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Polyhedral Model of Hyperbolic Surface

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Abstract:

In this paper, we determine the triangle (polyhedral) model of a hyperbolic surface, and we consider whether there is a closed polyhedron of the model of a hyperbolic surface.

1 Introduction

Thurston [Th] gives us an exercise: Construct a polyhedral model of a surface with negtive constant curvature using congruent triangles. In this article, we give the answer of this exercise, and consider whether there exists a closed polygon of this model.

It is very easy to get all polyhedral models of hyperbolic surfaces. We argue this in the section 2. In the section 3, we consider whether there exists a closed polyhedron in $\mathbf{R}^3$. We know well that there is no smoothly immersed surface in $\mathbf{R}^3$ with constant negative curvature. (Hilbert's theorem.) But how about the polyhedral case? Such polyhedron is closely related to a triangle group having a Fuchsian surface subgroup of genus 2. (See [KN2].) For example, we may consider a regular polyhedron with 28 regular triangles of genus 2. This model has a index $(7,7,7)$. (About index, see Proposition 2.4.) In this case, (if exists,) this polyhedron is related to the triangle group $(2,3,7)$. We can make a unfolding development figure of this polyhedron, but we cannot construct this (even we are allowed immersed case) in $\mathbf{R}^3$. Another example: When each face is a triangle with 54, 63, 63 degrees, we call this model *hyplane model.* This index of this model is $(6,6,7)$. The name *hyplane* is derived from the software 'Hyplane' developed by the author.

The author would like to thank Prof. Jeffrey Weeks, Prof. Yoshitake Hashimoto, and Prof. Nariya Kawazumi. Particularly Prof. Weeks shows that the hyplane model is a good model in the sense of Proposition 2.5. The author thanks him very much.

2 Polyhedral model of hyperbolic surface.

Let M be a polyhedron in $\mathbf{R}^3$. That is, M consists of some faces, some edges and some vertices in $\mathbf{R}^3$, such that (1) each face is a planer polygon, and that (2) for each edge at most 2 faces meet at the edge. If all edge have two faces then we call M *closed.*

First of all, we define *a curvature* at each vertex of M.

Definition 2.1

Suppose that a vertex v is contained in the interior of M. The curvature $\Delta(v)$ at v is given by:

$$\Delta(v) := 2\pi - \sum_{\text{angles at } v} (\text{radian of the angle})$$

In the sequel, we consider Eulidian geometry on $M^o := M \setminus \{\text{vertices}\}$. The following proposition is elementary.

Proposition 2.2

Let v be an interior vertex of M. If a triangle ΔABC on M^o satisfies that v is an only vertex in the area surrounded by $\Delta ABC \subset M^o \subset M$, then $\angle A + \angle B + \angle C = \pi + \Delta(v)$.

The proof is easy.

Next, we define *a triangle model of a hyperbolic surface* as follows.

Definition 2.3

Let M be a polyhedron. M is a triangle model of a hyperbolic surface if:

(1) Any two faces are congruent to each other. And the faces have mirror symmetry at each edges.

(2) For any vertex v in the interior of M, $\Delta(v)$ is equals to a negative constant Δ.

Here the faces have mirror symmetry if for each edge the two faces with the edge have mirror symmetry with respect to the edge. See Figure 1.

For any vertex, angles at the vertex are constant because of mirror symmetry. Let A, B, C be the angles of a face and let a, b, c be the numbers of angles A, B, C at a vertex respectively. From the condition (2), a, b, c must be constants. If a vertex v collects a angles of radian A, then $\Delta(v) = 2\pi - aA$. From the condition (2), we have

$$2\pi - aA = 2\pi - bB = 2\pi - cC < 0.$$

Since A, B, C are the angles of a triangle,

$$A + B + C = \pi$$

Let k be aA and we have

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{\pi}{k} < \frac{1}{2}.$$

So, we have the complete answer of the exercise by Thurston.

Theorem 2.4
Let (a, b, c) be a triple of positive integers satisfying:
(1) $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < \frac{1}{2}$
(2) If $a \neq b$ then c is even. If $b \neq c$ then a is even. If $c \neq a$ then b is even.
Then we have a triangle model of a hyperbolic surface. In fact,

$$k = \frac{abc\pi}{ab + bc + ca},$$

$$A = \frac{k}{a}, \quad B = \frac{k}{b}, \quad C = \frac{k}{c}.$$

We call (a, b, c) the index of the model.

The following two models are regarded as *good* ones.

Proposition 2.5
(1) If all faces are isosceles triangles then $\Delta = \frac{-\pi}{10}$ (the model of index $(6, 6, 7)$) is the maximum value of Δ. (This part is due to Prof. Weeks.)
(2) If each faces isn't isosceles, then $\Delta = \frac{-2\pi}{41}$ (the model of index $(4, 6, 14)$) is the maximum value of Δ.

Remark that the model $(6, 6, 7)$ is the hyplane model.

3 Closed polyhedron

In this section we argue whether there exists a closed polyhedron of a triangle model of a hyperbolic surface. The following two propositions are elementary.

Proposition 3.1 *Suppose M is a closed polyhedron of the model of index (a, b, c). Let g be the genus of M then we have $g \geq 2$.*

Proof:
Suppose that a, b, c are different with each other. It is easy to have

$$v = (\frac{1}{a} + \frac{1}{b} + \frac{1}{c})f, \quad e = \frac{3}{2}f,$$

where v, e, f are the numbers of all vertices, of all edges, and of all faces respectively. If the polyhedron is closed,

$$2 - 2g = v - e + f = (\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{1}{2})f < 0.$$

In the case $a = b \neq c$ and the case $a = b = c$, we get the same result in the similar way.

In the smooth case we know that the genus of a closed hyperbolic surface is more than 1. Proposition 3.1 is an analogue of this fact.

Proposition 3.2
Suppose that P is a $4g$-gon on M^o and that the sum of the internal angle of P is 2π. If v' is the number of vertices in the area surrounded by P, then $v = v'$.

Proof:
From Proposition 2.2, we have

$$2\pi = (4g - 2)\pi + v'\Delta.$$

After easy calculations, we obtain that

$$v = v' = \frac{2(ab + bc + ca)(2 - 2g)}{2(ab + bc + ca) - abc}.$$

In the triangle model, we may regard that vertices are placed uniformly on the polyhedron. It means that Proposition 2.2 is an analogue of Gauss-Bonnet theorem. Using Gauss-Bonnet theorem, we get the area of a closed smooth hyperbolic surface. From Proposition 3.2, we know that 'the area' of closed polyhedron doesn't contradict to the numbers of vertices.

So we cannot restrict the index (a, b, c) by the genus g using this proposition. But as mentioned in [NK2], if we fix g then the index (a, b, c) is restricted from the viewpoint of Fuchsian groups and triangle groups.

Theorem 3.3

(1) There are two polyhedron M_1 and M_2 of the model $(7, 7, 7)$ such that $M = M_1 \cup M_2$ can be a closed polyhedron of the model $(7, 7, 7)$ with genus 2.

(2) We cannot realize the above M in $\mathbf{R}^3$.

Proof:

(1) See Figure 2(1). This polyhedron is homeomorphic to a pants. If we consider two copies of this (and let them be M_1 and M_2) then we can construct a closed polyhedron of the index $(7, 7, 7)$. See Figure 2(2)

(2) The proof can be obtained only by a conbinatorial way and very complicated. So we omit the proof.

4 Reference

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Figure 1

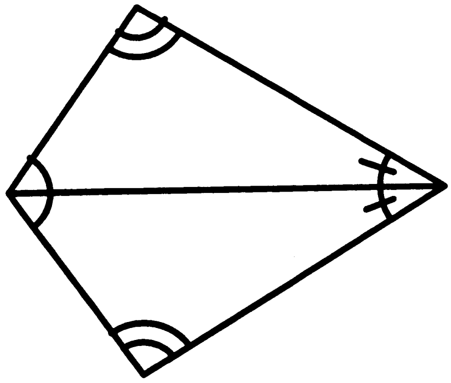


Figure 2(1)

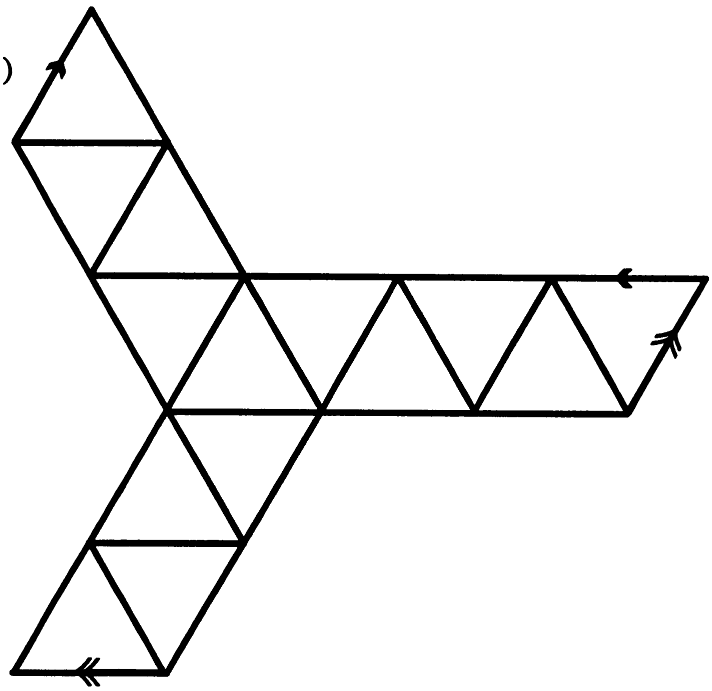
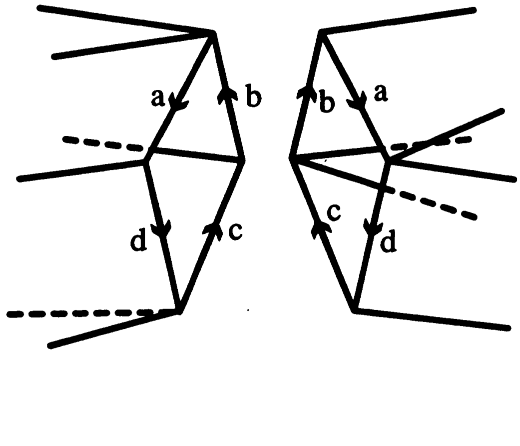


Figure 2(2)



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