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渦輪の線形不安定性: 曲率の効果

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概要

渦輪を軸対称のまま保つのは不可能で，形成直後から振動を開始し，ときに崩壊に至ることもある．不安定性のメカニズムとして,「Widnall の不安定」が有名である：

細い渦輪は局所的には円柱渦で，自身の上に誘導する局所歪み流によって断面がわずかに楕円形に変形している，と考える．断面が円形の渦は中立安定である．とくに，渦度分布が一様のものが「Rankine の渦」で，その上に立つ無限個の固有振動モードは「Kelvin 波」とよばれる．渦度ベクトルに垂直な断面上に渦中心を原点とする極座標 (r,θ) を導入しよう．歪み流は $\cos 2\theta$ によって特徴づけられる「四重極子流」である．この四重極子場を介して，Kelvin 波のうちの右巻き・左巻きの屈曲モード ($e^{i\theta}$, $e^{-i\theta}$ 成分) が共鳴を起こして指数関数的に成長する，というのがそのシナリオである．

「はたして，渦輪の不安定メカニズムはこれだけであろうか？」 Euler 方程式の漸近解を求めてみると，四重極子成分より「双極子 $(\cos\theta)$」 成分の方が卓越するのである．すなわち，渦核半径とリング半径の比 ϵ を微小パラメータとする展開において，$O(\epsilon^0)$ が Rankine 渦で，$O(\epsilon)$ の双極子場が続き，その後が $O(\epsilon^2)$ の四重極子場である．

ハミルトン力学系における Krein の理論によると，単独の Kelvin モードに微小摂動を加えても不安定化することはなく，少なくとも 2 個のモードの波数 k と周波数 ω が一致して始めて不安定化が可能になる．すなわち，(k,ω) 平面上に描かれた分散関係を表す 2 本以上の曲線が交差しなければならない．摂動が双極子流である場合には，$e^{im\theta}$, $e^{in\theta}$ 型の角度依存性をもつ 2 つの Kelvin モードが，関係 $|m-n|=1$ を満たすときに限って，分散曲線の交点上で共鳴を起こし得る．ちなみに，摂動が四重極子流の場合は，$|m-n|=2$ が共鳴に必要な条件になる．

本研究では，$O(\epsilon)$ までの定常解，すなわち,「Kelvin の渦輪」を基本流にとり，その上に加えられた「屈曲モード ($m=1$ または $m=-1$) と軸対称モード ($n=0$) の間で共鳴不安定が起こる」ことを示す．

1 Introduction

Vortex rings are invariably susceptible to wavy distortions, leading sometimes to disruption. We revisit the linear stability problem of a thin vortex ring. It is widely accepted that the *Widnall instability* is responsible for development of unstable waves. This is an instability for a straight vortex tube subjected to a straining field in a plane perpendicular to the tube axis (Moore & Saffman 1975, Tsai & Widnall 1976).

When viewed locally, a thin vortex ring looks like a straight tube. We confine ourselves to a circular core of uniform vorticity, that is, the *Rankine vortex.* This circular-cylindrical vortex tube supports a family of neutrally stable waves of infinitesimal amplitude, being well known as the *Kelvin waves.* The vortex ring induces, on itself, not only a local uniform flow but also a local straining field akin to a pure shear (Widnall & Tsai 1977). A pure shear with the principal axes perpendicular to the vorticity deforms the core into an ellipse. This is a quadrupole field proportional to $\cos 2\theta$ or $\sin 2\theta$, in terms of polar coordinates (r, θ) in the meridional plane, and is capable of mediating a parametric resonance between the bending waves of left- and right-handed. The Widnall instability has a wider applicability; the influence of neighbouring vortices is, in the leading-order approximation, represented by a linear shear flow.

However, the previous treatment has disregarded an ingredient peculiar to a curved vortex filament. The solution of the Navier-Stokes equations, obtained by using the matched asymptotic expansions in a small parameter ϵ, the ratio of the core to the ring radii, starts with a circular-cylindrical vortex tube, at $O(\epsilon^0)$. Then a ***dipole field*** proportional to $\cos\theta$ follows at $O(\epsilon^1)$. The quadrupole field proportional to $\cos 2\theta$ comes as a higher-order correction at $O(\epsilon^2)$ (Fukumoto & Moffatt 2000). The same is true of inviscid vortex rings. The dipole field does not have attracted as much attention as it deserves. This paper addresses a possible instability that the dipole field at $O(\epsilon)$ can trigger.

According to Krein's theory of parametric resonance in Hamiltonian systems (MacKay 1986), a single Kelvin mode cannot be fed by perturbations breaking the circular symmetry. An instability becomes permissible only for a superposition of at least two modes with the same wavenumber and the same frequency. Subjected to the dipole field, two Kelvin modes with angular dependence $e^{im\theta}$ and $e^{in\theta}$ can together be amplified at the intersection points of dispersion curves if the condition $|m-n| = 1$ is met and if the energies of the disturbance modes are of opposite signs, with one positive and the negative.

As a first step, we investigate a parametric resonance that may occur between *axisymmetric* ($m = 0$) and *bending* ($n = 1$ or $n = -1$) modes in the presence of the dipole field. In §2, we give a concise description of Kelvin's vortex ring and of the setting of linear stability analysis. In §3, the Kelvin waves are recalled. With this preliminary, equations of disturbances at $O(\epsilon)$ are written out in §4 and are solved in §5. In §6, the growth rate is calculated and a comparison is made with that of the Widnall instability.

2 Kelvin's vortex ring and setting of stability problem

We write down the flow field associated with Kelvin's vortex ring, a thin axisymmetric vortex ring with vorticity proportional to the distance from the axis of symmetry which propagates steadily in an incompressible inviscid fluid. The detail is found, for example, in Widnall & Tsai (1977). Our assumption reads that the ratio ϵ of the core radius σ to the ring radius R is very small:

$$\epsilon = \sigma / R \ll 1\,. \tag{2.1}$$

Introduce local cylindrical coordinates (r, θ) in the meridional plane, fixed to the ring, with the origin $r = 0$ maintained at the center of the circular core and with the angle θ measured from the direction parallel to the axis of symmetry.

The radial coordinate r is normalized by the core radius σ. The velocity is normalized by the maximum azimuthal velocity $\Gamma/2\pi\sigma$. Here Γ is the circulation carried by the ring. Let the r and θ components of velocity field be U and V, and the pressure be P inside the core $(r < 1)$. We denote the velocity potential for the exterior irrotational flow by Φ.

The basic flow is expanded in powers of ϵ, the first-order truncation of which provides us with Kelvin's vortex ring:

$$U = \epsilon U_1(r,\theta) + \cdots, \quad V = V_0(r) + \epsilon V_1(r,\theta) + \cdots, \tag{2.2}$$

$$P = P_0(r) + \epsilon P_1(r,\theta) + \cdots \quad \text{for } r < 1, \tag{2.3}$$

$$\Phi = \Phi_0(\theta) + \epsilon \Phi_1(r,\theta) + \cdots \quad \text{for } r > 1. \tag{2.4}$$

The leading-order flow is the Rankine vortex which is written, in dimensionless form, as

$$V_0 = r, \quad P_0 = \frac{1}{2}(r^2 - 1), \quad \Phi_0 = \theta. \tag{2.5}$$

At $O(\epsilon)$, the effect is curvature is called into play, and the flow field takes the following form:

$$U_1 = \frac{5}{8}(1 - r^2)\cos\theta, \quad V_1 = \left(-\frac{5}{8} + \frac{7}{8}r^2\right)\sin\theta, \quad P_1 = \left(-\frac{5}{8}r + \frac{3}{8}r^3\right)\sin\theta, \tag{2.6}$$

$$\Phi_1 = \left(\frac{1}{8}r - \frac{3}{8r} - \frac{1}{2}r\log r\right)\cos\theta. \tag{2.7}$$

To this order, the boundary shape remains to be circular $(r = 1)$. The pattern of streamlines in the exterior region $(r > 1)$ resembles that of the flow past a circular cylinder.

We inquire into evolution of three-dimensional disturbances of infinitesimal amplitude superposed on the above steady flow. We measure the centerline penetrating the torus with arclength s, and denote the toroidal component of disturbance velocity by w. Following the prescription of Moore & Saffman (1975) and Tsai & Widnall (1976), we pose the following form for disturbances velocity $\tilde{\boldsymbol{v}}$:

$$\tilde{\boldsymbol{v}} = (\boldsymbol{v}_0 + \epsilon \boldsymbol{v}_1 + \cdots)e^{i(ks - \omega t)}, \tag{2.8}$$

and in a similar way for disturbance pressure $\tilde{p}$. The wavenumber k and the frequency ω are also expanded in powers of ϵ as

$$k = k_0 + \epsilon k_1 + \cdots, \quad \omega = \omega_0 + \epsilon\omega_1 + \cdots. \tag{2.9}$$

The boundary of the core is disturbed as

$$r = 1 + \tilde{f}_0(\theta, s, t) + \epsilon \tilde{f}_1(\theta, s, t) + \cdots. \tag{2.10}$$

3 The Kelvin waves

At $O(\epsilon^0)$, the stability problem is reduced to oscillations of the Rankine vortex whose study is traced back to Kelvin (1880). The circular core of constant vorticity is neutrally stable. The waves of form $e^{i(m\theta+k_0 s-\omega_0 t)}$ on it, called the *Kelvin waves*, obeys the following dispersion relation:

$$a_m(\omega_0, k_0) = -i(\omega_0 - m)K_{|m|}(k_0)A_m + k_0 K'_{|m|}(k_0)J_{|m|}(\eta_m) = 0\,, \tag{3.1}$$

where $J_{|m|}$ and $K_{|m|}$ are, respectively, the Bessel function of the first kind and the modified Bessel function of the second kind, both with order $|m|$, a prime designates its differentiation, and

$$\eta_m^2 = \left[\frac{4}{(\omega_0 - m)^2} - 1\right] k_0^2\,, \tag{3.2}$$

$$A_m = \frac{i(\omega_0 - m)\eta_m J_{|m|-1}(\eta_m) + i|m|\left[-\omega_0 + m\left(1 - \frac{2}{|m|}\right)\right] J_{|m|}(\eta_m)}{4 - (\omega_0 - m)^2}\,. \tag{3.3}$$

For later use, we write down the eigenfunctions $\boldsymbol{v}_0 = \boldsymbol{v}_0^m(r)e^{im\theta}$, $p_0 = \pi_0^m(r)e^{im\theta}$, $\phi_0 = \phi_0^m(r)e^{im\theta}$ and $f_0 = f_0^m e^{im\theta}$ for the axisymmetric ($m = 0$) and the bending ($m = 1$) modes. Here the superscript m stands for azimuthal wavenumber.
For $m = 0$,

$$u_0^0 = -\frac{i\omega_0}{4 - \omega_0^2}\eta_0 J_1(\eta_0 r)\delta_0\,, \quad v_0^0 = -\frac{2}{4 - \omega_0^2}\eta_0 J_1(\eta_0 r)\delta_0\,, \quad w_0^0 = \frac{k_0}{\omega_0}J_0(\eta_0 r)\delta_0\,, \tag{3.4}$$

$$\pi_0^0 = J_0(\eta_0 r)\delta_0\,, \quad \phi_0^0 = K_0(k_0 r)\gamma_0\,, \quad f_0^0 = \frac{1}{4 - \omega_0^2}\eta_0 J_1(\eta_0)\delta_0\,, \tag{3.5}$$

where δ_0 and γ_0 are constants constrained by

$$\gamma_0 = -i\frac{J_0(\eta_0)}{\omega_0 K_0(k_0)}\delta_0\,, \tag{3.6}$$

but otherwise arbitrary.
For $m = 1$,

$$u_0^1 = \left\{-\frac{i}{2}\left(\frac{1}{\omega_0 - 1} + \frac{1}{\omega_0 - 3}\right)\eta_1 J_0(\eta_1 r) + \frac{i}{\omega_0 - 3}\frac{J_1(\eta_1 r)}{r}\right\}\beta_0\,, \tag{3.7}$$

$$v_0^1 = \left\{\frac{1}{2}\left(\frac{1}{\omega_0 - 1} - \frac{1}{\omega_0 - 3}\right)\eta_1 J_0(\eta_1 r) + \frac{1}{\omega_0 - 3}\frac{J_1(\eta_1 r)}{r}\right\}\beta_0\,, \tag{3.8}$$

$$w_0^1 = \frac{k_0}{\omega_0 - 1}J_1(\eta_1 r)\beta_0\,, \quad \pi_0^1 = J_1(\eta_1 r)\beta_0\,, \quad \phi_0^1 = K_1(k_0 r)\alpha_0\,, \tag{3.9}$$

$$f_0^1 = \frac{1}{\omega_0 - 3}\left\{\frac{1}{\omega_0 + 1}\eta_1 J_0(\eta_1) + \frac{1}{\omega_0 - 1}J_1(\eta_1)\right\}\beta_0\,, \tag{3.10}$$

where α_0 and β_0 are constants constrained by

$$\alpha_0 = -i\frac{J_1(\eta_1)}{(\omega_0 - 1)K_1(k_0)}\beta_0\,. \tag{3.11}$$

4 Equations for $O(\epsilon)$ disturbance field

The neutral stability of the Rankine vortex is attributed to the circular (S^1-) symmetry about the cylinder axis. At $O(\epsilon)$, the effect of curvature makes its appearance as the dipole field and this field breaks the S^1-symmetry. The disturbance velocity $\boldsymbol{v}_1 e^{i(ks-\omega t)}$ and the disturbance pressure $\pi_1 e^{i(ks-\omega t)}$ at $O(\epsilon)$ inside the core ($r<1$) are governed by

$$-i\omega_0 u_1 + \frac{\partial u_1}{\partial r} - 2v_1 + \frac{\partial \pi_1}{\partial r} = \left(i\omega_1 - \frac{\partial U_1}{\partial r}\right) u_0 - U_1\frac{\partial u_0}{\partial r} - \frac{V_1}{r}\frac{\partial u_0}{\partial \theta} - \left(\frac{1}{r}\frac{\partial U_1}{\partial \theta} - \frac{2V_1}{r}\right) v_0\,, \tag{4.1}$$

$$\begin{aligned}-i\omega_0 v_1 + 2u_1 + \frac{\partial v_1}{\partial \theta} + \frac{1}{r}\frac{\partial \pi_1}{\partial \theta} &= \left(i\omega_1 - \frac{1}{r}\frac{\partial V_1}{\partial \theta} - \frac{U_1}{r}\right) v_0 - \left(\frac{\partial V_1}{\partial r} + \frac{V_1}{r}\right) u_0 \\ &\quad - U_1\frac{\partial v_0}{\partial r} - \frac{V_1}{r}\frac{\partial v_0}{\partial \theta}\,,\end{aligned} \tag{4.2}$$

$$-i\omega_0 w_1 + \frac{\partial w_1}{\partial \theta} + ik_0\pi_1 = -ik_1\pi_0 + (i\omega_1 - r\cos\theta)w_0 - \frac{V_1}{r}\frac{\partial w_0}{\partial \theta} - U_1\frac{\partial w_0}{\partial r} + ik_0 r\sin\theta\pi_0\,, \tag{4.3}$$

$$\frac{\partial u_1}{\partial r} + \frac{u_1}{r} + \frac{1}{r}\frac{\partial v_1}{\partial \theta} + ik_0 w_1 = -\sin\theta u_0 - \cos\theta v_0 + ik_0 r\sin\theta w_0\,. \tag{4.4}$$

The last one is the equation of continuity. The velocity potential $\phi_1 e^{i(ks-\omega t)}$ for the disturbance flow outside the core ($r>1$) satisfies, at $O(\epsilon)$,

$$\frac{\partial^2 \phi_1}{\partial r^2} + \frac{1}{r}\frac{\partial \phi_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi_1}{\partial r^2} - k_0^2\phi_1 = 2k_0k_1\phi_0 - \sin\theta\frac{\partial \phi_0}{\partial r} - \frac{\cos\theta}{r}\frac{\partial \phi_0}{\partial \theta} - 2k_0^2 r\sin\theta\phi_0\,. \tag{4.5}$$

The boundary conditions require that the normal component of velocity and the pressure be continuous across the interface ($r=1$) of the core:

$$u_1 = \frac{\partial \phi_1}{\partial r}\,, \tag{4.6}$$

$$\pi_1 - i(\omega_0 - m)\phi_1 = i\omega_1\phi_0 - \frac{\partial \Phi_0}{\partial r}\frac{\partial \phi_0}{\partial r}\,. \tag{4.7}$$

In view of the Fourier modes $\cos\theta$ and $\sin\theta$ characterizing the dipole field U_1 and V_1, the disturbance fields of the modes $e^{im\theta}$ and $e^{in\theta}$ can afford to cooperate with each other to grow themselves if the difference of the azimuthal wavenumber $|m-n|=1$, and both the frequency and the axial wavenumber coincide with each other.

It is illustrative to carry throuth a calculation for the case of $m=0$ and $n=1$. The leading-order disturbance velocity $\boldsymbol{v}_0 e^{i(k_0 s-\omega_0 t)}$, thus the disturbance pressure $\pi_0 e^{i(k_0 s-\omega_0 t)}$ and the disturbance-velocity potential ϕ_0 as well, consist of a superposition of the axisymmetric and right-handed bending waves:

$$\boldsymbol{v}_0 = \boldsymbol{v}_0^0\delta_0 + \boldsymbol{v}_0^1 e^{i\theta}\beta_0\,, \quad \pi_0 = \pi_0^0\delta_0 + \pi_0^1 e^{i\theta}\beta_0\,, \qquad \text{for } r<1 \tag{4.8}$$

$$\phi_0 = K_0(k_0 r)\gamma_0 + K_1(k_0 r)e^{i\theta}\alpha_0\,, \qquad \text{for } r>1 \tag{4.9}$$

It follows from (4.1)–(4.4) that four Fourier modes with 1, $e^{\pm i\theta}$ and $e^{2i\theta}$ are excited at $O(\epsilon)$. The value of ω_1, depending on being non-real or real, tells us whether the parametric resonance instability occurs or not. To this aim, it suffices to look into, at $O(\epsilon)$, again the axisymmetric ($m=0$) and the bending ($m=1$) modes.

5 Waves at $O(\epsilon)$

Upon substituting (2.6) and (4.8) into (4.1)–(4.4), we obtain equations for the axisymmetric and bending waves at $O(\epsilon)$. The axisymmrtric wave at $O(\epsilon)$ is denoted by $\boldsymbol{v}_1^0=(u_1^0,v_1^0,w_1^0)$, π_1^0 $(r<1)$ and ϕ_1^0 $(r>1)$. The bending wave at $O(\epsilon)$ is denoted by $\boldsymbol{v}_1^1e^{i\theta}=(u_1^1,v_1^1,w_1^1)e^{i\theta}$, $\pi_1^1e^{i\theta}$ $(r<1)$ and $\phi_1^1e^{i\theta}$ $(r>1)$. A general solution of the velocity potential ϕ_1^0 and ϕ_1^1 is readily available. The Euler equations for $r<1$ are reduced to a second-order ordinary differential equation with inhomogeneous terms for π_1^0 and π_1^1. A general solution is obtainable in terms of the Bessel functions, from which the velocity components are manipulated. We omit the detail of a lengthy calculation and present the general solution such that the disturbance velocity is finite at $r=0$ and vanishes at infinity.

For $m=0$,

$$\begin{aligned}u_1^0=&\frac{i\omega_0}{\omega_0^2-4}\eta_0J_1(\eta_0r)\delta_1-i\left\{\frac{i\omega_1}{\omega_0^2-4}\left[\frac{4k_0^2}{\omega_0^2}rJ_0(\eta_0r)+\frac{\omega_0^2+4}{\omega_0^2-4}\eta_0J_1(\eta_0r)\right]+k_1\frac{k_0}{\omega_0}rJ_0(\eta_0r)\right\}\delta_0\\&+\frac{1}{16}\left\{\frac{(\omega_0-1)(9\omega_0^4-18\omega_0^3-17\omega_0^2-6\omega_0+8)}{(\omega_0+1)(\omega_0-3)(2\omega_0-1)}r\eta_1J_0(\eta_1r)\right.\\&\quad+\left[\frac{9\omega_0^8-54\omega_0^7+82\omega_0^6+16\omega_0^5-87\omega_0^4+54\omega_0^3+36\omega_0^2-56\omega_0+16}{2(\omega_0-3)(2\omega_0-1)^2}\right.\\&\quad\left.\left.+\frac{5k_0^2}{\omega_0-1}(r^2-1)\right]J_1(\eta_1r)\right\}\beta_0,\end{aligned}\tag{5.1}$$

$$\begin{aligned}v_1^0=&\frac{2}{\omega_0^2-4}\eta_0J_1(\eta_0r)\delta_1-\left\{\frac{4\omega_1}{\omega_0^3(\omega_0^2-4)}\left[2k_0^2rJ_0(\eta_0r)+\frac{\omega_0^4}{\omega_0^2-4}\eta_0J_1(\eta_0r)\right]+2k_1\frac{k_0}{\omega_0^2}rJ_0(\eta_0r)\right\}\delta_0\\&-\frac{i(\omega_0-1)}{8(\omega_0-3)}\left\{\frac{9\omega_0^2-27\omega_0+10}{2\omega_0-1}r\eta_1J_0(\eta_1r)\right.\\&\quad\left.+\left[\frac{9\omega_0^6-45\omega_0^5+37\omega_0^4+53\omega_0^3-34\omega_0^2+20\omega_0-8}{2(\omega_0-3)(2\omega_0-1)^2}+\frac{5(\omega_0-3)k_0^2}{(\omega_0-3)^3}(r^2-1)\right]J_1(\eta_1r)\right\}\beta_0,\end{aligned}\tag{5.2}$$

$$\begin{aligned}w_1^0=&\frac{k_0}{\omega_0}J_0(\eta_0r)\delta_1+\left\{\frac{\omega_1k_0}{\omega_0^2}\left[\frac{4k_0^2}{\omega_0^2\eta_0}rJ_1(\eta_0r)-J_0(\eta_0r)\right]-\frac{k_1}{\omega_0}\left[r\eta_0J_1(\eta_0r)-J_0(\eta_0r)\right]\right\}\delta_0\\&+\frac{i}{16}\left\{\left[\frac{\omega_0(\omega_0-1)^2(\omega_0+2)(9\omega_0^3-27\omega_0^2+28\omega_0-8)}{2(2\omega_0-1)^2}+\frac{5k_0^2}{\omega_0-1}(r^2-1)\right]\frac{\eta_1}{k_0}J_0(\eta_1r)\right.\\&\quad\left.+\frac{9\omega_0^4-18\omega_0^3-17\omega_0^2+30\omega_0-10}{(\omega_0-1)(2\omega_0-1)}k_0rJ_1(\eta_1r)\right\}\beta_0,\end{aligned}\tag{5.3}$$

$$
\begin{aligned}
\pi_1^0 &= J_0(\eta_0 r)\delta_1 + \left\{\frac{4k_0^2}{\omega_0^3}\omega_1 + \left(1 - \frac{4}{\omega_0^2}\right)k_0 k_1\right\}\frac{r}{\eta_0}J_1(\eta_0 r)\delta_0 \\
&+ \frac{i}{16}\left\{\left[\frac{\omega_0^2(\omega_0-1)^2(\omega_0+2)(9\omega_0^3 - 27\omega_0^2 + 28\omega_0 - 8)}{2(2\omega_0-1)^2 k_0^2} + 5(r^2-1)\right]\eta_1 J_0(\eta_1 r)\right. \\
&\left. + \frac{9\omega_0^4 - 9\omega_0^3 - 26\omega_0^2 + 20\omega_0 - 2}{2\omega_0 - 1} r J_1(\eta_1 r)\right\}\beta_0 ,
\end{aligned} \tag{5.4}
$$

$$
\phi_1^0 = K_0(k_0 r)\gamma_1 - k_1 r K_1(k_0 r)\gamma_0 + \frac{i}{4}\left[rK_1(k_0 r) + k_0 r^2 K_0(k_0 r)\right]\alpha_0 . \tag{5.5}
$$

Imposition of the boundary conditions (4.6) and (4.7) brings in a relation that holds between γ_1 and δ_1:

$$
\begin{pmatrix} -k_0 K_1(k_0) & -\frac{i\omega_0}{\omega_0^2 - 4}\eta_0 J_1(\eta_0) \\ -i\omega_0 K_0(k_0) & J_0(\eta_0) \end{pmatrix}\begin{pmatrix} \gamma_1 \\ \delta_1 \end{pmatrix} = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}, \tag{5.6}
$$

where

$$
\begin{aligned}
G_1 &= -i\left\{\frac{\omega_1}{\omega_0^2 - 4}\left[\frac{4k_0^2}{\omega_0^2}J_0(\eta_0) + \frac{\omega_0^2+4}{\omega_0^2-4}\eta_0 J_1(\eta_0)\right] + k_1\left[\frac{k_0}{\omega_0}J_0(\eta_0) - \frac{\omega_0}{\omega_0^2-4}\frac{K_0(k_0)}{K_1(k_0)}\eta_0 J_1(\eta_0)\right]\right\}\delta_0 \\
&- \frac{1}{4}\left\{\frac{1}{\omega_0-1}\left[\frac{9\omega_0^4 - 18\omega_0^3 - 17\omega_0^2 - 6\omega_0 + 8}{4(2\omega_0-1)} - k_0^2 + \frac{k_0(1+k_0^2)K_0(k_0)}{K_1(k_0) + k_0 K_0(k_0)}\right]k_0^2\frac{J_1(\eta_1)}{\eta_1}\right. \\
&- \frac{1}{\omega_0 - 3}\left[\frac{9\omega_0^8 - 54\omega_0^7 + 82\omega_0^6 + 16\omega_0^5 - 87\omega_0^4 + 54\omega_0^3 + 36\omega_0^2 - 56\omega_0 + 16}{8(2\omega_0-1)^2} + k_0^2\right. \\
&\left.\left. - \frac{k_0(1+k_0^2)K_0(k_0)}{K_1(k_0)+k_0K_0(k_0)}\right]J_1(\eta_1)\right\}\beta_0 ,
\end{aligned} \tag{5.7}
$$

$$
\begin{aligned}
G_2 &= \left\{-\frac{\omega_1}{\omega_0}\left[\frac{4}{\omega_0^2} + \frac{K_0(k_0)}{k_0 K_1(k_0)}\right]k_0^2\frac{J_1(\eta_0)}{\eta_0} + 4k_1\frac{k_0}{\omega_0^2}\frac{J_1(\eta_0)}{\eta_0}\right\}\delta_0 \\
&+ \frac{i}{8}\left\{\left[\frac{\omega_0^2(\omega_0+1)(\omega_0+2)(\omega_0-3)(9\omega_0^3 - 27\omega_0^2 + 28\omega_0 - 8)}{4(2\omega_0-1)^2}\right.\right. \\
&\left. + \frac{k_0^2}{\omega_0-1}\left(2\omega_0 - 1 + \frac{k_0K_0(k_0)}{K_1(k_0)+k_0K_0(k_0)}\right)\right]\frac{J_0(\eta_1)}{\eta_1} \\
&\left. - \frac{1}{\omega_0-3}\left[\frac{9\omega_0^5 - 36\omega_0^4 + \omega_0^3 + 90\omega_0^2 - 54\omega_0 + 4}{2(2\omega_0-1)} - \frac{k_0K_0(k_0)}{K_1(k_0)+k_0K_0(k_0)}\right]J_1(\eta_1)\right\}\beta_0 .
\end{aligned} \tag{5.8}
$$

For $m = 1$,

$$
\begin{aligned}
u_1^1 &= -\frac{i}{\omega_0 - 3}\left\{\frac{\omega_0-1}{\omega_0+1}\eta_1 J_0(\eta_1 r) - \frac{J_1(\eta_1 r)}{r}\right\}\beta_1 \\
&- i\omega_1\left\{\frac{k_0^2(\omega_0^2 - 4\omega_0 + 7)}{(\omega_0-1)^3(\omega_0-3)}\frac{J_0(\eta_1 r)}{\eta_1} + \frac{1}{(\omega_0-3)^2}\left[\frac{4(\omega_0-3)}{(\omega_0-1)^2(\omega_0+1)}k_0^2 r + \frac{1}{r}\right]J_1(\eta_1 r)\right\}\beta_0
\end{aligned}
$$

$$-ik_1\frac{k_0}{(\omega_0-1)^2}\left[(\omega_0-1)rJ_1(\eta_1 r)-(\omega_0-3)\frac{J_0(\eta_1 r)}{\eta_1}\right]\beta_0$$
$$+\frac{1}{16}\left\{-\left[\frac{\omega_0-1}{2(2\omega_0-1)^2}(9\omega_0^6-27\omega_0^5+\omega_0^4+55\omega_0^3-12\omega_0^2-14\omega_0+4)+\frac{5k_0^2}{\omega_0}(r^2-1)\right]J_0(\eta_0 r)\right.$$
$$-\frac{1}{\omega_0-2}\left[\frac{9\omega_0^5-18\omega_0^4-17\omega_0^3+72\omega_0^2-11\omega_0-10}{(\omega_0+2)(2\omega_0-1)}r\right.$$
$$\left.\left.+\left(\frac{\omega_0^2(\omega_0-1)(\omega_0+1)}{2(2\omega_0-1)^2k_0^2}(9\omega_0^3-27\omega_0^2+28\omega_0-8)-5\right)\frac{1}{r}\right]\eta_0 J_1(\eta_0 r)\right\}\delta_0\,, \tag{5.9}$$

$$v_1^1=-\frac{1}{\omega_0-3}\left\{\frac{2}{\omega_0+1}\eta_1 J_0(\eta_1 r)-\frac{J_1(\eta_1 r)}{r}\right\}\beta_1$$
$$-\frac{\omega_1}{(\omega_0-1)^3(\omega_0-3)}\left\{4(\omega_0-2)k_0^2\frac{J_0(\eta_1 r)}{\eta_1}+\left[\frac{8}{\omega_0+1}k_0^2 r+\frac{(\omega_0-1)^3}{(\omega_0-3)r}\right]J_1(\eta_1 r)\right\}\beta_0$$
$$-k_1\frac{k_0}{(\omega_0-1)^2}\left[2rJ_1(\eta_1 r)+(\omega_0-3)\frac{J_0(\eta_1 r)}{\eta_1}\right]\beta_0$$
$$+\frac{i}{16}\left\{\left[\frac{(\omega_0-1)(9\omega_0^5-18\omega_0^4+\omega_0^3-11\omega_0^2-17\omega_0+6)}{(2\omega_0-1)^2}+\frac{10k_0^2}{\omega_0^2}(r^2-1)\right]J_0(\eta_0 r)\right.$$
$$+\frac{1}{\omega_0-2}\left[\frac{18\omega_0^4-18\omega_0^3-42\omega_0^2+55\omega_0-14}{(\omega_0+2)(2\omega_0-1)}r\right.$$
$$\left.\left.+\left(\frac{\omega_0^2(\omega_0-1)^2(\omega_0+1)}{2(2\omega_0-1)^2k_0^2}(9\omega_0^3-27\omega_0^2+28\omega_0-8)-5\right)\frac{1}{r}\right]\eta_0 J_1(\eta_0 r)\right\}\delta_0\,, \tag{5.10}$$

$$w_1^1=\frac{k_0}{\omega_0}J_0(\eta_0 r)\delta_1+\left\{\frac{\omega_1 k_0}{\omega_0^2}\left[\frac{4k_0^2}{\omega_0^2\eta_0}rJ_1(\eta_0 r)-J_0(\eta_0 r)\right]-\frac{k_1}{\omega_0}\left[r\eta_0 J_1(\eta_0 r)-J_0(\eta_0 r)\right]\right\}\delta_0$$
$$+\frac{i}{16}\left\{\left[\frac{\omega_0(\omega_0-1)^2(\omega_0+2)(9\omega_0^3-27\omega_0^2+28\omega_0-8)}{2(2\omega_0-1)^2}+\frac{5k_0^2}{\omega_0-1}(r^2-1)\right]\frac{\eta_1}{k_0}J_0(\eta_1 r)\right.$$
$$\left.+\frac{9\omega_0^4-18\omega_0^3-17\omega_0^2+30\omega_0-10}{(\omega_0-1)(2\omega_0-1)}k_0 rJ_1(\eta_1 r)\right\}\beta_0\,, \tag{5.11}$$

$$\pi_1^1=J_0(\eta_0 r)\delta_1+\left\{\frac{4k_0^2}{\omega_0^3}\omega_1+\left(1-\frac{4}{\omega_0^2}\right)k_0k_1\right\}\frac{r}{\eta_0}J_1(\eta_0 r)\delta_0$$
$$+\frac{i}{16}\left\{\left[\frac{\omega_0^2(\omega_0-1)^2(\omega_0+2)(9\omega_0^3-27\omega_0^2+28\omega_0-8)}{2(2\omega_0-1)^2k_0^2}+5(r^2-1)\right]\eta_1 J_0(\eta_1 r)\right.$$
$$\left.+\frac{9\omega_0^4-9\omega_0^3-26\omega_0^2+20\omega_0-2}{2\omega_0-1}rJ_1(\eta_1 r)\right\}\beta_0\,, \tag{5.12}$$

$$\phi_1^1=K_0(k_0 r)\gamma_1-k_1 rK_1(k_0 r)\gamma_0+\frac{i}{4}\left[rK_1(k_0 r)+k_0 r^2K_0(k_0 r)\right]\alpha_0\,. \tag{5.13}$$

Imposition of the boundary conditions (4.6) and (4.7) brings in a relation that holds between α_1 and β_1:

$$\begin{pmatrix} -[K_1(k_0)+k_0K_0(k_0)] & \frac{i}{\omega_0-3}\left[\frac{\omega_0-1}{\omega_0+1}\eta_1 J_0(\eta_1)-J_1(\eta_1)\right] \\ -i(\omega_0-1)K_1(k_0) & J_1(\eta_1) \end{pmatrix}\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}=\begin{pmatrix}F_1\\ F_2\end{pmatrix}, \tag{5.14}$$

where

$$\begin{aligned} F_1 = i\Bigg\{ & -\frac{\omega_1}{\omega_0-3}\left[\frac{\omega_0^2-4\omega_0+7}{(\omega_0-1)^3}k_0^2\frac{J_0(\eta_1)}{\eta_1}+\frac{1}{\omega_0-3}\left(\frac{4(\omega_0-3)}{(\omega_0-1)^2(\omega_0+1)}k_0^2+1\right)J_1(\eta_1)\right] \\ & +k_1\frac{k_0}{(\omega_0-1)(\omega_0-3)}\left[2J_1(\eta_1)+(\omega_0-3)\left(k_0^2+\frac{\omega_0-3}{\omega_0-1}\right)\frac{J_0(\eta_1)}{\eta_1}\right. \\ & \left.-\frac{1+k_0^2}{k_0}(\omega_0-1)\left(J_1(\eta_1)-\frac{\omega_0-1}{\omega_0+1}\eta_1 J_0(\eta_1)\right)\frac{k_0K_0(k_0)}{K_1(k_0)+k_0K_0(k_0)}\right]\Bigg\}\beta_0 \\ & -\frac{1}{16\omega_0(2\omega_0-1)}\Bigg\{\frac{\omega_0(\omega_0-1)}{2(2\omega_0-1)}(9\omega_0^6-27\omega_0^5+\omega_0^4+55\omega_0^3-12\omega_0^2-14\omega_0+4)J_0(\eta_0) \\ & -\Bigg[9\omega_0^4-18\omega_0^3-17\omega_0^2+46\omega_0-18+\frac{\omega_0(\omega_0-1)^2(\omega_0+1)(\omega_0+2)}{2(2\omega_0-1)k_0^2}(9\omega_0^3-27\omega_0^2+28\omega_0-8) \\ & +4(2\omega_0-1)\frac{1+k_0^2}{k_0}\frac{K_0(k_0)}{K_1(k_0)}\Bigg]k_0^2\frac{J_1(\eta_0)}{\eta_0}\Bigg\}\delta_0, \end{aligned} \tag{5.15}$$

$$\begin{aligned} F_2 = \Bigg\{ & \omega_1\left[\frac{\omega_0^2-2\omega_0+5}{(\omega_0-1)^3}k_0^2\frac{J_0(\eta_1)}{\eta_1}+\frac{J_1(\eta_1)}{\omega_0-3}\right. \\ & \left.+\frac{1}{\omega_0-3}\left(\frac{\omega_0-1}{\omega_0+1}\eta_1 J_0(\eta_1)-J_1(\eta_1)\right)\frac{k_0K_0(k_0)}{K_1(k_0)+k_0K_0(k_0)}\right] \\ & +k_1\left[-\frac{1}{k_0}\eta_1 J_0(\eta_1)+\frac{\omega_0-1}{\omega_0-3}\left(\frac{\omega_0-1}{\omega_0+1}\eta_1 J_0(\eta_1)-J_1(\eta_1)\right)\frac{k_0K_0(k_0)}{K_1(k_0)+k_0K_0(k_0)}\right]\Bigg\}\beta_0 \\ & -i\frac{\omega_0-1}{16\omega_0}\Bigg\{\frac{\omega_0(\omega_0-1)(9\omega_0^2-27\omega_0+10)}{2\omega_0-1}J_0(\eta_0) \\ & +\Bigg[\frac{\omega_0(\omega_0-3)(\omega_0-1)(\omega_0+1)(\omega_0+2)(9\omega_0^3-27\omega_0^2+28\omega_0-8)}{2(2\omega_0-1)k_0^2}-4 \\ & +\frac{4K_0(k_0)}{k_0K_1(k_0)}\Bigg]k_0^2\frac{J_1(\eta_0)}{\eta_0}\Bigg\}\delta_0. \end{aligned} \tag{5.16}$$

The linear stability problem is thus reduced to the systems (5.6) and (5.14) of linear algebraic equations. As is common, the matrices at $O(\epsilon)$ are identical with those at $O(\epsilon^0)$. In order for (5.6) and (5.14) to have non-trivial solutions for (γ_1,δ_1) and (α_1,β_1), (F_1,F_2) and (G_1,G_2) must belong to the spaces of the images of the corresponding matrices.

This condition postulates that

$$i\omega_0K_0(k_0)G_1-k_0K_1(k_0)G_2 = 0, \tag{5.17}$$

$$i(\omega_0-1)K_1(k_0)F_1-[k_0K_0(k_0)+K_1(k_0)]F_2 = 0. \tag{5.18}$$

Substituting from (5.7), (5.8), (5.15) and (5.16), the coupled system of (5.17) and (5.18), given k_1, constitutes an eigenvalue problem for ω_1. The requirement that they possess a nontrivial solution for (β_0, δ_0) gives rise to ω_1. Simultaneously, the wavenumber range k_1 of instability is determined by the non-reality condition of ω_1.

6 Numerical result

Figure 1 displays curves of the dispersion relation of the Kelvin waves for the axisymmetric ($m = 0$) and the bending ($m = -1$) modes of left-handed. Curves of $m = -1$ mode are drawn with solid lines, whereas those of $m = 0$ mode are drawn with dashed lines. Curves for the right-handed bending mode ($m = 1$) are readily available from curves for $m = -1$ simply by altering the sign $\omega_0 \to -\omega_0$.

The curves of the axisymmetric mode all start from $(\omega_0, k_0) = (0, 0)$. This mode has two types of branches symmetrically with respect to the horizontal axis $\omega_0 = 0$, either increasing or decreasing with k_0. Each type has an infinite number of branches. Among the curves of the bending mode, one branch is isolated from the other branches and is drawn with a thick solid line. This branch is called the *primary mode* or the *long-wave mode.* An infinite number of the remaining curves start from $(\omega_0, k_0) = (0, -1)$ and are called the *Bessel modes* or the *short-wave modes.* They are classified into two types, either increasing or decreasing with k_0. The increasing branches correspond to waves rotating slower than the basic circulatory flow, while the decreasing branches correspond to waves rotating faster than the basic flow.

By inspection, the local maximum of growth rate, if the instability occurs, is attained when $k_1 = 0$. With the choice of $k_1 = 0$, we computed the value of ω_1 at many of the intersection points of the dispersion curves. The primary branch of $m = -1$ has turned out to be totally irrelevant to the instability, and hence is ignored. The correction ω_1 of the frequency takes pure-imaginary values only at the intersection points between the decreasing branches of $m = 0$ and the increasing branches of $m = -1$. Among all the intersection points looked at so far, the maximum growth rate is attained at the intersection point with the smallest k_0, that is,

$$(k_0, \omega_0) \approx (0.813487, -0.59709)\,. \tag{6.1}$$

This exhibits a marked contrast with the Widnall instability. In the case of the latter, the growth rate is maintained to be large at large wavenumbers. On the point (6.1), the growth rate and the band width Δk_1 in k_1 of the instability are

$$|\mathrm{Im}[\omega_1]| \approx 0.054341\,, \quad \Delta k_1 \approx 0.102208\,. \tag{6.2}$$

Putting aside the primary branch of $m = -1$, this intersection is a collision between the first branches of $m = 0$ and $m = -1$. Relatively large growth rate is attained at the intersection points of the same (n-th) branches of $m = 0$ and $m = -1$.

We need to be cautious about the smallness of the value of $|\mathrm{Im}[\omega_1]|$. The growth rate $\epsilon|\mathrm{Im}[\omega_1]|$ of the resonance between $m = 0$ and $m = -1$ modes and the growth rate $\epsilon^2|\mathrm{Im}[\omega_2]|$ of the Widnall instability are highly competitive. Comparison with the result of Widnall & Tsai (1977) shows that the present mechanism predominates over Widnall's one when the vortex ring is very thin: $\epsilon < 0.028$.

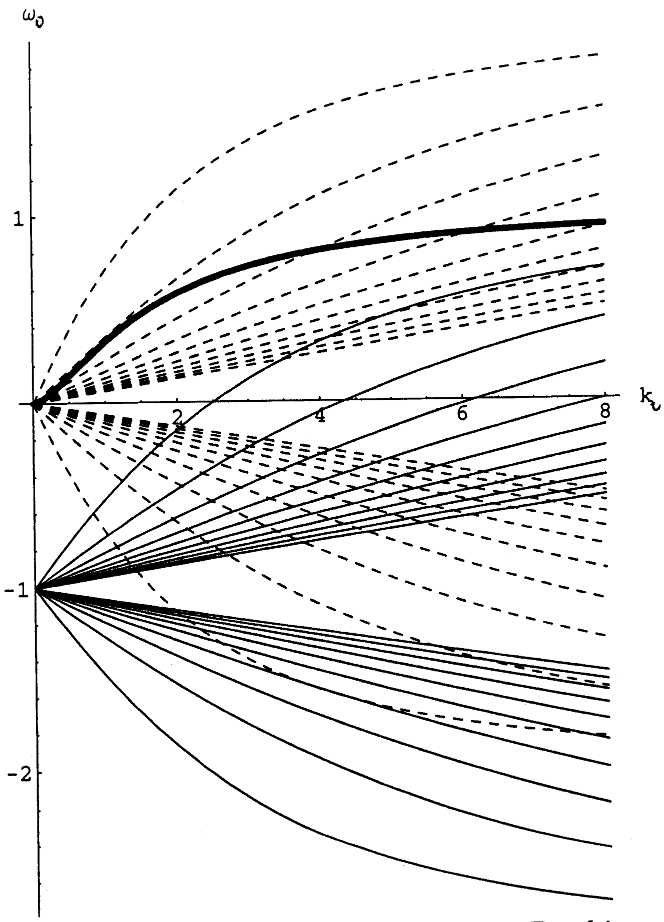

Fig. 1. Dispersion relation of the Kelvin waves on the Rankine vortex for axisymmetric mode $m = 0$ (dasshed lines) and bending mode $m = -1$ (solid lines).

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