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A subgrid model equation for two-dimensional homogeneous turbulence based on Lagrangian Spectral theory

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1 A brief review of the problem

Large Eddy Simulation (LES) for turbulent flow has been incereasing in importance in atmospheric, oceanographic and engineering research. Concept of LES is to project Navier Stokes turbulent flow onto grid scale flow and to numerically solve the coarse grained equation. In many cases in LES, Subgrid Scale (SGS) is treated as Smagorinsky type eddy viscosity. However, as seen in the non-equilibrium statistical mechanics, projection of some degrees of freedom onto retained degrees of freedom yields two effects, decay of memory(or damping) and random force, which are related to each other through so called fluctuation dissipation theorem. We can learn from the story that an analogy would hold for the turbulence, which is, however, far from thermal equilibrium. Kraichnan(1976) has discussed a model representaion of the DIA equations which consists of the nonlinear eddy damping and the random force. Inspired by this model, Bertoglio(1984) has derived a set of equations for LES, in which the SGS was treated as the eddy damping factor and the random force. Both are related so as to satisfy the EDQNM spectral equation. Similar approach has been used in Chasnov(1991). The random force is considered as back scatter, which represents the effects of back transfer of the energy from the subgrid scales to the grid scales. In the derivation they used the fact that the nonlinear energy transfer function in the EDQNM spectral equation consists of source and drain terms. The drain term goes to the eddy damping factor and the source term corresponds to the intensity of random force. There is no unique way to devide the nonlinear transfer function into the two terms. What they did are simply to use the EDQNM spectral equation as guide line to construct the SGS model equation, in other words, they sought what the SGS model equation obeys the EDQNM spectral equation. In deriving their formula, two time Eulerian correlation fucntions appear and are treated as if they have Lagrangian time scale $\bar{\epsilon}^{-1/3}k^{-2/3}$ in the inertial range, which is physically incosistent because the time scale of the Eulerian correlation is $(kU_0)^{-1}$, where $\bar{\epsilon}$ and U_0 are the mean rate of energy dissipation and the root mean square velocity, repsectively. There is a confusion about the Eulerian and Lagrangian time scales. In this sense, we consider that the LES modeling of the SGS needs more theoretical study.

In this note we present an LES modeling which is theoretically more consistent. Our fundamental spirit in the study is (1) to use the MLRA spectral equation for LES modeling, because the LRA and MLRA yield good results for the energy spectrum in both two and three dimensions without any ad hoc parameters (Kaneda 1981, Gotoh, Kaneda and Bekki, 1988), but the MLRA is easier to treat than the LRA, (2) to present a coarse graning method which is physically plausible and consistent with the MLRA equations, but (3) to use approximation to the models in order to obtain explicit analytical form of the model. The first one is equivalent to ask what the SGS model equation consistent with the MLRA equations is, in parallel to Bertoglio and Chasnov. The second means that when the MLRA is applied to the LES model obtained by the coarse graining method the resulting spectral equations are the same energy equation as that obtained by the coarse graining of the MLRA spectral equations. This needs more explanation. Let the coarse graining operation be $\mathcal{C}$, the Navier Stokes be $\mathrm{N}_s(\boldsymbol{u},\boldsymbol{u})$ and the MLRA operation be $\mathcal{L}$. Then we seek an operation $\mathcal{C}$ such that $\mathcal{L}\mathcal{C}(\mathrm{N}_s(\boldsymbol{u},\boldsymbol{u}))=\mathcal{C}\mathcal{L}(\mathrm{N}_s(\boldsymbol{u},\boldsymbol{u}))$, and the operation $\mathcal{C}$ is a renormalization of $\mathrm{N}_s(\boldsymbol{u},\boldsymbol{u})$ in the sense of the MLRA. The third is important to practical application of the LES modeling. Usually theoretical formulra of the LES model is sufficiently complicated so that it will be of no use for practical application without further simplification. Here we obtain explicit form of the eddy viscosity and the intensity of the random vorticity by approximation of taking two asymptotic forms for $k \ll k_m$ and $k_m - k \ll k \leq k_m$, where k_m is a wavenumber corresponding to the grid scale.

2 Basic equations in two dimensions

We consider two dimensional homogeneous isotropic turbulence in a steady state. The basic equation for an incompressible fluid in unbounded domain is

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nu \nabla^2 \boldsymbol{u}, \quad \nabla \cdot \boldsymbol{u} = 0, \tag{1}$$

where $\rho = 1$ is assumed for simplicity. For later use, we also write the vorticity equation:

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) \omega(\boldsymbol{k},t) = -\frac{1}{2} \sum_{\boldsymbol{p},\boldsymbol{q}}^{\Delta} \left(\frac{1}{p^2} - \frac{1}{q^2}\right) (\boldsymbol{p} \times \boldsymbol{q}) \omega(\boldsymbol{p},t) \omega(\boldsymbol{q},t). \tag{2}$$

3 LRA equations for two-dimensional turbulence

The MLRA equations for two-dimensional homogeneous isotropic turbulence are derived by using equation (1) (not (2)) as (Kaneda 1981, Gotoh, Kaneda and Bekki, 1988)

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) Q(k,t) = B(k,t), \tag{3}$$

$$\left(\frac{\partial}{\partial \tau} + \nu k^2 + \eta(k,t)\right) G(k,t,\tau) = 0, \quad \tau > 0, \tag{4}$$

$$G(k,t,\tau = 0) = 1, \tag{5}$$

where

$$B(k,t) = \iint_{\Delta} dp dq \; b_{kpq} \theta_{kpq}(t) Q(q,t)[Q(p,t) - Q(k,t)], \tag{6}$$

$$\theta_{kpq}(t) = \int_0^\infty G(k,t,\tau) G(p,t,\tau) G(q,t,\tau) d\tau, \tag{7}$$

$$\eta(k,t,\tau) = k \int_0^\infty q^2 J(q/k) Q(q,t) \int_0^\infty G(q,t,\tau) d\tau, \tag{8}$$

$$J(x) = J(1/x) = \frac{\pi}{2} \begin{cases} 3x - x^3, & x \leq 1 \\ 3/x - 1/x^3, & x > 1 \end{cases} \tag{9}$$

$$b_{kpq} = 4kp(xy - z + 2z^3)/(1-x^2)^{1/2} = 4k^{-2}(k^2 - q^2)(p^2 - q^2)(1-x^2)^{1/2}. \tag{10}$$

4 Coarse graining of equations

We now introduce a projection such that for a Fourier mode $A(\boldsymbol{k})$

$$A(\boldsymbol{k}) = \mathcal{P}A + \mathcal{P}'A \equiv A^{<}(\boldsymbol{k}) + A^{>}(\boldsymbol{k}), \tag{11}$$

$$\mathcal{P}A \equiv \begin{cases} A(\boldsymbol{k}) & |\boldsymbol{k}| \leq k_m \\ 0 & |\boldsymbol{k}| > k_m \end{cases}, \quad \mathcal{P}' = 1 - \mathcal{P}. \tag{12}$$

Correspondingly to this we have a decomposition of A in the physical space as

$$A(\boldsymbol{x}) = A^{<}(\boldsymbol{x}) + A^{>}(\boldsymbol{x}), \tag{13}$$

$$A^{<}(\boldsymbol{x}) = \int A^{<}(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}d\boldsymbol{k}, \quad A^{>}(\boldsymbol{x}) = \int A^{>}(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}d\boldsymbol{k}, \tag{14}$$

The velocity field $\boldsymbol{u}$ and the pressure p are decomposed as

$$\boldsymbol{u} = \boldsymbol{u}^{<} + \boldsymbol{u}^{>}, \quad p = p^{<} + p^{>}. \tag{15}$$

The equation for $\boldsymbol{u}^{<}$ is obtained as

$$\begin{aligned} \frac{\partial \boldsymbol{u}^{<}}{\partial t} &= -\mathcal{P}[(\boldsymbol{u}^{<} + \boldsymbol{u}^{>}) \cdot \nabla(\boldsymbol{u}^{<} + \boldsymbol{u}^{>})] - \nabla p^{<} + \nu\nabla^2\boldsymbol{u}^{<} \\ &= -\boldsymbol{u}^{<} \cdot \nabla\boldsymbol{u}^{<} - \nabla p^{<} + \nu\nabla^2\boldsymbol{u}^{<} + \boldsymbol{q}^{<}, \end{aligned} \tag{16}$$

$$\nabla \cdot \boldsymbol{u}^{<} = 0, \tag{17}$$

where $\boldsymbol{q}^{<}$ represents the grid-subgrid interaction:

$$\boldsymbol{q}^{<} = -\mathcal{P}[\boldsymbol{u}^{>} \cdot \nabla\boldsymbol{u}^{<} + (\boldsymbol{u}^{<} + \boldsymbol{u}^{>}) \cdot \nabla\boldsymbol{u}^{>}]. \tag{18}$$

We assume that $\boldsymbol{q}^{<}(\boldsymbol{x},t)$ is written as

$$\boldsymbol{q}^{<}(\boldsymbol{x},t) = \int \mu_e(\boldsymbol{x},\boldsymbol{y},t)\nabla_{\boldsymbol{y}}^2\boldsymbol{u}^{<}(\boldsymbol{y},t)d\boldsymbol{y} + \boldsymbol{f}^{<}(\boldsymbol{x},t) = \mu_e * \nabla^2\boldsymbol{u}^{<} + \boldsymbol{f}^{<}, \tag{19}$$

where μ_e is a deterministic function and $\boldsymbol{f}^{<}(\boldsymbol{x},t)$ is a random function and * denotes the convolution.

Next we define the Lagrangian position function $\psi^{<}$

$$\psi^{<}(\boldsymbol{y},t|\boldsymbol{x},s) = \delta(\boldsymbol{y} - \boldsymbol{z}^{<}(\boldsymbol{x},s|t)), \tag{20}$$

where $\boldsymbol{z}^{<}(\boldsymbol{x},s|t)$ is the position vector of a fluid particle, which is convected by the velocity $\boldsymbol{u}^{<}$, measured at time t and whose trajectory passes $\boldsymbol{x}$ at time s. This is important and possible whatever the velocity field is. Key point is that even when our knowledge of the velocity field is limitted to low resolution we can trace the particle with great accuracy. (For example, consider the case that we truck a fluid particle by cubic interpolation even when the velocity field has low resolution.) The equation for $\psi^{<}$ is given by

$$\frac{\partial\psi^{<}}{\partial t} + \boldsymbol{u}^{<} \cdot \nabla\psi^{<} = 0. \tag{21}$$

Next we define the coarse grained generalized velocity field as

$$\boldsymbol{v}^{<}(\boldsymbol{x},s|t) = \int \psi^{<}(\boldsymbol{y},t|\boldsymbol{x},s)\boldsymbol{u}^{<}(\boldsymbol{y},t)d\boldsymbol{y}. \tag{22}$$

Then using Eqs.(16) (17), (19) and (21) and noting that contributions from the boundaries vanish, we arrive at the following expression

$$\frac{\partial}{\partial t}\boldsymbol{v}^{<}(\boldsymbol{x},s|t) = -\int \psi^{<}\nabla p^{<}d\boldsymbol{y} + \int \psi^{<}(\nu + \mu_e *)\nabla^2\boldsymbol{u}^{<}d\boldsymbol{y} + \int \psi^{<}\boldsymbol{f}^{<}d\boldsymbol{y}. \tag{23}$$

The first term means the pressure force, and the second term the enhanced viscous force and the third term the random force acting on the fluid particle in the Lagrangian frame convected by the coarse graind velocity field. This equation is very similar to the one for the generalized velocity field

$\boldsymbol{v}(\boldsymbol{x}, s|t)$ in the MLRA. In eq.(23) μ_e represents the renormalized viscosity and the random force appears. Remember that in the MLRA the Lagrangian pressure gradient appears as memory decay of the fluid particle. The equation for $\boldsymbol{v}(\boldsymbol{x}, s|t)$ is physically equivalent to write

$$\frac{\partial}{\partial t}\boldsymbol{v}(\boldsymbol{x}, s|t) = -\eta(\boldsymbol{x}, s|t) * \boldsymbol{v}(\boldsymbol{x}, s|t) + \nu\nabla^2\boldsymbol{v}(\boldsymbol{x}, s|t) + \boldsymbol{f}(\boldsymbol{x}, s|t), \tag{24}$$

in the MLRA sense, from which the response equation for $\boldsymbol{G}$ in the MLRA is derived as

$$\frac{\partial}{\partial t}\boldsymbol{G}(\boldsymbol{x}, t|\boldsymbol{x}', s) = -\eta(\boldsymbol{x}, s|t) * \boldsymbol{G}(\boldsymbol{x}, t|\boldsymbol{x}', s) + \nu\nabla^2\boldsymbol{G}(\boldsymbol{x}, t|\boldsymbol{x}'s), \qquad \text{for } t \geq s. \tag{25}$$

The above story suggests that the response equation corresponding to eq.(23) would be of the form of

$$[\frac{\partial}{\partial \tau} + (\nu + \mu_e(k))k^2 + \eta^<(k)]G^<(k, \tau) = 0, \quad \tau = t - s, \tag{26}$$

$$\eta^<(k) = k\int_0^{k_m} q^2 J(q/k)Q(q)\int_0^\infty G^<(q, \tau)d\tau, \tag{27}$$

$$G^<(k, 0) = 1. \tag{28}$$

In fact, if we apply MLRA to the set of eqs.(16), (17), (19) and (20)-(23), we obtain the same equations (26)-(28). This means $\mathcal{LC}(\mathrm{N}_s(\boldsymbol{u}, \boldsymbol{u})) = \mathcal{CL}(\mathrm{N}_s(\boldsymbol{u}, \boldsymbol{u}))$.

For the SGS components we interpret that we have all the knowledge of the components, so that we can use full knowledge of equations of Navier-Stokes, continuity and Lagrangian position function. Correspondingly to this we can use the MLRA equations.

5 A Langevin equation model for two-dimensional turbulence

We have previously derived a simple Langevin equation model which contains the triple relaxation factor (7). Assuming $G(k, \tau) = \exp(-\eta(k)\tau)$ we obtained an approximate formula for θ_{kpq} as

$$\theta_{kpq} \;=\; (\eta(k) + \eta(p) + \eta(q))^{-1}, \tag{29}$$

$$\eta(k) \;\approx\; \left(\frac{3}{2}C'\int_0^k q^2 E(q)dq\right)^{1/2}, \tag{30}$$

where C' is the Kolmogorov constant in the inertial range. However, consideration in the previous section shows that η is altered to $\eta^<$ by the SGS effects and the viscosity is augumented by the renormalized eddy viscosity. This implies that the triple relaxation factor must be changed. In the following we present a new Langevin model for the LES equations.

Let us assume that the turbulence is steady and homogeneous and isotropic. The fundamental structure of the interactions between grid and subgrid scales is given by (19) and by the energy transfer function of the MLRA spectral equation (8). From eq.(26) the response is

$$G^<(k|k_m, \tau) = \exp\left(-(\nu k^2 + \mu_e(k|k_m)k^2 + \eta^<(k|k_m))\tau\right), \tag{31}$$

where the maximum wavenumber in grid scales k_m is explicitly written. Then, our Langevin equations for LES are

$$\left(\frac{\partial}{\partial t} + (\nu + \mu_e(k|k_m, t))k^2\right)\omega^<(\boldsymbol{k}, t) = -\frac{1}{2}\sum_{p \text{ and } q<k_m}^{\Delta}\left(\frac{1}{p^2} - \frac{1}{q^2}\right)(\boldsymbol{p}\times\boldsymbol{q})\omega^<(\boldsymbol{p}, t)\omega^<(\boldsymbol{q}, t) + f_\omega^>(\boldsymbol{k}, t), \tag{32}$$

$$\langle f_\omega^>(\boldsymbol{k}, t)f_\omega^>(-\boldsymbol{k}, s)\rangle = 2\delta(t - s)D(k|k_m, t), \tag{33}$$

where $f_\omega^{\gtrless}(k,t)$ is random vorticity source, and Gaussian and white in time. The function $D(k)$ is the spectral density function for the random vorticity source and should not be confused with the random force spectrum, which is $\pi k D(k)/k^2$. The eddy viscosity and random vorticity intensity are given by

$$\mu_e(k|k_m,t) = \frac{1}{4k^2}\left(\iint_{\Delta_1} dpdq\ \theta_{kpq}^{<><}(t) + \iint_{\Delta_2} dpdq\ \theta_{kpq}^{<>>}(t)\right)[b_{kpq}Q(q,t) + b_{kqp}Q(p,t)], \quad (34)$$

$$D(k|k_m,t) = \frac{1}{4}k^2\left(\iint_{\Delta_1} dpdq\ \theta_{kpq}^{<><}(t) + \iint_{\Delta_2} dpdq\ \theta_{kpq}^{<>>}(t)\right) a_{kpq}Q(p,t)Q(q,t), \quad (35)$$

$$a_{kpq} = b_{kpq} + b_{kqp},$$

$$\iint_{\Delta_1} dpdq = \int_{k_m}^{k_m+k} dp \int_{p-k}^{k_m} dq, \qquad \iint_{\Delta_2} dpdq = \int_{k_m}^{\infty} dp \int_{Min(k_m,p-k)}^{p} dq, \quad (36)$$

and

$$\theta_{kpq}^{<>>} = \left(\mu_e(k|k_m)k^2 + \eta^<(k|k_m) + \eta^>(p|k_m) + \eta^>(q|k_m)\right)^{-1}, \quad (37)$$

$$\theta_{kpq}^{<><} = \left(\mu_e(k|k_m)k^2 + \mu_e(q|k_m)q^2 + \eta^<(k|k_m) + \eta^>(p|k_m) + \eta^<(q|k_m)\right)^{-1}, \quad (38)$$

where time argument t is suppressed.

Our business is to fix $\eta^<(k|k_m), \eta^>(k|k_m)$ and $\mu_e(k|k_m)$. First consider $\eta^>(k|k_m)$. As previously stated that all the information for $v^>$ is assumed known, so that the response equation for $G^>(k,\tau)$ is given by that of the MLRA as

$$\begin{aligned} G^>(k,\tau) &= G_{MLRA}(k,\tau) \quad \text{for } k > k_m \\ &\approx \exp\left(-(\nu k^2 + \eta(k))\tau\right). \end{aligned} \quad (39)$$

Therefore $\eta^>(k|k_m)$ is identical to $\eta(k)$ for $k > k_m$, which means

$$\begin{aligned} \eta^>(k|k_m) &= k\int_0^\infty q^2 J(q/k)Q(q)\int_0^\infty G(q,\tau)d\tau, \\ &= k\left(\int_0^{k_m} q^2 J(q/k)Q(q)\int_0^\infty G^<(q,\tau)d\tau + \int_{k_m}^\infty q^2 J(q/k)Q(q)\int_0^\infty G^>(q,\tau)d\tau\right) \\ &\approx \frac{3}{2}\int_0^{k_m}\frac{q^2E(q)\ dq}{\mu_e(q|k_m)q^2 + \eta^<(q|k_m)} + \frac{3}{2}\int_{k_m}^{k}\frac{q^2E(q)\ dq}{\nu q^2 + \eta^>(q|k_m)} + \frac{3}{2}k^2\int_k^\infty \frac{E(q)\ dq}{\nu q^2 + \eta^>(q|k_m)}, \end{aligned} \quad (40)$$

where eq.(9) is used.

Next consider $\eta^<(k|k_m)$. From eqs.(27) and (31) we obtain

$$\begin{aligned} \eta^<(k|k_m) &\approx k\int_0^{k_m} J(q/k)\frac{q^2Q(q)}{\mu_e(q|k_m)q^2 + \eta^<(q|k_m)} \\ &\approx \frac{3}{2}\int_0^k \frac{q^2E(q)\ dq}{\mu_e(q|k_m)q^2 + \eta^<(q|k_m)} + \frac{3}{2}k^2\int_k^{k_m}\frac{E(q)\ dq}{\mu_e(q|k_m)q^2 + \eta^<(q|k_m)}. \end{aligned} \quad (41)$$

Functions $\eta^<(k|k_m), \eta^>(k|k_m)$ and $\mu_e(k|k_m)$ are given by the solution of equations of (34), (40) and (41) with (37) and (38). However, it is a very difficult task to find explicit from of the functions. By considering practical use and from our experience that the eddy viscosity $\mu_e(k|k_m)$ tends to a constant for $k \ll k_m$ while it has a cusp as k approaches k_m, it is expected that to approximate μ_e as an addition of two limitting cases could represent the fundamental nature of μ_e. In the following, therefore, we write μ_e as

$$\mu_e(k|k_m) \approx \mu_e^{nl}(k|k_m) + \mu_e^{l}(k|k_m), \tag{42}$$

$$\mu_e^{nl}(k|k_m) = \mu_e(k|k_m), \quad (\text{ for } k \ll k_m) \quad \mu_e^{l}(k|k_m) = \mu_e(k|k_m), \quad (\text{ for } \Delta k = k_m - k \ll k \leq k_m) \tag{43}$$

where superscripts ^{nl} denotes "non-local" in the sense that k is distant from k_m and ^{l} "local" for k close to k_m.

Non-local term

In this limit, $k \ll p \sim q$. We put $q = p + ku$ and expand the integrand of (34) in power of $k/p \ll 1$. It is found that to the leading order the integral becomes

$$\begin{aligned} \mu_e^{nl}(k|k_m) &\approx \frac{1}{4k^2} \iint_{\Delta_1+\Delta_2} dpdq\ \theta_{kpq}^{<>>} [b_{kpq}Q(q) + b_{kqp}Q(p)] \\ &\approx \frac{1}{4} \int_{k_m}^{\infty} \theta_{kpp}^{<>>} \frac{\partial}{\partial p} (pE(p))\ dp, \end{aligned} \tag{44}$$

which is similar to Kraichnan's (1976).

Local term

Since $(k_m - k =)\Delta k \ll k < k_m < p$, dominant contributions come from the term $b_{kpq}\theta_{kpq}^{<><}Q(q)$ in the integral over the domain Δ_1. The energy is contained at low q region. Expanding the metric in powers of q/k ($q \sim \Delta k$) and approximating as $\theta_{kpq}^{<><} \approx \theta_{kk_m0}^{<><}$ and doing tedious calculation, we obtain

$$\begin{aligned} \mu_e^{l}(k|k_m) &\approx \frac{1}{4k^2} \theta_{kk_m0}^{<><} \iint_{\Delta_1} dpdq\ b_{kpq}Q(q) \\ &\approx C_1 \left(\frac{k_m}{k}\right)^2 \theta_{kk_m0}^{<><} \int_{\Delta k}^{k_m} E(q)\ dq, \qquad C_1 = \frac{1}{2}\left(\frac{1}{\pi} - \frac{1}{6}\right). \end{aligned} \tag{45}$$

6 Enstrophy cascading range

6.1 Eddy viscosity and Eddy damping factor

Now we consider the eddy viscosity μ_e, the eddy damping factor $\eta^<$ and $\eta^>$, and the random force intensity in the enstrophy cascading range. The energy spectrum is of the form of $E(k) \propto k^{-\delta}, 3 < \delta \lesssim 4$. For sufficiently high Reynolds number the energy spectrum is given by

$$E(k) = C'\beta^{2/3}k^{-3}[\ln(k/k_0)]^{-1/3}, \quad C' = 1.81 \quad (\text{Kaneda 1987, Gotoh 1989}) \tag{46}$$

where β is the average rate of dissipation of the enstrophy and k_0 is the bottom wavenumber of the range. In this range, it is well known that the wavenumber dependency of the eddy damping factor η is weak and so does for θ_{kpq}. This means that

$$\mu_e(k|k_m) \approx -\frac{1}{4}k_m E(k_m)\theta_{kk_mk_m}^{<>>} + C_1 \left(\frac{k_m}{k}\right)^2 \theta_{kk_m0}^{<><} \int_{\Delta k}^{k_m} E(q)\ dq. \tag{47}$$

The first term is negative, the well known fact of the negative viscosity (Kraichnan 1976). But this value is very small when k_m is large, which is also consistent with Kraichnan's argument (Kraichnan 1976).

When $k \ll k_m$ such that the second term of (47) is negligible, μ_e becomes

$$\mu_e(k|k_m) \approx -\frac{1}{4}k_m E(k_m)\theta^{<>>}_{kk_mk_m}. \tag{48}$$

In order to find $\theta^{<>>}_{kk_mk_m}$, we consider first $\eta^>$. In eq.(40), the second and third terms are small compared to the first term (note that k in eq.(40) should be understood as $k \geq k_m$, so we write p in the following equation). Then we have

$$\eta^>(p|k_m) \approx \frac{3}{2}\int_0^{k_m} \frac{q^2 E(q)\,dq}{\mu_e(q|k_m)q^2 + \eta^<(q|k_m)} = \eta^>(k_m|k_m) = \eta^<(k_m|k_m), \tag{49}$$

which is a constant. Note that the right most equality of (49) is derived by using eq.(41). Next consider $\eta^<$. For small $k \ll k_m$, $\eta^<(k|k_m)$ is dominated by the second term of eq.(41) and the integral becomes finite, so that we can write

$$\eta^<(k|k_m) = k^2\zeta(k|k_m), \qquad \zeta(k|k_m) = \frac{3}{2}\int_k^{k_m} \frac{E(q)\,dq}{\mu_e(q|k_m)q^2 + \eta^<(q|k_m)}. \tag{50}$$

We see that $\zeta(k|k_m)$ tends to a constant and $\eta^<(k|k_m)$ vanishes as k tends to 0. Since $\mu_e(k|k_m)$ also tends to a constant $\mu_e(0|k_m)$, $\eta^<(k|k_m)$ can be written as

$$\eta^<(k|k_m) \approx \frac{3}{2}\frac{k^2}{\mu_e(0|k_m) + \zeta(0|k_m)}\int_k^{k_m} \frac{E(q)}{q^2}\,dq. \tag{51}$$

From eqs.(48) and (51) we have

$$\mu_e(0|k_m) \approx -\frac{1}{8}\frac{k_m E(k_m)}{\eta^<(k_m|k_m)} < 0. \tag{52}$$

Putting $k = 0$ in eq.(51) we obtain the quadratic equation for $\zeta(0|k_m)$. Since $\mu_e(0|k_m)$ is very small compared to the integral factor in eq.(51), $\zeta(0|k_m)$ is given by

$$\zeta(0|k_m) \approx \left(\frac{3}{2}\int_0^{k_m} \frac{E(q)}{q^2}\,dq\right)^{1/2}. \tag{53}$$

When $k_m - k \ll k \leq k_m$, the second term of eq.(47) is dominant:

$$\mu_e(k|k_m) \approx C_1\left(\frac{k_m}{k}\right)^2 \frac{1}{\mu_e(k|k_m)k^2 + \eta^<(k|k_m) + \eta^<(k_m|k_m)}\int_{\Delta k}^{k_m} E(q)\,dq. \tag{54}$$

Since $\eta^<(k|k_m)$ is finite and $\mu_e(k|k_m)$ becomes cusp and dominates $\eta^<$ as $k \to k_m$, we can write eq.(54) as

$$\mu_e(k|k_m) \approx C_1\left(\frac{k_m}{k}\right)^2 \frac{1}{\mu_e(k|k_m)k^2}\int_{\Delta k}^{k_m} E(q)\,dq, \tag{55}$$

from which we obtain

$$\mu_e(k|k_m) \approx \left(\frac{k_m}{k}\right)\left(C_1\int_{\Delta k}^{k_m} E(q)\,dq\right)^{1/2}. \tag{56}$$

Suppose that $E(k) \propto k^{-\delta}$. Then the integral is dominated by the lower boundary contributions, so that $\mu_e(k|k_m) \propto (\Delta k)^{(1-\delta)/2}(3 < \delta \lesssim 4)$, which is consistent with the cusp behavior of $\mu_e(k|k_m)$ for k close to k_m. For $\delta = 3$ we have the cusp exponent -1 which is identical to the one by Kraichnan (1976).

For $\eta^<(k|k_m)$, the first term in eq.(41) yields

$$\begin{aligned}\eta^<(k|k_m) &\approx \frac{3}{2}\int_0^k \frac{q^2E(q)\,dq}{\mu_e(q|k_m)q^2+\eta^<(q|k_m)}\\ &\approx \left(\int_0^{k-\Delta k} dq + \int_{k-\Delta k}^{k_m} dq\right) \times \frac{q^2E(q)}{\mu_e(q|k_m)q^2+\eta^<(q|k_m)}.\end{aligned} \tag{57}$$

Since $\mu_e(k|k_m)$ is very small for small k and rapidly increases near k_m, μ_e in the first integral is negligible while $\eta^<$ can be neglected in the second integral. We have seen that for the energy spectrum of $E(k) \propto k^{-\delta}$ $\mu_e(k|k_m) \propto (\Delta k)^{(1-\delta)/2}$. Therefore the second integral in eq.(57) is of the order of $(\Delta k)^{(1+\delta)/2}$ and negligible when compared to the first term. Since wavenumber dependency of $\eta^<(k|k_m)$ is weak, we may put $\eta^<$ outside the integral, finally we obtain

$$\eta^<(k|k_m) \approx \left(\frac{3}{2}\int_0^k q^2E(q)\,dq\right)^{1/2}. \tag{58}$$

6.2 Random vorticity intensity

The random vorticity intensity $D(k)$ is similarly computed as μ_e. Correspondingly to eqs.(42) and (43) we have

$$D(k|k_m) \approx D^{nl}(k|k_m) + D^l(k|k_m), \tag{59}$$

$$D^{nl}(k|k_m) = D(k|k_m), \quad (\text{ for } k \ll k_m), \qquad D^l(k|k_m) = D(k|k_m), \quad (\text{ for } k_m - k \ll k \le k_m) \tag{60}$$

Non-local term

For small $k \ll k_m$, the non-local term D^{nl} becomes

$$D^{nl} \approx \frac{1}{4}k^2 \iint_{\Delta_1+\Delta_2} dpdq\ a_{kpq}\theta_{kpq}^{<>>}Q(p)Q(q). \tag{61}$$

Substituting $q = p + ku$ into eq.(61) and expanding the resulting equation in the power of k/p, we obtain

$$\begin{aligned} D^{nl} &\approx \frac{1}{2\pi}k^4 \int_{k_m}^{\infty} \theta_{kpp}^{<>>}\frac{E^2(p)}{p}\,dp\\ &\approx \frac{1}{2\pi}k^4 \int_{k_m}^{\infty} \frac{1}{\mu_e(k|k_m)k^2+\eta^<(k|k_m)+2\eta^<(k_m|k_m)}\frac{E^2(p)}{p}dp\\ &\approx \frac{1}{4\pi}\frac{k^4}{\eta^<(k_m|k_m)}\int_{k_m}^{\infty}\frac{E^2(p)}{p}\,dp, \end{aligned} \tag{62}$$

where the fact was used that at low k, $\mu_e(k|k_m)k^2+\eta^<(k|k_m)$ becomes negligibly small when compared to $\eta^<(k_m|k_m)$. The integral is over the wavenumbers larger than k_m and the excitation in the range is significantly weak, so that the contribution of this non-local term to $D(k)$ is negligibly small. This is consistent with Kraichnan's results (1976).

Local term

For $k_m - k \ll k \le k_m$, the local term D^{nl} is

$$\begin{aligned} D^l &\approx \frac{1}{4}k^2\iint_{\Delta_1} dpdq\ a_{kpq}\theta_{kpq}^{<>>}Q(p)Q(q)\\ &\approx \frac{1}{4}k^2\int_{\Delta k}^{k_m}dq\int_{k_m}^{q+k}dp\frac{a_{kpq}Q(p)Q(q)}{\mu_e(k|k_m)k^2+\eta^<(k|k_m)+\eta^>(p|k_m)+\mu_e(q|k_m)q^2+\eta^<(q|k_m)}. \end{aligned} \tag{63}$$

Since the most contribution comes from $q \sim k_0 \ll k_m$ (k_0: the upper wavenumber below which most energy stays) and $p \geq k_m$ and $k \lesssim k_m$, $\mu_e(k|k_m)k^2$ term is dominant in the triple relaxation factor. Noting that $a_{kpq} \approx 4kp\sqrt{1-y^2}$, we approximate eq.(63) as

$$\begin{aligned} D^l &\approx \frac{1}{4\mu_e(k|k_m)}\int_{\Delta k}^{k_m} dq \int_{k_m}^{q+k} dp\ a_{kpq} Q(p)Q(q) \\ &\approx \frac{k_m}{\mu_e(k|k_m)}\int_{\Delta k}^{k_m} Q(q)dq \int_{k_m}^{q+k} \sqrt{1-y^2}\ pQ(p)\ dp. \end{aligned} \tag{64}$$

To further simplify the last integral in eq.(64), we write the integral over p as

$$\int_{k_m}^{q+k} p\sqrt{1-y^2}Q(p)\ dp \approx \frac{1}{\pi}\int_0^{\pi} \sin\alpha d\alpha \int_{k_m}^{2k_m} E(p)\ dp = \frac{2}{\pi}\int_{k_m}^{2k_m} E(p)\ dp. \tag{65}$$

Then the term D^l becomes

$$D^l \approx \frac{2}{\pi^2}\frac{k_m}{\mu_e(k|k_m)}\int_{\Delta k}^{k_m} \frac{E(q)}{q}\ dq \int_{k_m}^{2k_m} E(p)\ dp. \tag{66}$$

For the energy spectrum $E(k) \propto k^{-\delta}$, the integral over q increases as $(\Delta k)^{-\delta}$ for k near k_m, so that $D^l \propto (\Delta k)^{-(1+\delta)/2}$, again consistent with Kraichnan's results (1976).

7 Summary

Let us summarize the formula. In the enstrophy cascading range, our Langevin equation model is given by eqs.(32), (33), (34), (35), and

$$\begin{aligned} \mu_e(k|k_m) &= -\frac{1}{4}\frac{k_m E(k_m)}{\eta^<(k|k_m) + 2\eta^<(k_m|k_m)} + k_m^{-1}\left(C_1\int_{\Delta k}^{k_m} E(q)\ dq\right)^{1/2}, \qquad (67) \\ D(k|k_m) &= \frac{1}{4\pi}\frac{k^4}{\eta^<(k_m|k_m)}\int_{k_m}^{\infty}\frac{E^2(p)}{p}\ dp + \frac{2}{\pi^2}\frac{k_m}{\mu_e(k|k_m)}\int_{\Delta k}^{k_m}\frac{E(q)}{q}\ dq\int_{k_m}^{2k_m} E(p)\ dp, \qquad (68) \\ \eta^<(k|k_m) &= \frac{3}{2}\frac{k^2}{\mu_e(0|k_m)+\zeta(0|k_m)}\int_k^{k_m}\frac{E(q)}{q^2}\ dq + \left(\frac{3}{2}\int_0^k q^2 E(q)\ dq\right)^{1/2}, \qquad (69) \end{aligned}$$

and

$$\begin{aligned} \eta^<(k_m|k_m) &= \left(\frac{3}{2}\int_0^{k_m} q^2 E(q)\ dq\right)^{1/2}, \qquad (70) \\ \mu_e(0|k_m) &= -\frac{1}{8}\frac{k_m E(k_m)}{\eta^<(k_m|k_m)}, \qquad (71) \\ \zeta(0|k_m) &= \left(\frac{3}{2}\int_0^{k_m}\frac{E(q)}{q^2}dq\right)^{1/2}. \qquad (72) \end{aligned}$$

Note that all the quantities are given as the explicit form of the energy spectrum, which means that numerical implimentation is much easier than the Langevin equation models ever proposed. This is greatly useful for practical applications. In Bertogrio (1988) and Chasnov (1991), $\eta(k|k_m)$ is expressed by the total strain below the wavenumber k. In the present results, $\eta^<(k|k_m)$ has two contributions, the one from the excitation from lower wavenumbers than k and the other from higher wavenumbers than k. This is more physical.

References

J. P. Bertoglio: *Macroscopic modelling of Turbulent Flows* ed. by U. Frisch and J.B. Keller. pp.100 (1984) Springer.

J. R. Chasnov: Phys. Fluids **A 3** (1991) 188.

T. Gotoh, Y. Kaneda and N. Bekki: J. Phys. Soc. Jpn., **57** (1988) 866.

T. Gotoh: J. Phys. Soc. Japan, **58** (1989) 2365.

Y. Kaneda: J. Fluid Mech. **107** (1981) 131.

Y. Kaneda: Phys. Fluids **29** (1986) 701.

R. H. Kraichnan: J. Atmos. Sci. **33** (1976) 1521.