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Multi-Variable White Noise Functions: Standard Setup Revisited

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1 Introduction

In the recent development of white noise theory the framework proposed by Cochran–Kuo–Sengupta [4] has become more important for their characterization theorems, see also [1]. In fact, much attention has been paid to characterization theorems for the test functions $\mathcal{W}$, for the generalized functions $\mathcal{W}^*$, for white noise operators $\mathcal{L}(\mathcal{W},\mathcal{W}^*)$ and for $\mathcal{L}(\mathcal{W},\mathcal{W})$. As was pointed out first by Chung–Chung–Ji [2], those characterization theorems are related each other, however, the statements are not unified because their objects are different so far as we are concerned with a single CKS-space over a particular underlying Gelfand triple. In this paper, using the *standard setup of white noise calculus* proposed by Hida–Obata–Saitô [6] and by Obata [12], we unify those characterization theorems into a single statement.

Let $\mathcal{S}_A(U)$ and $\mathcal{S}_B(V)$ be countable Hilbert nuclear spaces constructed from $L^2(U)$ and $L^2(V)$ in the standard manner, respectively, see §2. Then, for two weight sequences $\alpha=\{\alpha(n)\}_{n=0}^\infty$ and $\beta=\{\beta(n)\}_{n=0}^\infty$ we consider CKS-spaces of test white noise functions defined by

$$\mathcal{U}\equiv\Gamma_\alpha(\mathcal{S}_A(U)), \qquad \mathcal{V}\equiv\Gamma_\beta(\mathcal{S}_B(V)).$$

We assume condition (H1) for both $\mathcal{S}_A(U)$ and $\mathcal{S}_B(V)$ and conditions (A1)–(A4) for α and β. These conditions are described in §§2–3. Then the main result is stated in the following

Theorem 1.1 *For a continuous operator $\Xi\in\mathcal{L}(\mathcal{U},\mathcal{V})$ put*

$$\Theta(\xi,\eta)=\langle\langle\Xi\phi_\xi,\,\phi_\eta\rangle\rangle, \qquad \xi\in\mathcal{S}_A(U), \quad \eta\in\mathcal{S}_B(V), \tag{1}$$

where ϕ_ξ and ϕ_η are exponential vectors in $\mathcal{U}$ and $\mathcal{V}$, respectively. Then, Θ satisfies the following two conditions:

(O1) *Θ is a Gâteaux-entire function on $\mathcal{S}_A(U)\times\mathcal{S}_B(V)$;*

(O2) *for any $p\ge 0$ there exist $q\ge 0$ and $C\ge 0$ such that*

$$|\Theta(\xi,\eta)|^2\le CG_\alpha(|\,\xi\,|_{p+q}^2)G_{1/\beta}(|\,\eta\,|_{-p}^2), \qquad \xi\in\mathcal{S}_A(U), \quad \eta\in\mathcal{S}_B(V),$$

where G_α and $G_{1/\beta}$ are exponential generating functions of the weight sequences α and $1/\beta$, respectively.

Conversely, if a $\mathbf{C}$*-valued function* Θ *defined on* $\mathcal{S}_A(U) \times \mathcal{S}_B(V)$ *fulfills conditions (O1) and (O2), there exists a unique continuous operator* $\Xi \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ *satisfying* (1).

Specializing the underlying spaces $\mathcal{S}_A(U)$ and $\mathcal{S}_B(V)$, we obtain the characterization theorems mentioned at the beginning as corollaries of our main theorem. Moreover, our theorem yields characterization theorems for multi-variable white noise functions in a highly general form. It is noteworthy that multi-variable white noise functions have become more important in applications [13]. Further development is expected together with another type of characterization theorems based on Bargmann–Segal spaces, which has been also extensively studied along with complex white noise [7].

2 Standard Construction of an Underlying Gelfand Triple

We assemble some notions and results in [12]. Let U be a topological space equipped with a σ-finite Borel measure ν and consider the complex Hilbert space $L^2(U) = L^2(U, \nu)$. Let A be a selfadjoint operator in $L^2(U)$ such that $\inf \mathrm{Spec}\,(A) > 0$. With each $p \in \mathbf{R}$ we associate a Hilbert space $E_p(U)$ with a norm defined by

$$|\xi|_p = |A^p \xi|_{L^2(U)}, \qquad p \in \mathbf{R}.$$

More precisely, for $p \geq 0$, $E_p(U)$ consists of $\xi \in L^2(U)$ satisfying $|\xi|_p < \infty$ and $E_{-p}(U)$ is the completion of $L^2(U)$ with respect to the norm $|\cdot|_{-p}$. Thus we come to a countable Hilbert space:

$$\mathcal{S}_A(U) = \operatorname*{proj\,lim}_{p\to\infty} E_p(U).$$

The strong dual space of $\mathcal{S}_A(U)$ is identical to the inductive limit:

$$\mathcal{S}_A^*(U) = \operatorname*{ind\,lim}_{p\to\infty} E_{-p}(U)$$

and we come to a rigging:

$$\mathcal{S}_A(U) \subset L^2(U) \subset \mathcal{S}_A^*(U). \tag{2}$$

The canonical $\mathbf{C}$-bilinear form on $\mathcal{S}_A^*(U) \times \mathcal{S}_A(U)$ is denoted by $\langle \cdot, \cdot \rangle$. Then by definition the norm of $L^2(U)$ is given by $|\xi|_0^2 = \langle \bar{\xi}, \xi \rangle$.

Lemma 2.1 *A countable Hilbert space* $\mathcal{S}_A(U)$ *defined as above is nuclear if and only if there exists* $r > 0$ *such that* A^{-r} *is of Hilbert-Schmidt type.*

As for a spectral property of A we consider

(H1) $\inf \mathrm{Spec}\,(A) > 1$ and A^{-r} is of Hilbert–Schmidt type for some $r > 0$.

The first condition is taken into account beforehand. It follows from (H1) that

$$\| A^{-1} \|_{\mathrm{OP}} < 1, \qquad \lim_{r\to\infty} \| A^{-r} \|_{\mathrm{HS}} = 0. \tag{3}$$

Definition A countable Hilbert nuclear space $\mathcal{S}_A(U)$ constructed from $L^2(U)$ by means of a selfadjoint operator A satisfying (H1) is called *standard*.

As is suggested by (2), it is natural to consider $\mathcal{S}_A(U)$ and $\mathcal{S}_A^*(U)$ are spaces of test functions and generalized functions (or distributions) on U, respectively. However, it is not clear at all whether the delta function (evaluation map) is a member of $\mathcal{S}_A^*(U)$. On the contrary, continuity of a test function does not follow automatically. By construction each element of $\mathcal{S}_A(U)$ is merely a function on U which is determined up to ν-null functions. We thus come to:

(H2) for each function $\xi \in \mathcal{S}_A(U)$ there exists a unique continuous function $\widetilde{\xi}$ on U such that $\xi(u) = \widetilde{\xi}(u)$ for ν-a.e. $u \in U$.

Once this condition is satisfied, we consider $\mathcal{S}_A(U)$ always as a space of continuous functions on U and we do not use the exclusive symbol $\widetilde{\xi}$. Under (H2) we put two more hypotheses to keep a delta function in $\mathcal{S}_A^*(U)$:

(H3) for each $u \in U$ the evaluation map $\delta_u : \xi \mapsto \xi(u)$, $\xi \in \mathcal{S}_A(U)$, is a continuous linear functional, i.e., $\delta_u \in \mathcal{S}_A^*(U)$;

(H4) the map $u \mapsto \delta_u \in \mathcal{S}_A^*(U)$, $u \in U$, is continuous with respect to the strong dual topology of $\mathcal{S}_A^*(U)$.

Conditions similar to (H1)–(H4) were also discussed by Kubo–Takenaka [11]. These hypotheses are essential for topological arguments of $\mathcal{S}_A(U)$. Here we recall the following

Proposition 2.2 *Let $\mathcal{S}_A(U)$ be a standard countable Hilbert space and let $\xi_n \in \mathcal{S}_A(U)$, $n = 1, 2, \cdots$, be a sequence converging to 0 in $\mathcal{S}_A(U)$. If (H2) and (H3) are satisfied, then the sequence converges pointwisely, i.e., $\lim_{n\to\infty} \xi_n(u) = \xi(u)$ for any $u \in U$. Moreover, if (H4) is satisfied in addition, the pointwise convergence is uniform on every compact subset of U.*

Proposition 2.3 *Let $\mathcal{S}_A(U)$ be a standard countable Hilbert space satisfying conditions (H2) and (H3). Then,*

$$\| A^{-r} \|_{\mathrm{HS}}^2 = \int_U | \delta_u |_{-r}^2 \, \nu(du) = \sum_{j=0}^{\infty} \lambda_j^{-2r} < \infty,$$

where $1 < \lambda_1 \le \lambda_2 \le \cdots$ are the eigenvalues of A.

Recall that for two selfadjoint operators A_i in a Hilbert space H_i $(i = 1, 2)$ their tensor product $A_1 \otimes A_2$ becomes a selfadjoint operator in $H_1 \otimes H_2$ in a canonical way. Moreover, if $\inf \mathrm{Spec}\,(A_i) > 0$ for $i = 1, 2$, then $\inf \mathrm{Spec}\,(A_1 \otimes A_2) > 0$ as well.

Proposition 2.4 *Let $\mathcal{S}_A(U)$ and $\mathcal{S}_B(V)$ be standard countable Hilbert nuclear spaces. Then the canonical isomorphism $L^2(U \times V) \cong L^2(U) \otimes L^2(V)$ induces a topological isomorphism:*

$$\mathcal{S}_{A\otimes B}(U \times V) \cong \mathcal{S}_A(U) \otimes \mathcal{S}_B(V). \tag{4}$$

Moreover, if both $\mathcal{S}_A(U)$ and $\mathcal{S}_B(V)$ satisfy hypotheses (H2)–(H4), so does $\mathcal{S}_{A\otimes B}(U \times V)$.

Useful sufficient conditions for (H2)–(H4) are known, see [12, §1.4]. In fact, these conditions are essential to formulate quantum white noise $\{a_u, a_u^* \,;\, u \in U\}$, however, in this paper we do not go into this subject.

3 Conditions for Weight Sequences

After Asai–Kubo–Kuo [1] we introduce some general notion for positive sequences. A sequence $\alpha = \{\alpha(n)\}_{n=0}^{\infty}$ of positive numbers is called *log-concave* if

$$\alpha(n)\alpha(n+2) \le \alpha(n+1)^2, \qquad n = 0, 1, 2, \dots.$$

Two positive sequences $\alpha = \{\alpha(n)\}_{n=0}^{\infty}$ and $\beta = \{\beta(n)\}_{n=0}^{\infty}$ are called *equivalent* if there exist positive constants $K_1, K_2, M_1, M_2 > 0$ such that

$$K_1 M_1^n \alpha(n) \le \beta(n) \le K_2 M_2^n \alpha(n), \qquad n = 0, 1, 2, \dots.$$

For a positive sequence $\alpha = \{\alpha(n)\}_{n=0}^{\infty}$ we consider the following conditions:

(A1) $\alpha(0) = 1$ and $\inf_{n\ge 0} \sigma^n \alpha(n) > 0$ for some $\sigma \ge 1$;

(A2) $\lim_{n\to\infty} \left\{ \dfrac{\alpha(n)}{n!} \right\}^{1/n} = 0$;

(A3) α is equivalent to a positive sequence $\gamma = \{\gamma(n)\}$ such that $\left\{ \dfrac{\gamma(n)}{n!} \right\}$ is log-concave;

(A4) α is equivalent to another positive sequence $\gamma = \{\gamma(n)\}$ such that $\left\{ \dfrac{1}{n!\gamma(n)} \right\}$ is log-concave.

For example, $(n!)^{\beta}$ with $0 \le \beta < 1$ and the Bell numbers of order k satisfy the above conditions [4].

The exponential generating functions of α and $1/\alpha$ are defined by

$$G_\alpha(t) = \sum_{n=0}^{\infty} \frac{\alpha(n)}{n!} t^n, \qquad G_{1/\alpha}(t) = \sum_{n=0}^{\infty} \frac{1}{n!\alpha(n)} t^n,$$

respectively. Both are entire holomorphic functions by (A1) and (A2). We next define

$$\widetilde{G}_\alpha(t) \equiv \sum_{n=0}^{\infty} t^n \frac{n^{2n}}{n!\alpha(n)} \left\{ \inf_{\tau>0} \frac{G_\alpha(\tau)}{\tau^n} \right\},$$

$$\widetilde{G}_{1/\alpha}(t) \equiv \sum_{n=0}^{\infty} t^n \frac{n^{2n}\alpha(n)}{n!} \left\{ \inf_{\tau>0} \frac{G_{1/\alpha}(\tau)}{\tau^n} \right\}.$$

It is known [1] that (A3) and (A4) are necessary and sufficient conditions respectively for $\widetilde{G}_\alpha$ and for $\widetilde{G}_{1/\alpha}$ to have positive radiai of convergence. These functions will play a crucial role in norm estimates. Moreover, the next fact is known [1].

Lemma 3.1 *Assume that $\alpha = \{\alpha(n)\}$ satisfies (A1)–(A4).*

(1) *There exists a constant $C_{\alpha 1} > 0$ such that*

$$\alpha(n)\alpha(m) \le C_{\alpha 1}^{n+m} \alpha(n+m), \qquad n, m = 0, 1, 2, \dots.$$

(2) *There exists a constant $C_{\alpha 2} > 0$ such that*

$$\alpha(n+m) \le C_{\alpha 2}^{n+m} \alpha(n)\alpha(m), \qquad n, m = 0, 1, 2, \dots.$$

(3) *There exists a constant $C_{\alpha 3} > 0$ such that*

$$\alpha(n) \leq C_{\alpha 3}^m \alpha(m), \qquad 0 \leq n \leq m.$$

Then, by a simple calculation we have

Proposition 3.2 *Let $\alpha = \{\alpha(n)\}$ be as above and $G_\alpha(t)$ the exponential generating function. Then, for $s, t \geq 0$ we have:*

(1) $G_\alpha(0) = 1$ *and* $G_\alpha(s) \leq G_\alpha(t)$ *for* $s \leq t$.

(2) $G_\alpha(s)G_\alpha(t) \leq G_\alpha(C_{\alpha 1}(s+t))$.

(3) $G_\alpha(s+t) \leq G_\alpha(C_{\alpha 2}s)G_\alpha(C_{\alpha 2}t)$.

(4) $e^s G_\alpha(t) \leq G_\alpha(C_{\alpha 3}(s+t))$, *in particular,* $e^t \leq G_\alpha(C_{\alpha 3}t)$.

4 Standard CKS-Space

Suppose we are given a standard, countable Hilbert nuclear space:

$$\mathcal{S}_A(U) = \operatorname*{proj\,lim}_{p\to\infty} E_p(U)$$

and a positive sequence $\alpha = \{\alpha(n)\}$ satisfying (A1)–(A4). We first form a weighted Fock space:

$$\Gamma_\alpha(E_p(U)) = \left\{ \phi = (f_n)_{n=0}^\infty \,;\, f_n \in E_p(U)^{\widehat{\otimes} n},\ \| \phi \|_p^2 \equiv \sum_{n=0}^\infty n!\, \alpha(n) \,|\, f_n \,|_p^2 < \infty \right\},$$

where $E_p(U)^{\widehat{\otimes} n}$ stands for the n-fold symmetric tensor power, and then take its projective limit:

$$\mathcal{U} = \Gamma_\alpha(\mathcal{S}_A(U)) = \operatorname*{proj\,lim}_{p\to\infty} \Gamma_\alpha(E_p(U)).$$

Since $\mathcal{U}$ becomes a countable Hilbert nuclear space, we obtain a Gelfand triple:

$$\mathcal{U} = \Gamma_\alpha(\mathcal{S}_A(U)) \subset \Gamma(L^2(U)) \subset \Gamma_\alpha(\mathcal{S}_A(U))^* = \mathcal{U}^*, \tag{5}$$

where the middle space is the usual Boson Fock space, i.e., a weighted Fock space with weight sequence $\alpha(n) \equiv 1$. We may consider (5) as a variant of "second quantization" of an underlying Gelfand triple:

$$\mathcal{S}_A(U) \subset L^2(U) \subset \mathcal{S}_A^*(U).$$

We call (5) a *standard CKS-space* after [4].

The topology of $\mathcal{U}$ is defined by the family of norms:

$$\| \phi \|_{p,+}^2 = \sum_{n=0}^\infty n!\, \alpha(n) \,|\, f_n \,|_p^2, \qquad \phi = (f_n) \in \mathcal{U}, \quad p \geq 0.$$

The canonical **C**-bilinear form on $\mathcal{U}^* \times \mathcal{U}$ is denoted by $\langle\langle\cdot,\,\cdot\rangle\rangle$. Then

$$\langle\langle\Phi,\,\phi\rangle\rangle = \sum_{n=0}^{\infty} n!\,\langle F_n,\,f_n\rangle\,, \qquad \Phi = (F_n) \in \mathcal{U}^*, \quad \phi = (f_n) \in \mathcal{U},$$

and it holds that

$$|\,\langle\langle\Phi,\,\phi\rangle\rangle\,| \le \|\Phi\|_{-p,-}\|\phi\|_{p,+},$$

where

$$\|\Phi\|_{-p,-}^2 = \sum_{n=0}^{\infty} \frac{n!}{\alpha(n)}\,|F_n|_{-p}^2, \qquad \Phi = (F_n) \in \mathcal{U}^*.$$

We also note that

$$\Gamma_\alpha(\mathcal{S}_A(U))^* \cong \operatorname*{ind\,lim}_{p\to\infty} \Gamma_{1/\alpha}(E_{-p}(U)),$$

where $\Gamma_\alpha(\mathcal{S}_A(U))^*$ carries the strong dual topology and $\cong$ stands for a topological linear isomorphism.

Conditions (A1)–(A4) are sorted out by Asai–Kubo–Kuo [1] from many similar ones that have been introduced to keep "nice" properties of a CKS-space. On the other hand, it is also possible to start with a generating function G_α or another function controlling growth rate. This reversed approach is concise and useful for some questions [1], [5], [8], however, we prefer to the explicit description for our later calculation.

By the Wiener–Itô–Segal isomorphism the Boson Fock space $\Gamma(L^2(U))$ is realized as an L^2-space over a Gaussian space. In that sense $\Gamma_\alpha(\mathcal{S}_A(U))$ is a space of functions on the Gaussian space. By a parallel argument with [12, §3.2] one can show easily that $\Gamma_\alpha(\mathcal{S}_A(U))$ satisfies conditions (H2)–(H4). Thus, for example, white noise delta functions are defined as white noise distributions.

5 Proof of Main Theorem

First we recall some notation used in the statement of Theorem 1.1. For each $\xi \in \mathcal{S}_A(U)$ we put

$$\phi_\xi = \left(1, \frac{\xi}{1!}, \frac{\xi^{\otimes 2}}{2!}, \dots, \frac{\xi^{\otimes n}}{n!}, \dots\right).$$

Then $\phi_\xi \in \mathcal{U}$ and is called an *exponential vector* or a *coherent vector.* Let $\mathfrak{X}_1, \dots, \mathfrak{X}_n$ be complex topological vector spaces in general. Then a **C**-valued function F defined on $\mathfrak{X}_1 \times \cdots \times \mathfrak{X}_n$ is called *Gâteaux-entire* if the function

$$z \mapsto F(\xi_1, \dots, \xi_k + z\xi', \dots, \xi_n)$$

is entire holomorphic in $z \in \mathbf{C}$ for any choice of $\xi_1 \in \mathfrak{X}_1, \dots, \xi_n \in \mathfrak{X}_n$ and $\xi' \in \mathfrak{X}_k$, $1 \le k \le n$.

Now we start with the following

Lemma 5.1 *Let $F : \mathcal{S}_A(U) \to \mathbf{C}$ be a Gâteaux-entire function. Assume that there exist an entire function G on $\mathbf{C}$ and $p \in \mathbf{R}$ such that*

$$|F(\xi)|^2 \le G(|\xi|_p^2), \qquad \xi \in \mathcal{S}_A(U).$$

Then for any $n \ge 0$ the Gâteaux derivative

$$F_n(\xi_1, \dots, \xi_n) = \frac{1}{n!} D_{\xi_1} \dots D_{\xi_n} F(0) \tag{6}$$

becomes a continuous n-linear form on $\mathcal{S}_A(U)$ satisfying

$$|F_n|_{-(p+s)}^2 \le \left(\frac{n^n}{n!}\right)^2 \left\{\inf_{\tau>0} \frac{G(\tau)}{\tau^n}\right\} \| A^{-s} \|_{\mathrm{HS}}^{2n}. \tag{7}$$

PROOF. 1°. It is a standard result that F_n is a (not necessarily continuous) n-linear form on $\mathcal{S}_A(U)$.

2°. The Taylor expansion of an entire function $z \mapsto F(z\xi)$ is given by

$$F(z\xi) = \sum_{n=0}^{\infty} F_n(\xi, \dots, \xi) z^n, \qquad \xi \in \mathcal{S}_A(U).$$

3°. By Cauchy's integral formula we obtain

$$|F_n(\xi, \dots, \xi)| \le |\xi|_p^n \left\{\inf_{\tau>0} \frac{G(\tau)}{\tau^n}\right\}^{1/2}.$$

4°. By the polarization formula we have

$$\sup\{|F_n(\xi_1, \dots, \xi_n)|\,;\, |\xi_1|_p \le 1, \dots, |\xi_n|_p \le 1\} \le \frac{n^n}{n!} \left\{\inf_{\tau>0} \frac{G(\tau)}{\tau^n}\right\}^{1/2}.$$

5°. Let $1 < \lambda_0 \le \lambda_1 \le \dots$ be the eigenvalues of A and $\{e_j\}_{j=0}^{\infty}$ the corresponding eigenvectors which form a complete orthonormal basis of $L^2(U)$. Then

$$\begin{aligned}
|F_n|_{-(p+s)}^2 &= \sum_{j_1, \dots, j_n = 0}^{\infty} |F_n(e_{j_1}, \dots, e_{j_n})|^2 \lambda_{j_1}^{-2(p+s)} \dots \lambda_{j_n}^{-2(p+s)} \\
&= \sum_{j_1, \dots, j_n = 0}^{\infty} |F_n(\lambda_{j_1}^{-p} e_{j_1}, \dots, \lambda_{j_n}^{-p} e_{j_n})|^2 \lambda_{j_1}^{-2s} \dots \lambda_{j_n}^{-2s} \\
&\le \left(\frac{n^n}{n!}\right)^2 \left\{\inf_{\tau>0} \frac{G(\tau)}{\tau^n}\right\} \| A^{-s} \|_{\mathrm{HS}}^{2n}.
\end{aligned}$$

This completes the proof. ∎

Lemma 5.2 *Let $F : \mathcal{S}_A(U) \to \mathbf{C}$ be a Gâteaux-entire function. Assume that there exist constants $C \ge 0$ and $p \in \mathbf{R}$ such that*

$$|F(\xi)|^2 \le C G_\alpha(|\xi|_p^2), \qquad \xi \in \mathcal{S}_A(U).$$

For each $n \ge 0$ denote by F_n the n-th Gâteaux derivative defined by (6). Then $\Phi = (F_n) \in \Gamma_\alpha(\mathcal{S}_A(U))^*$ and we have

$$\| \Phi \|_{-(p+s),-}^2 \le C \widetilde{G}_\alpha(\| A^{-s} \|_{\mathrm{HS}}^2).$$

PROOF. By the definition of norms and (7) we have

$$\begin{aligned} \| \Phi \|_{-(p+s),-}^2 &= \sum_{n=0}^{\infty} \frac{n!}{\alpha(n)} \, | F_n |_{-(p+s)}^2 \\ &\le \sum_{n=0}^{\infty} \frac{n!}{\alpha(n)} \left(\frac{n^n}{n!}\right)^2 \left\{ \inf_{\tau>0} \frac{CG_\alpha(\tau)}{\tau^n} \right\} \| A^{-s} \|_{\mathrm{HS}}^{2n} \\ &= C\widetilde{G}_\alpha(\| A^{-s} \|_{\mathrm{HS}}^2), \end{aligned}$$

as desired. ∎

Recall that $\widetilde{G}_\alpha$ has a positive radius of convergence. Then $\widetilde{G}_\alpha(\| A^{-s} \|_{\mathrm{HS}}^2) < \infty$ for all sufficiently large $s > 0$ because $\lim_{s\to\infty} \| A^{-s} \|_{\mathrm{HS}} = 0$ by (3). In a similar manner we have

Lemma 5.3 *Let $F : \mathcal{S}_A(U) \to \mathbf{C}$ be a Gâteaux-entire function. Assume that there exist constants $C \ge 0$ and $p \in \mathbf{R}$ such that*

$$|F(\xi)|^2 \le CG_{1/\alpha}(| \xi |_p^2), \qquad \xi \in \mathcal{S}_A(U).$$

For each $n \ge 0$ let F_n be the n-th Gâteaux derivative defined by (6). Then $\Phi = (F_n) \in \Gamma_\alpha(\mathcal{S}_A(U))^*$ and we have

$$\| \Phi \|_{-(p+s),+}^2 \le C\widetilde{G}_{1/\alpha}(\| A^{-s} \|_{\mathrm{HS}}^2).$$

Remark Lemma 5.2 is the so-called characterization theorem for white noise distributions first shown by Potthoff–Streit [15] for a particular CKS-space called the Hida–Kubo–Takenaka space. The essence of their proof, however, remains invariable though a few variants have been discussed during the recent development of white noise theory. Lemma 5.3 implies immediately characterization theorem for white noise test functions $\Gamma_\alpha(\mathcal{S}_A(U))$. In this connection see also Theorem 6.1.

PROOF OF THEOREM 1.1. Fix $\eta \in \mathcal{S}_B(V)$ and we consider

$$F_\eta(\xi) = \Theta(\xi, \eta), \qquad \xi \in \mathcal{S}_A(U).$$

Then by Lemma 5.2 there exists $\Phi_\eta \in \mathcal{U}^*$ such that

$$F_\eta(\xi) = \langle\!\langle \Phi_\eta, \phi_\xi \rangle\!\rangle, \quad \text{i.e.,} \quad \Theta(\xi, \eta) = \langle\!\langle \Phi_\eta, \phi_\xi \rangle\!\rangle,$$

and

$$\| \Phi_\eta \|_{-(p+q+s),-}^2 \le C\widetilde{G}_\alpha(\| A^{-s} \|_{\mathrm{HS}}^2) G_{1/\beta}(| \eta |_{-p}^2).$$

Next, for a fixed $\phi \in \mathcal{U}$ we consider

$$H_\phi(\eta) = \langle\!\langle \Phi_\eta, \phi \rangle\!\rangle, \qquad \eta \in \mathcal{S}_B(V).$$

Obviously,

$$\begin{aligned} |H_\phi(\eta)|^2 &\le \| \Phi_\eta \|_{-(p+q+s),-}^2 \, \| \phi \|_{p+q+s,+}^2 \\ &\le C\widetilde{G}_\alpha(\| A^{-s} \|_{\mathrm{HS}}^2) G_{1/\beta}(| \eta |_{-p}^2) \, \| \phi \|_{p+q+s,+}^2 . \end{aligned}$$

Moreover, $H_\phi(\eta+z\eta')$ is a compact uniform limit of a sequence of entire functions which are linear combinations of

$$H_{\phi_\xi}(\eta+z\eta') = \langle\langle \Phi_{\eta+z\eta'},\, \phi_\xi\rangle\rangle = \Theta(\xi,\eta+z\eta'),$$

where ξ runs over $\mathcal{S}_A(U)$, and hence H_ϕ is Gâteaux-entire. Applying Lemma 5.3 to H_ϕ we find $\Psi_\phi \in \mathcal{V}^*$ such that

$$H_\phi(\eta) = \langle\langle \Psi_\phi,\, \phi_\eta\rangle\rangle\,, \qquad \eta \in \mathcal{S}_B(V)$$

and

$$\| \Psi_\phi \|^2_{-(-p+t),+} \le C\widetilde{G}_\alpha(\| A^{-s} \|^2_{\mathrm{HS}})\widetilde{G}_{1/\beta}(\| A^{-t} \|^2_{\mathrm{HS}})\, \| \phi \|^2_{p+q+s,+}\,. \tag{8}$$

Define a linear map $\Xi : \mathcal{U} \to \mathcal{V}$ by $\Xi\phi = \Psi_\phi$. Then (8) implies that

$$\| \Xi\phi \|^2_{p-t,+} \le C\widetilde{G}_\alpha(\| A^{-s} \|^2_{\mathrm{HS}})\widetilde{G}_{1/\beta}(\| A^{-t} \|^2_{\mathrm{HS}})\, \| \phi \|^2_{p+q+s,+}\,,$$

which proves that Ξ is continuous. Since

$$\langle\langle \Xi\phi_\xi,\, \phi_\eta\rangle\rangle = H_{\phi_\xi}(\eta) = \langle\langle \Phi_\eta,\, \phi_\xi\rangle\rangle = \Theta(\xi,\eta),$$

this $\Xi \in \mathcal{L}(\mathcal{U},\mathcal{V})$ is what we searched for. ∎

In fact, as the above proof is a simple combination of [12, §3.6] and [2], nothing new is required essentially. A few variants of the proof are easily obtained following the arguments [1], [3], [4], [5], [9], [10].

6 Unification of Traditional Characterization Theorems

Since the exponential vectors $\{\phi_\xi\,;\,\xi \in E_{\mathbf{C}}\}$ are linearly independent and span a dense subspace of $\mathcal{U}$, they play a fundamental role in specifying white noise functions and white noise operators. In practice, the most important are the *S-transform* of $\Phi \in \mathcal{U}^*$ defined by

$$S\Phi(\xi) = \langle\langle \Phi,\, \phi_\xi\rangle\rangle\,, \qquad \xi \in \mathcal{S}_A(U),$$

and the *symbol* of $\Xi \in \mathcal{L}(\mathcal{U},\mathcal{U}^*)$ defined by

$$\widehat{\Xi}(\xi,\eta) = \langle\langle \Xi\phi_\xi,\, \phi_\eta\rangle\rangle\,, \qquad \xi,\eta \in \mathcal{S}_A(U).$$

The traditional characterization theorems are mentioned as follows.

Theorem 6.1 *A* **C***-valued function F defined on $\mathcal{S}_A(U)$ is the S-transform of some $\Phi \in \mathcal{U}^*$ if and only if*

(F1) *F is Gâteaux-entire;*

(F2) *there exist $C \ge 0$ and $p \ge 0$ such that*

$$|F(\xi)|^2 \le CG_\alpha(|\,\xi\,|^2_p), \qquad \xi \in E_{\mathbf{C}}.$$

Moreover, $\Phi \in \mathcal{U}$ *if and only if (F1) and*

(F3) *for any* $p \geq 0$ *there exist* $C \geq 0$ *such that*

$$|F(\xi)|^2 \leq CG_{1/\alpha}(|\,\xi\,|_{-p}^2), \qquad \xi \in E_{\mathbf{C}}.$$

Theorem 6.2 *A* **C***-valued function* Θ *defined on* $\mathcal{S}_A(U) \times \mathcal{S}_A(U)$ *is the symbol of an operator* $\Xi \in \mathcal{L}(\mathcal{U},\mathcal{U}^*)$ *if and only if*

(O1) Θ *is Gâteaux-entire;*

(O2) *there exist constant numbers* $C \geq 0$ *and* $p \geq 0$ *such that*

$$|\Theta(\xi,\eta)|^2 \leq CG_\alpha(|\,\xi\,|_p^2)G_\alpha(|\,\eta\,|_p^2), \qquad \xi,\eta \in \mathcal{S}_A(U).$$

Moreover, $\Xi \in \mathcal{L}(\mathcal{U},\mathcal{U})$ *if and only if (O1) and*

(O3) *for any* $p \geq 0$ *there exist constant numbers* $C \geq 0$ *and* $q \geq 0$ *such that*

$$|\Theta(\xi,\eta)|^2 \leq CG_\alpha(|\,\xi\,|_{p+q}^2)G_{1/\alpha}(|\,\eta\,|_{-p}^2), \qquad \xi,\eta \in \mathcal{S}_A(U).$$

We may regard the zero space $\{0\}$ also as a standard countable Hilbert space though there is no underlying topological space U or rather we may understand that $\mathcal{S}(\emptyset) = \{0\}$. Then $\Gamma_\alpha(\mathcal{S}(\emptyset))$ is one-dimensional space consisting only scalar multiples of the vacuum vector ϕ_0. As for Theorem 6.1, if we set $\mathcal{S}_B(V) = \{0\}$ in Theorem 1.1, the statement is characterization of $\mathcal{S}_A(U)^*$. If we set $\mathcal{S}_A(U) = \{0\}$, the statement is characterization of $\mathcal{S}_B(V)$. The second part of Theorem 6.2 is immediate from Theorem 1.1 by setting $\mathcal{S}_A(U) = \mathcal{S}_B(V)$. We need some discussion to derive the first part of Theorem 6.2.

Lemma 6.3 *If* $\mathcal{S}_A(U)$ *and* $\mathcal{S}_B(V)$ *are standard countable Hilbert nuclear spaces, so is* $\mathcal{S}_{A\oplus B}(U \cup V)$ *and* $\mathcal{S}_{A\oplus B}(U \cup V) \cong \mathcal{S}_A(U) \oplus \mathcal{S}_B(V)$.

Lemma 6.4 *If* $\mathcal{S}_A(U)$ *is a standard countable Hilbert nuclear space, so is* $\mathcal{S}_{A\otimes I}(U \times T)$ *for any finite discrete space* T *equipped with counting measure. Moreover,* $\mathcal{S}_{A\otimes I}(U\times T) \cong \mathcal{S}_A(U) \otimes \mathbf{C}^n$*, where* $|T| = n$.

The above results are straightforward. In general, for two Hilbert spaces H_1 and H_2 there is a canonical unitary isomorphism

$$\mathcal{T} : \Gamma(H_1) \otimes \Gamma(H_2) \to \Gamma(H_1 \oplus H_2)$$

specified by the correspondence of exponential vectors $\mathcal{T}(\phi_\xi \otimes \phi_\eta) = \phi_{\xi\oplus\eta}$. In fact, for

$$\phi_{\xi_1,\dots,\xi_j} = (0,\dots,0,\xi_1\widehat{\otimes}\dots\widehat{\otimes}\xi_j,0,\dots), \quad \psi_{\eta_1,\dots,\eta_k} = (0,\dots,0,\eta_1\widehat{\otimes}\dots\widehat{\otimes}\eta_k,0,\dots),$$

$\mathcal{T}(\phi_{\xi_1,\dots,\xi_j} \otimes \psi_{\eta_1,\dots,\eta_k})$ is given as

$$\mathcal{T}(\phi_{\xi_1,\dots,\xi_j} \otimes \psi_{\eta_1,\dots,\eta_k}) = (0,\dots,0,h_{j+k},0,\dots),$$

where

$$h_{j+k} = (\xi_1 \oplus 0)\widehat{\otimes}\dots\widehat{\otimes}(\xi_j \oplus 0)\widehat{\otimes}(0 \oplus \eta_1)\widehat{\otimes}\dots\widehat{\otimes}(0 \oplus \eta_k).$$

For arbitrary $\phi = (0, \dots, 0, f_j, 0, \dots) \in \Gamma(H_1)$ and $\psi = (0, \dots, 0, g_k, 0, \dots) \in \Gamma(H_2)$, $\mathcal{T}(\phi \otimes \psi)$ is given by a bilinear map $h_{j,k} : H_1^{\widehat{\otimes} j} \times H_2^{\widehat{\otimes} k} \to (H_1 \oplus H_2)^{\widehat{\otimes}(j+k)}$ in such a way that

$$\mathcal{T}(\phi \otimes \psi) = (0, \dots, 0, h_{j,k}(f_j, g_k), 0, \dots). \tag{9}$$

For this $h_{j,k}$ we have by Fourier expansion

$$| h_{j,k}(f_j, g_k) |^2_{(H_1 \oplus H_2)^{\otimes(j+k)}} = \frac{j!k!}{(j+k)!} | f_j |^2_{H_1^{\otimes j}} | g_k |^2_{H_2^{\otimes k}} . \tag{10}$$

Lemma 6.5 *The canonical isomorphism* $\Gamma(L^2(U) \oplus L^2(V)) \cong \Gamma(L^2(U)) \otimes \Gamma(L^2(V))$ *induces a topological isomorphism:*

$$\Gamma_\alpha(\mathcal{S}_A(U) \oplus \mathcal{S}_B(V)) \cong \Gamma_\alpha(\mathcal{S}_A(U)) \otimes \Gamma_\alpha(\mathcal{S}_B(V)).$$

PROOF. Let $\mathcal{T}$ be the canonical isomorphism from $\Gamma(L^2(U)) \otimes \Gamma(L^2(V))$ onto $\Gamma(L^2(U) \oplus L^2(V))$ described above. For $\phi = (f_j) \in \Gamma(L^2(U))$ and $\psi = (g_k) \in \Gamma(L^2(V))$, we set $\mathcal{T}(\phi \otimes \psi) = (h_n) \in \Gamma(L^2(U) \oplus L^2(V))$. Then by (9) we have

$$h_n = \sum_{j+k=n} h_{j,k}(f_j, g_k).$$

Since the right hand side is an orthogonal sum, using (10) we come to

$$| h_n |_p^2 = \sum_{j+k=n} | h_{j,k}(f_j, g_k) |_p^2 = \sum_{j+k=n} \frac{j!k!}{(j+k)!} | f_j |_p^2 | g_k |_p^2 .$$

Hence

$$\| \mathcal{T}(\phi \otimes \psi) \|_{p,+}^2 = \sum_{n=0}^{\infty} n!\, \alpha(n) | h_n |_p^2 = \sum_{j,k=0}^{\infty} j!k!\, \alpha(j+k) | f_j |_p^2 | g_k |_p^2 . \tag{11}$$

Then by Lemma 3.1 (2),

$$\| \mathcal{T}(\phi \otimes \psi) \|_{p,+}^2 \le \sum_{j=0}^{\infty} j!\, \alpha(j) C_{\alpha 2}^j | f_j |_p^2 \sum_{k=0}^{\infty} k!\, \alpha(k) C_{\alpha 2}^k | g_k |_p^2 . \tag{12}$$

Choose $q \ge 0$ such that $C_{\alpha 2} \| A^{-1} \|_{\mathrm{OP}}^{2q} \le 1$ and $C_{\alpha 2} \| B^{-1} \|_{\mathrm{OP}}^{2q} \le 1$. Then (12) becomes

$$\| \mathcal{T}(\phi \otimes \psi) \|_{p,+}^2 \le \sum_{j=0}^{\infty} j!\, \alpha(j) | f_j |_{p+q}^2 \sum_{k=0}^{\infty} k!\, \alpha(k) | g_k |_{p+q}^2 = \| \phi \|_{p+q,+}^2 \| \psi \|_{p+q,+}^2 ,$$

that is,

$$\| \mathcal{T}(\phi \otimes \psi) \|_{p,+} \le \| \phi \|_{p+q,+} \| \psi \|_{p+q,+} . \tag{13}$$

In a similar manner, applying Lemma 3.1 (1) to (11), we obtain

$$\| \mathcal{T}(\phi \otimes \psi) \|_{p,+} \ge \| \phi \|_{p-r,+} \| \psi \|_{p-r,+} , \tag{14}$$

where $r \ge 0$ is taken in such a way that $C_{\alpha 1} \| A^{-1} \|_{\mathrm{OP}}^{2r} \le 1$ and $C_{\alpha 1} \| B^{-1} \|_{\mathrm{OP}}^{2r} \le 1$. The assertion then follows from (13) and (14). ∎

Remark For Hilbert spaces H_1 and H_2 there is no isomorphism between $\Gamma_\alpha(H_1 \oplus H_2)$ and $\Gamma_\alpha(H_1) \otimes \Gamma_\alpha(H_2)$ for a general α.

Lemma 6.6 *Let $\mathcal{S}_A(U)$ be a standard countable Hilbert nuclear space and let T be a discrete space of n points with counting measure. Then the isomorphism*

$$\Gamma(L^2(U \times T)) \cong \Gamma(L^2(U)) \otimes \cdots \otimes \Gamma(L^2(U)) \quad (n \text{ times})$$

induces a topological isomorphism

$$\Gamma_\alpha(\mathcal{S}_{A\otimes I}(U \times T)) \cong \Gamma_\alpha(\mathcal{S}_A(U)) \otimes \cdots \otimes \Gamma(\mathcal{S}_A(U)) \quad (n \text{ times}).$$

PROOF. Note first that

$$L^2(T) = \mathcal{S}_I(T) \cong \mathbf{C}^n.$$

It then follows from Proposition 2.4 that the canonical isomorphism

$$L^2(U \times T) \cong L^2(U) \otimes L^2(T) \cong L^2(U) \oplus \cdots \oplus L^2(U) \quad (n \text{ times}) \tag{15}$$

induces a topological isomorphism:

$$\mathcal{S}_{A\otimes I}(U \times T) \cong \mathcal{S}_A(U) \otimes \mathcal{S}_I(T) \cong \mathcal{S}_A(U) \oplus \cdots \oplus \mathcal{S}_A(U) \quad (n \text{ times}).$$

On the other hand, from (15) we see that

$$\Gamma(L^2(U \times T)) \cong \Gamma(L^2(U)) \otimes \cdots \otimes \Gamma(L^2(U)) \quad (n \text{ times})$$

and from Lemma 6.5

$$\Gamma_\alpha(\mathcal{S}_{A\otimes I}(U \times T)) \cong \Gamma_\alpha(\mathcal{S}_A(U)) \otimes \cdots \otimes \Gamma(\mathcal{S}_A(U)) \quad (n \text{ times}).$$

The assertion is then clear. ∎

We go back to Theorem 1.1. By Lemma 6.6 the statement of Theorem 1.1 remains valid if U is replaced with $U \times \{1,2\}$ and V with $\emptyset$. Moreover,

$$\begin{aligned}\mathcal{L}(\Gamma_\alpha(\mathcal{S}_{A\otimes I}(U \times \{1,2\})), \mathbf{C}) &\cong \mathcal{L}(\Gamma_\alpha(\mathcal{S}_A(U)) \otimes \Gamma_\alpha(\mathcal{S}_A(U)), \mathbf{C}) \\ &\cong \{\Gamma_\alpha(\mathcal{S}_A(U)) \otimes \Gamma_\alpha(\mathcal{S}_A(U))\}^* \\ &\cong \mathcal{L}(\Gamma_\alpha(\mathcal{S}_A(U)), \Gamma_\alpha(\mathcal{S}_A(U))^*).\end{aligned}$$

Thus, in this case, Theorem 1.1 is reduced to characterizaion of white noise operators $\mathcal{L}(\Gamma_\alpha(\mathcal{S}_A(U)), \Gamma_\alpha(\mathcal{S}_A(U))^*)$.

In order to complete the reduction we need to discuss conditions (O1) and (O2), see Theorem 6.2. Let $T = \{1, 2, \ldots, n\}$. By the isomorphism $\mathcal{S}_{A\otimes I}(U \times T) \cong \mathcal{S}_A(U) \otimes \mathbf{C}^n$ described in Lemma 6.4 we come to

$$\mathcal{S}_{A\otimes I}(U \times T) \cong \mathcal{S}_A(U) \oplus \cdots \oplus \mathcal{S}_A(U) \quad (n\,\text{times}).$$

For $(\xi_1, \ldots, \xi_n) \in \mathcal{S}_A(U) \oplus \cdots \oplus \mathcal{S}_A(U)$ the corresponding element $\boldsymbol{\xi} \in \mathcal{S}_{A\otimes I}(U \times T)$ is given by

$$\boldsymbol{\xi}(u, j) = \xi_j(u)$$

Then, there is a one-to-one correspondence between functions on $\mathcal{S}_{A\otimes I}(U \times T)$ and on $\mathcal{S}_A(U) \times \cdots \times \mathcal{S}_A(U)$ (n times) given by

$$\Theta(\boldsymbol{\xi}) = F(\xi_1, \ldots, \xi_n).$$

Lemma 6.7 *Notations being as above, Θ is Gâteaux-entire on $\mathcal{S}_{A\otimes I}(U \times T)$ if and only if so is F on $\mathcal{S}_A(U) \times \cdots \times \mathcal{S}_A(U)$ (n times).*

PROOF. We need only to recall Hartogs' theorem of holomorphy. ∎

Lemma 6.8 *Notations being as above, if there exist $C \ge 0$ and $p \in \mathbf{R}$ such that*

$$|\Theta(\boldsymbol{\xi})|^2 \le C G_\alpha(|\boldsymbol{\xi}|_p^2), \qquad \boldsymbol{\xi} \in \mathcal{S}_{A\otimes I}(U \times T),$$

then there exists $q \ge 0$ such that

$$|F(\xi_1, \dots, \xi_n)|^2 \le C \prod_{j=1}^{n} G_\alpha(|\xi_j|_{p+q}^2), \qquad \xi_1, \dots, \xi_n \in \mathcal{S}_A(U).$$

Conversely, if there exist $C \ge 0$ and $p \in \mathbf{R}$ such that

$$|F(\xi_1, \dots, \xi_n)|^2 \le C \prod_{j=1}^{n} G_\alpha(|\xi_j|_p^2), \qquad \xi_1, \dots, \xi_n \in \mathcal{S}_A(U),$$

then there exists $q \ge 0$ such that

$$|\Theta(\boldsymbol{\xi})|^2 \le C G_\alpha(|\boldsymbol{\xi}|_{p+q}^2), \qquad \boldsymbol{\xi} \in \mathcal{S}_{A\otimes I}(U \times T).$$

PROOF. This is a simple consequence of Proposition 3.2 (2) and (3). ∎

With the help of Lemmas 6.7 and 6.8 we see immediately that conditions (O1) and (O2) in Theorem 1.1 in the case where U and V are replaced with $U \times \{1, 2\}$ and $\emptyset$, respectively, coincide with the usual ones in Theorem 6.2.

7 Characterization Theorems for Multi-Variable Case

Let us start with a single CKS-space:

$$\mathcal{U} = \Gamma_\alpha(\mathcal{S}_A(U)) \subset \Gamma(L^2(U)) \subset \Gamma_\alpha(\mathcal{S}_A(U))^* = \mathcal{U}^*. \tag{16}$$

We are interested in multi-variable functions defined on $\mathcal{S}_A(U) \times \cdots \times \mathcal{S}_A(U)$ (n-times), in particular, of the forms:

$$F(\xi_1, \dots, \xi_m) = \langle\!\langle \Phi, \phi_{\xi_1} \otimes \cdots \otimes \phi_{\xi_m} \rangle\!\rangle, \tag{17}$$

$$G(\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_n) = \langle\!\langle \Xi(\phi_{\xi_1} \otimes \cdots \otimes \phi_{\xi_m}), \phi_{\eta_1} \otimes \cdots \otimes \phi_{\eta_n} \rangle\!\rangle, \tag{18}$$

where $\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_n \in \mathcal{S}_A(U)$, and $\Phi \in (\mathcal{U}^{\otimes m})^*$, $\Xi \in \mathcal{L}(\mathcal{U}^{\otimes m}, (\mathcal{U}^{\otimes n})^*)$. It is clear that the functions defined in (17) and (18) are Gâteaux-entire.

The following results are immediate corollaries of Theorem 1.1 with the help of Lemmas 6.7 and 6.8.

Theorem 7.1 *A Gâteaux-entire function $F : \mathcal{S}_A(U)^m \to \mathbf{C}$ is expressed in the form (17) with $\Phi \in (\mathcal{U}^{\otimes m})^*$ if and only if there exist constant numbers $C \ge 0$ and $p \ge 0$ such that*

$$|F(\xi_1, \dots, \xi_m)|^2 \le C \prod_{j=1}^{m} G_\alpha(|\xi_j|_p^2), \qquad \xi_1, \dots, \xi_m \in \mathcal{S}_A(U).$$

Theorem 7.2 *A Gâteaux-entire function $F : \mathcal{S}_A(U)^m \to \mathbf{C}$ is expressed in the form* (17) *with $\Phi \in \mathcal{U}^{\otimes m}$ if and only if for any $p \geq 0$ there exists $C \geq 0$ such that*

$$|F(\xi_1, \ldots, \xi_m)|^2 \leq C \prod_{j=1}^{m} G_{1/\alpha}(|\,\xi_j\,|_{-p}^2), \qquad \xi_1, \ldots, \xi_m \in \mathcal{S}_A(U).$$

Theorem 7.3 *A Gâteaux-entire function $G : \mathcal{S}_A(U)^{m+n} \to \mathbf{C}$ is expressed in the form* (18) *with $\Xi \in \mathcal{L}(\mathcal{U}^{\otimes m}, (\mathcal{U}^{\otimes n})^*)$ if and only if there exist $C \geq 0$ and $p \geq 0$ such that*

$$|G(\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n)|^2 \leq C \prod_{j=1}^{m} G_\alpha(|\,\xi_j\,|_p^2) \prod_{k=1}^{n} G_\alpha(|\,\eta_k\,|_p^2),$$

for $\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n \in \mathcal{S}_A(U)$.

In that case, since $\mathcal{L}(\mathcal{U}^{\otimes m}, (\mathcal{U}^{\otimes n})^*) \cong (\mathcal{U}^{\otimes(m+n)})^*$, we may choose an operator Ξ from $\mathcal{L}(\mathcal{U}^{\otimes m'}, (\mathcal{U}^{\otimes n'})^*)$ whenever $m' + n' = m + n$.

Theorem 7.4 *A Gâteaux-entire function $G : \mathcal{S}_A(U)^{m+n} \to \mathbf{C}$ is expressed in the form* (18) *with $\Xi \in \mathcal{L}(\mathcal{U}^{\otimes m}, \mathcal{U}^{\otimes n})$ if and only if for any $p \geq 0$ there exist $C \geq 0$ and $q \geq 0$ such that*

$$|G(\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n)|^2 \leq C \prod_{j=1}^{m} G_\alpha(|\,\xi_j\,|_{p+q}^2) \prod_{k=1}^{n} G_{1/\alpha}(|\,\eta_k\,|_{-p}^2),$$

for $\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n \in \mathcal{S}_A(U)$.

References

[1] N. Asai, I. Kubo and H.-H. Kuo: *General characterization theorems and intrinsic topologies in white noise analysis*, to appear in Hiroshima Math. J. **31** (2001).

[2] D. M. Chung, T. S. Chung and U. C. Ji: *A simple proof of analytic characterization theorem for operator symbols*, Bull. Korean Math. Soc. **34** (1997), 421–436.

[3] D. M. Chung, U. C. Ji and N. Obata: *Higher powers of quantum white noises in terms of integral kernel operators*, Infinite Dimen. Anal. Quantum Probab. Rel. Top. **1** (1998), 533–559.

[4] W. G. Cochran, H.-H. Kuo and A. Sengupta: *A new class of white noise generalized functions*, Infinite Dimen. Anal. Quantum Prob. **1** (1998), 43–67.

[5] R. Gannoun, R. Hachaichi, H. Ouerdiane and A. Rezgui: *Un théoreme de dualité entre espaces de fonctions holomorphes à croissance exponentielle*, J. Funct. Anal. **171** (2000), 1–14.

[6] T. Hida, N. Obata and K. Saitô: *Infinite dimensional rotations and Laplacians in terms of white noise calculus*, Nagoya Math. J. **128** (1992), 65–93.

[7] U. C. Ji and N. Obata: *A role of Bargmann–Segal spaces in characterization and expansion of operators on Fock space*, preprint, 2000.

[8] U. C. Ji, N. Obata and H. Ouerdiane: *Analytic characterization of generalized Fock space operators as two-variable entire functions with growth condition*, to appear in Infinite Dimen. Anal. Quantum Probab. Rel. Top.

[9] I. Kubo and H.-H. Kuo: *Simple proof of Hida distribution characterization theorem*, in "Exploring Stochastic Laws (A. V. Skorokhod and Yu. V. Borovskikh, Eds.)," pp. 243–250, VSP, Utrecht, 1995.

[10] I. Kubo, H.-H. Kuo and A. Sengupta: *White noise analysis on a new space of Hida distributions*, Infinite Dimen. Anal. Quantum Probab. Rel. Top. **2** (1999), 315–335.

[11] I. Kubo and S. Takenaka: *Calculus on Gaussian white noise I–IV*, Proc. Japan Acad. **56A** (1980), 376–380; 411–416; **57A** (1981), 433–437; **58A** (1982), 186–189.

[12] N. Obata: "White Noise Calculus and Fock Space," Lect. Notes in Math. Vol. 1577, Springer–Verlag, 1994.

[13] N. Obata: *Inverse S-transform, Wick product and overcompleteness of exponential vectors*, to appear in "Quantum Information IV (T. Hida and K. Saitô, Eds.)," World Scientific.

[14] H. Ouerdiane: *Fonctionnelles analytiques avec condition de croissance, et application à l'analyse gaussienne*, Japan. J. Math. **20** (1994), 187–198.

[15] J. Potthoff and L. Streit: *A characterization of Hida distributions*, J. Funct. Anal. **101** (1991), 212–229.