope*, and such slopes are parameterized by $\mathbb{Q} \cup \{\infty\}$. With respect to this coordinate, the range of exceptional surgery slopes is *unbounded.* Some specific examples were given in [4, Section 5].

In this article, we will give some bounds on the range of exceptional surgery slopes with respect to the coordinate above in terms of the genera of knots. In the sequel, let $K(r)$ be the closed 3-manifold obtained by a Dehn surgery on a hyperbolic knot K in S^3 along a slope $r \neq \infty$ and g denote the genus of K. Our first theorem which is based on [10] is the following.

Theorem 1. *If $|r| > 3 \cdot 2^{7/4}g$, then $K(r)$ is an irreducible 3-manifold with infinite and word-hyperbolic fundamental group.*

Remark that an approximate value of $3 \cdot 2^{7/4}$ is 10.09. It is known that the Thurston's Geometrization Conjecture would imply that irreducible 3-manifolds with infinite and word-hyperbolic fundamental group are actually hyperbolic. We will briefly review on this fact in the end of Section 3.

Next, we restrict knots to some special classes and give more sharper bounds. One class which we will consider is that of amphicheiral knots. Remark that an approximate value of $1 - 2^{-1/2}$ is 0.29.

Theorem 2. *If K is amphicheiral and $|r| > 3 \cdot 2^{7/4}\{g - (1 - 2^{-1/2})\}$, then $K(r)$ is an irreducible 3-manifold with infinite and word-hyperbolic fundamental group.*

The next result is based on a joint work with Makoto Ozawa [13]. In the study of exceptional surgeries, a fruitful method is to consider some surfaces in a knot complement or a surgered manifold. In this article, we consider closed essential (i.e., incompressible and not ∂-parallel) surfaces in a knot complement which admit an annulus connecting the surface and the knot. Note that when the knot is hyperbolic such a surface corresponds to a surface subgroup contains accidental parabolics in the knot group.

Theorem 3. *Suppose that the complement of K contains a closed essential surface admitting an annulus connecting the surface and K. Under this assumption, if $|r| \geq 4g+1$, then $K(r)$ is hyperbolic.*

By virtue of the Thurston's Uniformization Theorem [23] (see [18] for detail), the proof of Theorem 3 which we will give is purely topological. Also will be used the results on such surfaces by Ozawa and the author [11], [12].

Concerning the surgeries yielding lens spaces, the following conjecture was proposed by Goda and Teragaito in [6].

Conjecture 1.1. *If a Dehn surgery on a hyperbolic knot in S^3 along a slope $r \neq \infty$ yields a lens space, then the knot is fibered and $2g+8 \leq |r| \leq 4g-1$, where g denotes the genus of the knot.*

They gave an upper bound $12g-7$ and proved that no such surgeries can occur for genus one knots. Our theorems give some new bounds which are sharper than theirs in certain cases. The conjecture above and their results is one of the motivations of our work.

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2 Preliminaries

The notations used throughout the article are as follows. For a topological space X, $\mathrm{Int}(X)$, ∂X, $|X|$ and $\chi(X)$ denote the interior, the boundary, the number of connected components and the Euler characteristic of

X, respectively. For a subset Y of X, $\mathrm{Ext}(Y)$ denotes the exterior of Y in X, that is, the closure of $X - \mathrm{N}(Y)$, where $\mathrm{N}(Y)$ denotes the regular neighborhood of Y in X.

A 3-manifold M is called *irreducible* if every 2-sphere embedded in M bounds a 3-ball.

By a *Dehn filling*, we mean the operation of attaching solid tori to a 3-manifold with toral boundaries. A *Dehn surgery* on a link in a 3-manifold means the following operation. Remove the open regular neighborhood of the link and then perform a Dehn filling. It is well-known that every closed, orientable 3-manifold is obtained by a Dehn surgery on a link in the 3-sphere S^3 [16].

We call the isotopy class of a non-trivial simple closed curve on a torus a *slope*. When a generator system for the first homology of a torus is fixed, slopes on the torus are parameterized by $\mathbb{Q} \cup \{\infty\}$. In the sequel, we always fix the standard meridian-longitude system for the first homology of the peripheral torus of a knot in S^3 [21].

A Dehn surgery on a knot K is determined by its *surgery slope*. That is, the slope of the meridian of the attached solid torus uniquely determines the homeomorphism type of the resultant 3-manifold.

The 3-manifold obtained from a 3-manifold M by Dehn filling along a slope r is denoted by $M(r)$. The 3-manifold obtained by a Dehn surgery on a knot K in S^3 along a slope r is denoted by $K(r)$.

3 Proofs of Theorem 1 and Theorem 2

In this section, M denotes a 3-manifold with a single toral boundary ∂M. Suppose that $\mathrm{Int}(M)$ admits a complete hyperbolic structure of finite volume. One can take a horoball neighborhood C of the cusp of $\mathrm{Int}(M)$ and then identify ∂M with the boundary ∂C of C. Since ∂C is regarded as a Euclidean torus as demonstrated in [22], the length of a curve on ∂M can be defined. The *length* of a slope r on ∂M is defined as the minimum of the lengths of simple closed curves with slope r, and we denote it by $L(r)$. Note that this length depends upon the choice of C.

Let us prepare the following three lemmas. The next lemma was shown by Agol [2], which was also obtained by Lackenby [17].

Lemma 3.1 ([2, Lemma 6.1]). *If the length of a slope r on ∂M is greater than 6, then the surgered manifold $M(r)$ is irreducible and its fundamental group is infinite and word-hyperbolic.* □

One can take a particular horoball neighborhood C as follows. Take a maximal one among those having no overlapping interior, and then slightly shrink it. The next lemma holds for this C, which was given in [1].

Lemma 3.2 ([1, Theorem 5.3]). *Every slope on ∂M has the length greater than $2^{1/4}$, if M is neither the figure-eight knot exterior, the exterior of the knot 5_2 in the knot table [21] nor the manifold obtained by (2,1)-Dehn-filling on the Whitehead link exterior.* □

A properly immersed surface in M is called *essential* if the immersion induces injective maps of the fundamental groups and of the relative fundamental groups. In [2], Agol proved the following.

Lemma 3.3 ([2, Lemma 5.1]). *Suppose that an essential surface S with boundary in M is given. Let $r_1, \dots, r_n$ be the slopes of boundary components of S. Then $\sum_{i=1}^{n} L(r_i) \leq 6|\chi(S)|$.* □

Proof of Theorem 1. We first assume that K is the figure eight knot in S^3. In this case, it is shown in [22] that if $K(r)$ is non-hyperbolic and $r \neq \infty$ then $|r| \leq 4 = 4g$.

Next, in the case that K is the knot 5_2 in S^3, it is also shown in [3] that if $K(r)$ is non-hyperbolic and $r \neq \infty$ then $|r| \leq 4 = 4g$.

Now, we consider a hyperbolic knot K in S^3 neither the figure eight knot nor the knot 5_2. Let M denote the exterior of K. Let p/q be a slope on ∂M, where p, q are coprime integers and $q \neq 0$. Suppose that

$|p| > 3 \cdot 2^{7/4} g|q|$. By virtue of Lemma 3.1, we only need to show that $L(p/q) > 6$.

We choose a horoball neighborhood C as above and identify ∂M with ∂C. Let $\widetilde{\partial C}$ be a component of the preimage of ∂C in the universal cover of $\mathrm{Int}(M)$. The preimage of a point on ∂C gives a lattice on $\widetilde{\partial C}$. By fixing the base point O, each primitive lattice point corresponds to a slope on ∂C, and the distance between O and a primitive lattice point is equal to the length of the corresponding slope.

Take a lattice point P such that the path OP is projected to the $|q|$ multiple of the longitude. We can take another primitive lattice point Q corresponding to the slope p/q such that the path PQ is projected to $|p|$ multiple of the meridian. Then, the triangle inequality gives that

$$|p|L(\infty) = PQ < OP + OQ = |q|L(0) + L(p/q) .$$

This implies that

$$L(p/q) > |p|L(\infty) - |q|L(0) .$$

Let g be the genus of K, that is, the minimum of the genera of Seifert surfaces for K. Since a minimal genus Seifert surface is essential, $L(0) \leq 6(2g-1)$ holds by Lemma 3.3.

Combining this and Lemma 3.2, we conclude

$$L(p/q) > 3 \cdot 2^{7/4} g|q| 2^{1/4} - |q| 6(2g-1) > 6 .$$

□

A knot in S^3 is called *amphicheiral* if it is ambient isotopic to its mirror image.

Proof of Theorem 2. In the same way as the proof above, we only need to consider a hyperbolic knot K in S^3 neither the figure eight knot nor the knot 5_2 and will show that $L(p/q) > 6$. We use the same notations as the proof above and let C_1, C_2 be $3 \cdot 2^{7/4}$, $1 - 2^{-1/2}$, respectively.

The key fact is that if K is amphicheiral then the geodesics represented by meridian and the longitude are orthogonal in a horoball neighborhood [20]. From this fact, the path OP is orthogonal to PQ, and so the angle POQ is less than $\pi/2$. Thus, one have

$$|p|^2 L(\infty)^2 = PQ^2 > OP^2 + OQ^2 = |q|^2 L(0)^2 + L(p/q)^2 ,$$

and

$$L(p/q)^2 > |p|^2 L(\infty)^2 - |q|^2 L(0)^2 .$$

Together with the assumption that $|r| = |p/q| > C_1(g - C_2)$, $q \geq 1$ and the facts that $L(\infty) > 2^{1/4}$, $L(0) \leq 6(2g-1)$, one has the next inequalities.

$$\begin{aligned} L(p/q)^2 &> \{\sqrt{2}{C_1}^2(g-C_2)^2 - 36(2g-1)^2\}|q|^2 \\ &\geq 144(g-C_2)^2 - 36(2g-1)^2 \\ &= 36(1-2C_2)(4g-2C_2-1) \\ &= 36(\sqrt{2}-1)(4g-3+\sqrt{2}) \\ &\geq 36 \end{aligned}$$

Consequently, we have that $L(p/q) > 6$. $\square$

As we remarked in Section 1, the Thurston's Geometrization Conjecture would imply that irreducible 3-manifolds with infinite and word-hyperbolic fundamental group are actually hyperbolic.

Here, let us give a definition of the word-hyperbolic group. Let G be a finitely presented group. Fix a finite presentation of G and let Γ be the Cayley graph of G with respect to the presentation. One can regard Γ as a metric space by setting that each edge has length one. Then G is called *word-hyperbolic* if there exists a positive constant δ such that for every geodesic triangle in Γ, each one edge is contained in a δ-neighborhood of the other two edges. It can be proved that this definition does not depend on the choice of presentations [8]. It is shown that the fundamental group of a negatively curved manifold is word-hyperbolic. Conversely, for 3-manifolds, the following is conjectured.

Conjecture 3.1. *If a closed, irreducible 3-manifold has infinite fundamental group which is word hyperbolic, then it is hyperbolic.*

It is known that if a closed, irreducible 3-manifold has infinite fundamental group which is word hyperbolic, then it is neither toroidal nor Seifert fibered. A 3-manifold is called *toroidal* if it contains an embedded essential torus, and it called *Seifert fibered* if it admits a foliation by circles. Therefore, if the well-known Thurston's Hyperbolization Conjecture, which says that such manifolds are actually hyperbolic, is affirmatively solved, then the consequence of our theorems is rewritten as that $K(r)$ is hyperbolic.

4 Proof of Theorem 3

In this section, let K be a hyperbolic knot in S^3, M the exterior of K in S^3.

Suppose that M contains a closed, essential, that is, incompressible and not ∂-parallel, embedded surface S which admits an annulus connecting S and K.

This assumption is also stated in the following way. If K is hyperbolic, $\pi_1(M)$ is identified with a kleinian group G, and if S is essential, the inclusion map $i : S \to M$ induces the monomorphism $i_* : \pi_1(S) \to G$. If this $i_*(S)$ contains a parabolic element, then one can find an annulus which runs from S to ∂M by the annulus theorem. Then, by [5, Lemma 2.5.3], such an annulus determines either meridional or integral slope. In the case that the slope is integral, such an annulus is regarded to be running from S to the knot K. Moreover, in [12], such an annulus is uniquely determined in that case up to isotopy. Consequently, the assumption above is equivalent to that there exist a closed, essential embedded surface S such that $i_*(S)$ contains a parabolic element other than that represented by the meridian of K.

To prove Theorem 3, we use the following lemma. By an annulus-compression, one obtains from S an essential surface properly embedded in M. We denote it by S'. The boundary $\partial S'$ determines a slope α on ∂M.

Lemma 4.1 (**[13, Lemma 3.1]**). *Let F be an essential surface and f the slope determined by ∂F. Then, $\Delta(\alpha, f) \leq -2\chi(F)/|\partial F|$, where $\Delta(\alpha, f)$ denotes the minimal geometric intersection number of the slopes.*

Here, we only describe the outline of its proof. See [13] for detail.

The key to prove this lemma is the fact that at least one component of $\partial\mathrm{Ext}(S)$ is essential in $\mathrm{Ext}(S)$. By using this fact and the analysis of the graph appearing as $F \cap S'$, one obtains an upper bound on the number of components of $F \cap S'$.

Proof of Theorem 3. By virtue of Thurston's Uniformization Theorem [23], we only need to show that $K(r)$ is irreducible, S remains essential in $K(r)$, $K(r)$ is not Seifert fibered and $K(r)$ contains no essential tori.

We set that $r = p/q$ and $\alpha = p'/q'$, where p and q, p' and q' are coprime integers, and assume that q, $q' \neq 0$. Then, their minimal geometric intersection number $\Delta(\alpha, r)$ are given as $|pq' - p'q|$.

First, let us show that $K(r)$ is irreducible if $|r| = |p/q| \geq 4g$. Suppose that $K(r)$ is reducible. Since K is hyperbolic, the exterior M is irreducible. Hence, the reducing sphere must intersect the attached solid torus, and one can find an essential planer surface F properly embedded in M. Note that the slope r is represented by the boundary of F. Let m be the number of components of ∂F. Then, the next follows from Lemma 4.1.

$$\Delta(\alpha, r) \leq \frac{-2(2-m)}{m} = 2 - \frac{4}{m} < 2\,.$$

On the other hand, by considering a minimal genus Seifert surface, whose boundary represents the slope 0, we have that

$$\Delta(\alpha, 0) \leq -2(2 - 2g - 1) = 4g - 2\,.$$

This implies that $|p'1 - 0q'| = |p'| \leq 4g - 2$ and that

$$\Delta(\alpha, r) = |pq' - p'q| \geq ||p||q'| - |p'||q|| \geq ||p||q'| - (4g-2)|q||\,.$$

Since the assumption is that $|p/q| \geq 4g$, it follows that

$$\Delta(\alpha, r) \geq ||p||q'| - (4g-2)|q|| \geq |4g|q||q'| - (4g-2)|q|| \geq |(4g|q'| - 4g + 2)|q||$$

Consequently, by $|q|, |q'| > 0$, we conclude that $\Delta(\alpha, r) \geq 2$. This is a contradiction.

Next, we show that S remains essential in $K(r)$. This is an immediate corollary of [5, Theorem 2.4.3]. It says that if $\Delta(\alpha, r) \geq 2$, then S remains essential in $K(r)$. As we showed above, in fact, $\Delta(\alpha, r) \geq 2$ holds.

Now, since K is hyperbolic, the genus of S is greater than one, and since K is a knot in S^3, S is separating in $K(r)$. This implies that $K(r)$ is not Seifert fibered [14, Theorem VI.34].

Finally, we show that $K(r)$ contains no essential tori if $|r| = |p/q| \geq 4g + 1$. The argument to show this is almost same as that to show $K(r)$ is irreducible. Suppose that $K(r)$ contains an essential torus. Since K is hyperbolic, one can find an essential punctured torus F properly embedded in M. Note that the slope r is represented by the boundary of F. Let m be the number of components of ∂F. Then, the next also follows from Lemma 4.1.

$$\Delta(\alpha, r) \leq \frac{-2(-m)}{m} = 2\ .$$

On the other hand, again by using that $|p'| \leq 4g - 2$, we have

$$\Delta(\alpha, r) = |pq' - p'q| \geq ||p||q'| - |p'||q|| \geq ||p||q'| - (4g-2)|q||\ .$$

Since the assumption is that $|p/q| \geq 4g + 1$ and $|q|, |q'| > 0$, it follows that

$$\Delta(\alpha, r) \geq ||p||q'| - (4g-2)|q|| \geq |(4g+1)|q||q'| - (4g-2)|q|| \geq 3\ .$$

This is a contradiction, and completes the proof. □

References

[1] C. C. Adams, *Waist size for cusps in hyperbolic 3-manifolds*, preprint.

[2] I. Agol, *Bounds on exceptional Dehn filling*, Geom. Topol. **4** (2000), 431–449.

[3] M. Brittenham and Y.-Q. Wu, *The classification of exceptional Dehn surgeries on 2-bridge knots*, Comm. Anal. Geom. **9** (2001), 97–113.

[4] S. A. Bleiler and C. D. Hodgson, *Spherical space forms and Dehn filling*, Topology **35** (1996), no. 3, 809–833.

[5] M. Culler, C. McA. Gordon, J. Luecke, and P. B. Shalen, *Dehn surgery on knots*, Ann. of Math. **125** (1987), 237-300.

[6] H. Goda and M. Teragaito, *Dehn surgeries on knots which yield lens spaces and genera of knots*, Math. Proc. Cambridge Philos. Soc. **129** (2000), no. 3, 501–515.

[7] C. McA. Gordon, *Dehn filling: a survey*, in: Knot theory (Warsaw, 1995), 129–144, Banach Center Publ., **42**, Polish Acad. Sci., Warsaw, 1998.

[8] M. Gromov, *Hyperbolic groups*, in: Essays in group theory, 75–263, Math. Sci. Res. Inst. Publ., **8**, Springer, New York, 1987

[9] C. D. Hodgson and S. P. Kerckhoff, *The shape of hyperbolic Dehn surgery space*, in preparation.

[10] K. Ichihara, *Exceptional surgeries and genera of knots*, to appear in . Proc. Japan Acad. Ser. A Math. Sci., Vol. 77, No.4.

[11] K. Ichihara and M. Ozawa, *Hyperbolic knot complements without closed embedded totally geodesic surfaces*, J. Austral. Math. Soc. Ser. A **68** (2000), no. 3, 379–386.

[12] K. Ichihara and M. Ozawa, *Accidental surfaces in knot complements*, J. Knot Theory Ramifications **9** (2000), no. 6, 725–733.

[13] K. Ichihara and M. Ozawa, *Accidental surfaces and exceptional surgeries*, to appear in Osaka J. Math. .

[14] W. Jaco, *Lectures on three-manifold topology*, CBMS Regional Conference Series in Mathematics, **43**, American Mathematical Society, Providence, R.I., 1980.

[15] S. P. Kerckhoff, *Deformations of hyperbolic cone manifolds*, in: Topology and Teichmuller spaces (Katinkulta, 1995), 101–114, World Sci. Publishing, River Edge, NJ, 1996.

[16] W.B.R. Lickorish, *A presentation of orientable combinatorial 3-manifolds*, Ann. of Math. (2) **76** (1962), 531-540.

[17] M. Lackenby, *Word hyperbolic Dehn surgery*, Invent. Math. **140** (2000), 243-282.

[18] J. W. Morgan, *On Thurston's uniformization theorem for three-dimensional manifolds*, in: The Smith conjecture (New York, 1979), 37–125, Pure Appl. Math., **112**, Academic Press, Orlando, FL, 1984.

[19] C. Petronio and J. Porti, *Negatively oriented ideal triangulations and a proof of Thurston's hyperbolic Dehn filling theorem*, Expo. Math. **18** (2000), no. 1, 1–35.

[20] R. Riley, *Seven excellent knots*, in: Low-dimensional topology (Bangor, 1979), 81–151, London Math. Soc. Lecture Note Ser., **48**, Cambridge Univ. Press, Cambridge-New York, 1982.

[21] D. Rolfsen, *Knots and Links*, Publish or Perish, Berkeley, Ca, 1976.

[22] W. P. Thurston, *The geometry and topology of 3-manifolds*, Lecture notes, Princeton University, 1978.

[23] W. P. Thurston, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. (N.S.) **6** (1982), no. 3, 357–381.