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DIRAC OPERATORS AND HYPERELLIPTIC MAPPING CLASS GROUPS

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1. INTRODUCTION

Meyer defined a 2-cocycle on the mapping class group of a closed oriented surface of genus greater than 0 using the signature of 4-manifolds. It is called Meyer's signature 2-cocycle [15, 1, 2, 7, 12, 16]. It defines a nontrivial class in the second cohomology group of the mapping class group with coefficients in $\mathbb{Z}$. In the case that the genus of the surface is 1 or 2, it is a torsion class, hence is trivial over $\mathbb{Q}$. Since the first cohomology group over $\mathbb{Q}$ of the mapping class group vanishes, there is a unique rational valued function on the mapping class group of genus 1 or 2 whose coboundary is the Meyer's signature cocycle. This function is called Meyer function. Since Meyer's signature cocycle is defined in a geometrical manner, it is thought that there is a geometric interpretation of the Meyer function. In fact, in the case of genus 1, using the fact that the mapping class group is $SL(2,\mathbb{Z})$, Atiyah gave various geometric interpretations of it in terms of the following: Hirzebruch's signature defect, Dedekind η-function, Quillen's determinant line bundle, Shimizu L-function, Atiyah-Patodi-Singer η-invarinat and the adiabatic limit of η-invariant [1].

In higher genus cases, Meyer's signature 2-cocycle defines a nontrivial class over $\mathbb{Q}$. Thus, on the whole mapping class group, the same doesn't go well, but if we consider only the subgroup of it called the hyperelliptic mapping class group, the same situation occurs. Therefore we have a unique function whose coboundary is Meyer's signature 2-cocycle on the subgroup, which is also called the Meyer function.

Since hyperelliptic mapping class groups and Meyer's signature 2-cocycles are geometrical objects, Meyer functions ought to have some geometric interpretations or some relations to other geometrical objects like the case of genus 1. In fact, there are some works in this direction. See [7, 10, 14, 16] for genus $\geqq 2$.

In this note, we define some functions on subgroups of the hyperelliptic mapping class groups of surfaces using η-invariants of the Dirac operator and the signature one and show

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a relation of them to the Meyer function on the hyperelliptic mapping class group (see also [11]).

2. η-INVARIANTS OF THREE MANIFOLDS

In this section we recall the definition of η-invariants of 3-manifolds and some properties of them [3].

Let M be a closed oriented spin manifold of dimension 3. If a Riemannian metric on M is given, then the Dirac operator

$$D\colon \Gamma(S_M) \to \Gamma(S_M)$$

on the spinor bundle S_M over M is defined. It is a self adjoint elliptic operator. The function

$$\eta_D(s) = \sum_{\lambda \neq 0} \frac{sign\lambda}{|\lambda|^s},$$

where λ runs over the nonzero eigenvalues of the Dirac operator D with multiplicities, is holomorphic for $Re(s) > -\frac{1}{2}$ and extends to a meromorphic function on the whole s-plane with a finite value at $s = 0$. The η-invariant η_D of the Dirac operator D is defined by the value $\eta_D(0)$ of this function at the origin.

It is known that any closed oriented spin 3-manifold is realized as the boundary of a compact oriented spin 4-manifold. For the spin 3-manifold M, let Z be such a spin 4-manifold. We give a Riemannian metric on Z such that its restriction to a product neighborhood $(-1,0) \times M \subset Z$ of the boundary $\partial Z = M$ is the product metric of the one on M with the standard one on $(-1,0]$. Then the Dirac operator

$$D^+\colon \Gamma(S_Z^+) \to \Gamma(S_Z^-)$$

on the half spinor bundles is defined. Here $S_Z^\pm$ denote the positive and the negative half spinor bundles over Z. On the product neighborhood $(-1,0] \times M$ of the boundary, we have

$$D^+ = e_1 \cdot (\frac{\partial}{\partial t} - D),$$

where t is the coordinate of $(-1,0]$ and $e_1 \cdot$ is the Clifford multiplication by $\partial/\partial t$. We remark that the orientation of $(-1,0] \times M$, namely of Z is given by $\frac{\partial}{\partial t} \wedge$(orientation of M) in this note.

Let P be the projection of $\Gamma(S_M)$ onto the space spanned by the eigenfunctions of D for nonnegative eigenvalues. Let $\Gamma(S_Z^+; P)$ be the subspace of $\Gamma(S_Z^+)$ consisting of the sections u which satisfy the condition $P(u|_{0\times M}) = 0$. The operator

$$D^+\colon \Gamma(S_Z^+; P) \to \Gamma(S_Z^-)$$

has a finite index, which is denoted by $\operatorname{ind} D^+$.

Theorem 1 (Atiyah-Patodi-Singer [3]). *Under the above setting, the equality*

$$\operatorname{ind} D^+ = -\frac{1}{24}\int_Z p_1 - \frac{h_D + \eta_D}{2}$$

holds. Here p_1 *is the first Pontrjagin form of the Riemannian metric on* Z *and* $h_D :=$ $\dim\ker D$ *is the dimension of the harmonic spinors on* M *with respect to the metric.*

Similarly we have the following theorem, which doesn't need spin structures.

Theorem 2 (Atiyah-Patodi-Singer [3]). *The equality*

$$\operatorname{sign} Z = \frac{1}{3}\int_Z p_1 - \eta_B$$

holds. Here $\operatorname{sign} Z$ *is the signature of the 4-manifold* Z *and* η_B *is the* η*-invariant of the signature operator*

$$B\colon \Omega^{even}(M;\mathbb{C}) \ni \phi \mapsto (-1)^{\frac{\deg\phi}{2}}(*d - d*)\phi \in \Omega^{even}(M;\mathbb{C}),$$

where $*$ *is the Hodge* $*$*-operator with respect to the Riemannian metric on* M.

Put

$$F_M^\sigma(m) := 4\eta_D + \eta_B,$$

where m and σ are the Riemannian metric and the spin structure on M considered above respectively. Theorem 1 and 2 imply

$$F_M^\sigma(m) = -8\operatorname{ind} D^+ - \operatorname{sign} Z - 4h_D.$$

It is known that η_B is continuous on the space $Met(M)$ of the Riemannian metrics on M and that so is η_D on the subspace $Met_0(M) := \{m \in Met(M) | h_{D(m)} = 0\}$ of $Met(M)$, where $D(m)$ is the Dirac operator with respect to a Riemannian metric m. This implies that, on $Met_0(M)$, $F_M^\sigma(m)$ is locally constant and $F_M^\sigma(m) = -8\operatorname{ind} D^+ - signZ$. We remark that the above result holds also in the case that the 3-manifold M is not connected. We also remark that the invariant $F_M^\sigma(m)$ has appeared in the Seiberg-Witten theory [18, 19].

3. Bismut and Cheeger's proposition

In this section, we partially extend Proposition 4.41 in [6] by Bismut and Cheeger to the case that a manifold admits boundaries.

Let Σ be a closed oriented smooth manifold of even dimension l and B compact oriented smooth manifold of even dimension k possibly with boundary. We consider a fiber bundle

$\pi\colon Z \to B$ with fiber Σ. Near the boundary of the fibration, we may identify it with the product $id \times (\pi|_{\partial Z})\colon (-\delta, 0] \times \partial Z \to (-\delta, 0] \times \partial B$ for some $\delta > 0$. Take a splitting $TZ = T^H Z \oplus T^V Z$ of the tangent bundle over Z satisfying $\mathbb{R}\frac{\partial}{\partial t} \subset T^H Z$, where t denotes the standard coordinate of $(-\delta, 0]$. Here $T^V Z$ denotes the tangent bundle along the fiber. We assume that both $T^V Z$ (, hence Σ) and B have spin structures. Then a spin structure on $T^H Z$ is induced from that of B via $\pi\colon Z \to B$, hence that of TZ, namely of Z is also defined (see [13]). In this paper, such a spin structure on a fiber bundle is called a *decomposed* spin structure.

We consider a Riemannian metric

$$m_Z = \pi^* m_B \oplus m^V$$

on Z such that the above splitting of TZ is orthogonal, where m_B is a Rimannian metric on B and m^V is a fiber metric on $T^V Z$. Moreover we assume $m_B = dt^2 \oplus (m_B|_{\partial B})$ on $(-\delta, 0] \times \partial B$ and $m_Z = dt^2 \oplus \pi^*(m_B|\partial B) \oplus (m^V|_{\partial Z})$ on $(-\delta, 0] \times \partial Z$. Thus the boundary $\partial Z \to \partial B$ also is in the same situation.

For any $\varepsilon > 0$, put

$$m_{Z,\varepsilon} = (\frac{1}{\varepsilon}\pi^* m_B) \oplus m^V,$$

then we have a 1-parameter family of Riemannian metrics on Z.

Thus we can consider a 1-parameter family of Dirac operators

$$D_{Z,\varepsilon}\colon \Gamma(S_{Z,\varepsilon}) \to \Gamma(S_{Z,\varepsilon}),$$

where ε presents the dependence on the metrics.

We can consider the Dirac operators

$$D_{Z,\varepsilon}\colon \Gamma(S_Z, P_\varepsilon) \to \Gamma(S_Z)$$

with the Atiyah-Patodi-Singer boundary condition as in section 2. We note that, for each $b \in B$, we have the Dirac operator $D_{\pi^{-1}(b)}(m_Z|_{\pi^{-1}(b)})$ on $\pi^{-1}(b)$ with respect to the induced Riemannian metric $m_Z|_{\pi^{-1}(b)}$.

Proposition 3. *Under the above situation, assume that the Dirac operator $D_{\pi^{-1}(b)}(m_Z|_{\pi^{-1}(b)})$ is invertible for any $b \in B$. Then, for any sufficiently small $\varepsilon > 0$, the kernel of the Dirac operator $D_{Z,\varepsilon}\colon \Gamma(S_Z, P_\varepsilon) \to \Gamma(S_Z)$ vanishes.*

We can prove this proposition in the same way as the proof by Bismut and Cheeger in

Corollary 4. *Under the assumption of Proposition 3, the kernels, the cokernels and the indices of the Dirac operators* $D^+_{Z,\varepsilon}\colon \Gamma(S^+_{Z,\varepsilon}, P^+_\varepsilon) \to \Gamma(S^-_{Z,\varepsilon})$ *and* $D_{\partial Z,\varepsilon}\colon \Gamma(S_{\partial Z,\varepsilon}) \to \Gamma(S_{\partial Z,\varepsilon})$ *vanish for any sufficiently small* $\varepsilon > 0$.

The statement on $D_{\partial Z,\varepsilon}$ in this corollary is a result of Bismut and Cheeger's proposition [6].

4. The hyperelliptic mapping class groups and the Meyer functions

In this section, we recall the definitions of the hyperelliptic mapping class group and of the Meyer function on it.

Let Σ_g be a closed oriented surface of genus $g \geqq 1$ and $\mathcal{M}_g$ its mapping class group consisting of the isotopy classes of orientation preserving diffeomorphisms of Σ_g. We denote the 2-sphere with 3-holes by P. For any $a, b \in \mathcal{M}_g$, let $N_{a,b}$ be the Σ_g-bundle over P with monodromies a^{-1} and b^{-1}.

Meyer's signature 2-cocycle

$$sign_g\colon \mathcal{M}_g \times \mathcal{M}_g \to \mathbb{Z}$$

is defined by $sign_g(a,b)\colon = \text{sign}(N_{a,b})$, where $\text{sign}(N_{a,b})$ is the signature of the 4-manifold $N_{a,b}$ [1, 15]. Novikov additivity for the signature of manifolds shows that $sign_g$ satisfies the cocycle condition.

Let ι be the involution on Σ_g with $2g+2$ fixed points depicted in Figure 1.

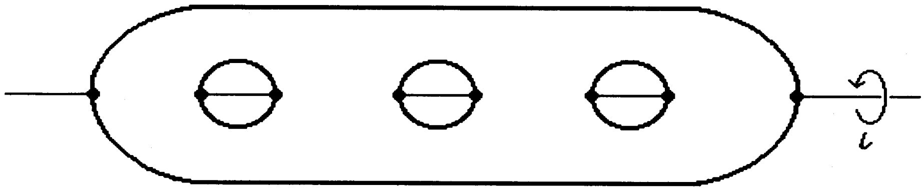

Figure 1. An involution ι on Σ_g with $2g+2$ fixed points.

The hyperelliptic mapping class group $\mathcal{H}_g$ of Σ_g is the subgroup of $\mathcal{M}_g$ consisting of elements which commute with the class of ι. It is known that $\mathcal{M}_1 = \mathcal{H}_1 = SL(2,\mathbb{Z})$, $\mathcal{M}_2 = \mathcal{H}_2$ and that $\mathcal{H}_g (g \geqq 3)$ is a subgroup of $\mathcal{M}_g$ of infinite index.

Meyer's signature cocycle $sign_g$ defines a nontrivial class of the second cohomology group $H^2(\mathcal{M}_g, \mathbb{Z})$ of $\mathcal{M}_g$ with coefficients in $\mathbb{Z}$ and its restriction to $\mathcal{H}_g$ is also nontrivial. But it is trivial in $H^2(\mathcal{H}_g, \mathbb{Q})$. Thus there exists a function or 1-cochain

$$\phi_g\colon \mathcal{H}_g \to \mathbb{Q}$$

such that $sign_g = \delta\phi_g$, where δ denotes the coboundary operator defined by $\delta\phi_g(a,b) = \phi_g(b) - \phi_g(ab) + \phi_g(a)$ for $a, b \in \mathcal{H}_g$. It follows that ϕ_g is unique from the fact of $H^1(\mathcal{H}_g, \mathbb{Q}) = \{0\}$. This function ϕ_g is called the Meyer function. It is known to be conjugacy invariant. Its values are contained in $\frac{1}{2g+1}\mathbb{Z}$ and concrete values on Lickorish generators and BSCC maps are calculated by Endo [7], Matsumoto [14] and Morifuji [16].

In the case of $g = 1$, under the identification $\mathcal{M}_1 \cong \mathcal{H}_1 \cong SL(2,\mathbb{Z})$, Meyer [15] and Atiyah [1] gave an explicit expression of the Meyer function using the Dedekind sums (see also [12]). Thus we can compute the values of it. Moreover Atiyah [1] put various geometric interpretations on the values of ϕ_1 on hyperbolic elements.

There is another description of the hyperelliptic mapping class group as follows, which is needed in this note.

We consider the subgroup $Diff_+^\iota(\Sigma_g)$ of the group $Diff_+(\Sigma_g)$ of orientation preserving diffeomorphisms of Σ_g consisting of the elements which commute with ι. Birman and Hilden [5] proved that the quotient group of this subgroup modulo its identity component is isomorphic to the hyperelliptic mapping class group $\mathcal{H}_g$.

In this note we let a hyperelliptic fibration mean a Σ_g-bunlde with structure group $Diff_+^\iota(\Sigma_g)$. Since it is known that the identity component of $Diff_+^\iota(\Sigma_g)$ is contractible, that we consider hyperelliptic fibrations is equivalent to that we consider representations of the fundamental groups of their base spaces to the hyperelliptic mapping class group $\mathcal{H}_g$.

5. A result of Bär and Schmutz for Dirac operators on surfaces

In this section we recall a result [4] of Bär and Schmutz for the Dirac operators on hyperelliptic Riemann surfaces.

Let Σ_g be a closed oriented surface of genus $g \geq 2$.

For any spin structure and any Riemannian metric on Σ_g, we have the Dirac operator

$$D\colon \Gamma(S_{\Sigma_g}) \to \Gamma(S_{\Sigma_g}),$$

where S_{Σ_g} is the spinor bundle over Σ_g with respect to the spin structure and the Riemannian metric on Σ_g.

We are interested in the behavior of the dimension $\dim\ker D$ of the space of the harmonic spinors under deformation of metrics. On a surface, since the dimensions of the spaces of the positive and the negative harmonic spinors agree, we have only to know the behavior of the dimension h^0 of the positive spinors. If we consider only metrics inducing a hyperelliptic complex structure, it has been completely described by C. Bär and P. Schmutz [4] as follows.

Theorem 5 (C. Bär and P. Schmutz [4]). *Let Σ_g be a hyperelliptic Riemann surface of odd genus g with Weierstrass points $p_1, \ldots, p_{2g+2}$. Then the 2^{2g} divisors*

$$(g-1)p_1,\ (g-2k)p_{i_1} + p_{i_2} + \cdots + p_{i_{2k}}\ (k = 1, 2, \ldots, \frac{g-1}{2}),\ -p_1 + p_{i_2} + \cdots + p_{i_{g+1}},$$

where $i_\nu < i_\mu$ for $\nu < \mu$, are the pairwise inequivalent square roots of the canonical divisor, hence these give the spin structures of Σ_g.

Moreover, for the spin structures corresponding to the above divisors, the dimensions h^0 of the positive harmonic spinors are given by

$$\frac{g+1}{2},\ \frac{g-2k+1}{2}\ (k = 1, 2, \ldots, \frac{g-1}{2}),\ 0$$

respectively.

Similarly in the case of even genus g, the 2^{2g} divisors are given by

$$(g-(2k+1))p_{i_1} + p_{i_2} + \cdots + p_{i_{2k+1}}\ (k = 0, 1, \ldots, \frac{g-2}{2}),\ -p_1 + p_{i_2} + \cdots + p_{i_{g+1}}$$

and the corresponding dimensions h^0 are given by

$$\frac{g-(2k+1)+1}{2}\ (k = 0, 1, \ldots, \frac{g-2}{2}),\ 0$$

respectively.

Let $\mathcal{S}(\Sigma_g)$ be the set of the spin structures on Σ_g, then we have $\sharp\mathcal{S}(\Sigma_g) = 2^{2g}$.

Let ι be the involution in section 4 and $Met(\Sigma_g)^\iota$ the space of ι-invariant Riemannian metrics on Σ_g. Then we can obtain the following corollary from Theorem 5 and some elementary facts about hyperelliptic Riemann surfaces.

Corollary 6. *For any fixed spin structure on Σ_g, the dimension* $\dim\ker D$ *of the harmonic spinors on Σ_g is constant on $Met(\Sigma_g)^\iota$. Moreover put $\mathcal{S}_0(\Sigma_g) = \{\sigma \in \mathcal{S}(\Sigma_g) |$* $\dim\ker D = 0$ *on $Met(\Sigma_g)^\iota\}$, then the number $\sharp\mathcal{S}_0(\Sigma_g)$ is $\binom{2g+1}{g}$.*

Clearly the subset $\mathcal{S}_0(\Sigma_g)$ is preserved by the action of $\mathcal{H}_g$.

We remark that this corollary holds also for $g = 0, 1(, 2)$ by a result [9] of Hitchen. In this case, it holds on the space of all Riemannian metrics.

6. Some functions on subgroups of hyperelliptic mapping class groups

In this section we define some functions on subgroups of hyperelliptic mapping class groups and state our main theorem.

For any spin structure $\sigma \in \mathcal{S}(\Sigma_g)$, let $\mathcal{H}_g^\sigma$ be the subgroup of $\mathcal{H}_g$ consisting of the elements which preserve σ.

Let $* \in D^2 \subset \Sigma_g$ be a base point and an embedded disk in Σ_g. Let $\mathcal{M}_{g,1}$ be the group of all isotopy classes relative to D^2 of diffeomorphisms of Σ_g which restrict to the identity on D^2. Then there is a natural homomorphism $j\colon \mathcal{M}_{g,1} \to \mathcal{M}_g$. Let $\mathcal{H}^\sigma_{g,1}$ be the subgroup of $\mathcal{M}_{g,1}$ given by $j^{-1}(\mathcal{H}^\sigma_g)$.

Let σ_{S^1} be the spin structure on $S^1 = \partial D^2$ induced from the unique one on D^2.

For any $a \in \mathcal{H}^\sigma_{g,1}$, we define a Σ_g-bundle M_a over S^1 by $M_a = \Sigma_g \times [0,1]/(x,0) \sim (a(x),1)$. Moreover we have the identification i of Σ_g with the fiber of M_a at the base point $1 \in S^1$. Here we remark that we can confuse diffeomorphisms on Σ_g with their mapping classes since surface bundles are determined by their holonomies in $\mathcal{M}_{g,1}$ for $g \geq 1$.

Lemma 7. *A decomposed spin structure σ_a on M_a is uniquely constructed for each $a \in \mathcal{H}^\sigma_{g,1}$.*

The decomposed spin structure σ_a in this lemma is defined as follows. Take a splitting $TM_a = T^V M_a \oplus T^H M_a$, where $T^V M_a$ be the tangent bundle of the Σ_g-bundle M_a along the fiber. A spin structure on $T^H M_a$ is given by the pullback of the spin structure σ_{S^1} on S^1 via the projection $\pi\colon M_a \to S^1$. Let $P_{GL_+}(T^V M_a)$ be the $GL_+(2,\mathbb{R})$-bundle over M_a associated with $T^V M_a$. It can be regarded also as a bundle over S^1 with fiber $P_{GL_+}(T\Sigma_g)$ which is the $GL_+(2,\mathbb{R})$-bundle associated with $T\Sigma_g$. We note that a spin structure on $T^V M_a$ is corresponding to a homomorphism from $\pi_1(P_{GL_+}(T^V M_a))$ to $\mathbb{Z}_2$ with the non-trivial value on the class of $SO(2)$ in the fiber $GL_+(2,\mathbb{R})$. If we take an oriented basis $b = \{b_1, b_2\}$ for $T_*\Sigma_g$ at the base point, then since any element of $\mathcal{H}^\sigma_{g,1}$ preserves the basis b for $T_*\Sigma_g$, the bundle $P_{GL_+}(T^V M_a)$ over S^1 has the section $\bar{b}$ obtained from the basis b. We give a spin structure on $T^V M_a$ by the homomorphism on $\pi_1(P_{GL_+}(T^V M_a))$ whose restriction to the fiber is corresponding to σ and whose value on S^1, which is the image of $\bar{b}$, is trivial.

These spin structures induce a spin structure σ_a on TM_a. This is the required one.

Next we replace the representative of the class $a \in \mathcal{H}^\sigma_{g,1} \subset \mathcal{M}^\sigma_{g,1}$ by that of $j(a) \in \mathcal{H}^\sigma_g$ which is taken in $Diff^\iota_+(\Sigma_g)$. Then we can obtain a structure of a hyperelliptic fibration on M_a. Moreover this fibration has a decomposed spin structure induced from σ_a using an isotopy between old and new representatives.

From now on, we assume $\sigma \in \mathcal{S}_0(\Sigma_g)$. Let $m_a = \pi^* m_{S^1} \oplus m^V$ be a metric on M_a satisfying the same conditions as in Proposition 3 and $m_{a,\varepsilon} = (\varepsilon^{-1}\pi^* m_{S^1}) \oplus m^V$ a 1-parameter family of Riemannian metrics on M_a with $\varepsilon > 0$. Thus we have the 1-parameter family of the Dirac operators $D_{M_{a,\varepsilon}}\colon \Gamma(S_{M_{a,\varepsilon}}) \to \Gamma(S_{M_{a,\varepsilon}})$ on the 3-manifold M_a with the spin structure σ_a for $\varepsilon > 0$. By Corollary 4 and the fact that the condition of

$\dim\ker D_{M_a} = 0$ is an open one on the space of the Riemannian metrics, the function

$$F_{\sigma,1}\colon \mathcal{H}_{g,1}^{\sigma} \to \mathbb{Z}$$

defined by

$$F_{\sigma,1}(a) := \lim_{\varepsilon \to +0} F_{M_a}^{\sigma_a}(m_{a,\varepsilon}),$$

where $F_{M_a}^{\sigma_a}(m_{a,\varepsilon})$ was defined in section 2, is well defined since any two metrics on M_a satisfying the above conditions can be connected by a path of metrics with the same conditions.

Lemma 8. *For any $\sigma \in \mathcal{S}_0(\Sigma_g)$, the following holds:*

1. $F_{\sigma,1}(1) = 0$,
2. $F_{\sigma,1}(a^{-1}) = -F_{\sigma,1}(a)$,
3. $F_{(f^{-1})^*\sigma,1}(faf^{-1}) = F_{\sigma,1}(a)$,
4. $j^*sign_g = -\delta F_{\sigma,1}$ *on* $\mathcal{H}_{g,1}^{\sigma}$,

where $a \in \mathcal{H}_{g,1}^{\sigma}$, $f \in \mathcal{H}_{g,1}$, 1 is the identity element of $\mathcal{H}_{g,1}^{\sigma}$ and δ is the coboundary operator.

For any $\sigma \in \mathcal{S}_0(\Sigma_g)$, let

$$\psi_{\sigma,1}\colon \mathcal{H}_{g,1}^{\sigma} \to \mathbb{Q}$$

be the function defined by

$$\psi_{\sigma,1} := F_{\sigma,1} + j^*\phi_g.$$

Since the Meyer function ϕ_g has similar properties to those in Lemma 8, we have the following corollary.

Corollary 9. *For any $\sigma \in \mathcal{S}_0(\Sigma_g)$, $\psi_{\sigma,1}$ is a homomorphism on $\mathcal{H}_{g,1}^{\sigma}$. Moreover the equality $\psi_{(f^{-1})^*\sigma,1}(faf^{-1}) = \psi_{\sigma,1}(a)$ holds for all $a \in \mathcal{H}_{g,1}^{\sigma}$ and $f \in \mathcal{H}_{g,1}$.*

The next proposition shows that the functions $F_{\sigma,1}$ and $\psi_{\sigma,1}$ are obtained from functions on $\mathcal{H}_g^{\sigma}$ by the pullback via $j\colon \mathcal{H}_{g,1}^{\sigma} \to \mathcal{H}_g^{\sigma}$.

Proposition 10. *The functions $F_{\sigma,1}$ and $\psi_{\sigma,1}$ descend to $F_{\sigma}\colon \mathcal{H}_g^{\sigma} \to \mathbb{Z}$ and $\psi_{\sigma}\colon \mathcal{H}_g^{\sigma} \to \mathbb{Q}$ respectively. Moreover F_{σ} and ψ_{σ} have similar properties to those in Lemma 8 and Corollary 9 respectively*

Now we state our main theorem. Put $\mathcal{H}_g^{\mathcal{S}_0} := \bigcap_{\sigma \in \mathcal{S}_0(\Sigma_g)} \mathcal{H}_g^{\sigma}$, then it is a subgroup of $\mathcal{H}_g$ of finite index and all of funcions F_{σ} are defined on it.

Theorem 11. *The equality $\phi_g = -\frac{1}{\sharp\mathcal{S}_0(\Sigma_g)} \sum_{\sigma \in \mathcal{S}_0(\Sigma_g)} F_{\sigma}$ holds on $\mathcal{H}_g^{\mathcal{S}_0}$.*

From this theorem and explicit values of ϕ_g (see [7, 16]), we can find that the functions F_σ and ψ_σ are nontrivial on $\mathcal{H}_g^{\mathcal{S}_0}$, hence on $\mathcal{H}_g^\sigma$ for any $\sigma \in \mathcal{S}_0(\Sigma_g)$.

References

[1] M. F. Atiyah, *The logarithm of the Dedekind η-function*, Math. Ann. 278(1987), 335-380.

[2] M. F. Atiyah, *On framings of* 3*-manifolds*, Topology 29(1990), 1-7.

[3] M. F. Atiyah, V. K. Patodi and I. M. Singer, *Spectral asymmetry and Riemannian geometry I*, Math. Proc. Camb. Phil. Soc. 77(1975), 43-69.

[4] C. Bär and P. Schmuts, *Harmonic Spinors on Riemann Surfaces*, Ann. Glob. Anal. and Geom. 10(1992), 263-273.

[5] J. Birman and H. Hilden, *On the mapping class groups of closed surfaces as covering spaces*, in: Advances in the Theory of Riemann Surfaces, Ann. of Math. Stud. 66(1971), 81-115.

[6] J.-M. Bismut and J. Cheeger, *η-invariants and their adiabatic limits*, J. Amer. Math. Soc. 2(1989), 33-70.

[7] H. Endo, *Meyer's signature cocycle and hyperelliptic fibrations*, Math. Ann. 316(2000), 237-257.

[8] H. Farkas and I. Kra, Riemann surfaces, Springer-Verlag Berlin-Heidelberg-New York, 1980.

[9] N. Hitchin, *Harmonic spinors*, Adv. in Math. 14(1974), 1-55.

[10] R. Kasagawa, *On a function on the mapping class group of a surface of genus* 2, Topology Appl. 102(2000), 219-237.

[11] R. Kasagawa, *Dirac operators and hyperelliptic mapping class groups*, preprint.

[12] R. Kirby and P. Melvin, *Dedekind sums, μ-invariants and the signature cocycle*, Math. Ann. 299(1994), 231-267.

[13] H. B. Lawson and M.-L. Michelsohn, Spin geometry, Princeton Math. Series, No 38, Princeton Univ. Press, Princeton, NJ, 1989.

[14] Y. Matsumoto, *Lefshetz fibrations of genus two; -a topological approach-*, in Proceedings of the 37th Taniguchi symposium on topology and Teichmüller spaces, ed. by Sadayoshi Kojima et al. 1996, World Scientific Publishing Co. pp123-148.

[15] W. Meyer, *Die Signatur von Flächenbündeln*, Math. Ann. 201(1973), 239-264.

[16] T. Morifuji, *On Meyer's function of hyperelliptic mapping class groups*, preprint, 1998.

[17] S. Morita, *Characteristic classes of surface bundles*, Invent. Math. 90(1987), 551-577.

[18] L. I. Nicolaescu, *Lattice points, Dedekind-Rademacher sums and a conjecture of Kronheimer and Mrowka*, math. DG/9801030.

[19] L. I. Nicolaescu, *Seiberg-Witten Theoretic Invariants of Lens Spaces*, math. DG/9901071.

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