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A construction of compact matrix quantum groups and description of the related C*-algebras *

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Abstract

A construction of compact matrix quantum groups is given. The construction is based on Woronowicz's theory. A fundamental role in the construction is played by a generalized determinant, related to permutation groups. Description of the C*-algebras related to the quantum groups is given in terms of irreducible *-representations on Hilbert spaces.

1 Introduction

In [SLW2] Woronowicz presented the following idea of compact matrix quantum groups (c.f. the proof of Theorem 1.1). Let $G \subset M_N(\mathbb{C})$ be a compact group of N x N complex matrices. An element $g \in G$ is then a matrix with entries g_{jk} and the entries' functions $w_{jk} : G \ni g \mapsto g_{jk} = w_{jk}(g) \in \mathbb{C}$ form a collection $\{w_{jk} : 1 \leq j,k \leq N\}$ of N^2 continuous functions on the group G. In terms of these functions we can describe various algebraic properties of the group. The idea is that we can reflect algebraic group properties as properties of the *-algebra generated by these functions, where $*$ is the complex conjugation.

Let us first consider the multiplication in G. When two matrices $g,h \in G$ are multiplied, the standard rule of multiplication of entries is expressed by the entries' functions as $w_{jk}(g \cdot h) = \sum_{r=1}^{N} w_{jr}(g) \cdot w_{rk}(h) = \sum_{r=1}^{N}(w_{jr} \otimes w_{rk})(g \otimes h)$.

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Hence the transformation $\Phi(w_{jk}) = \sum_{r=1}^{N} w_{jr} \otimes w_{rk}$ reflects the multiplication in G. This transformation is therefore called *co-multiplication.*

Now let us consider the inverse in G, which is the transformation $G \ni g \mapsto g^{-1} \in G$. This can also be expressed in terms of the entries' functions. Namely, by a change of the scalar product $\langle , \rangle$ in $\mathbb{C}^N$ we can obtain unitary representation of G, so the inverse matrix will become the conjugate matrix. If a strictly positive matrix M gives the change of the scalar product into the new one $[,]$, so that $[x, y] := \langle Mx, y\rangle$ for $x, y \in \mathbb{C}^N$, then $M = g * Mg$ and $g^{-1} = M^{-1}g * M$. Since $w_{jk}(g*) = \overline{w_{kj}(g)}$ is a complex conjugate combined with the transposition, it follows that

$$w_{jk}(g^{-1}) = \sum_{r,s=1}^{N} (M^{-1})_{jr} w_{rs}(g*)(M)_{sk} = \sum_{r,s=1}^{N} (M^{-1})_{jr} \overline{w_{rs}(g)}(M)_{sk}$$

Hence the transformation

$$\kappa(w_{jk}) := \sum_{r,s=1}^{N} (M^{-1})_{jr}(M)_{sk}\overline{w_{rs}} \tag{1.1}$$

reflects taking the inverse in G. This transformation κ is therefore called *co-inverse.* The equality above shows, that $\kappa(w_{jk})$ can be expressed as a linear combination of complex conjugations of of the entries' functions, so it is an element of the *-algebra generated by these functions.

Let us now look at the properties of the group identity. Let $e \in G$ be the group identity, which is the N x N identity matrix. Then for any $g \in G$ we have

$$\delta_{jk} = w_{jk}(e) = w_{jk}(gg^{-1}) = \textstyle\sum_{r=1}^{N} w_{jr}(g) w_{rk}(g^{-1})$$
$$= \textstyle\sum_{r=1}^{N} w_{jr}(g)\kappa(w_{rk})(g) = \sum_{r=1}^{N}(w_{jr} \cdot \kappa(w_{rk}))(g)$$

This yields the equalities for the entries' functions

$$\delta_{jk} \cdot I = \sum_{r=1}^{N} w_{jr}\kappa(w_{rk}) = \sum_{r=1}^{N} \kappa(w_{jr})w_{rk}$$

These identities reflect the properties of the identity matrix in the group G, so they constitute the properties of the so called *co-unit.* This way we see that, without having the group G given itself, we can "recover" it from the properties of a associated co-structure. This co-structure is what one calls the *quantum group.*

The notion of a *compact matrix pseudogroup*, later renamed for *compact matrix quantum group*, was introduced by Woronowicz in [SLW2], to name a C^*-algebraic structure which reflects group properties on the C^*-algebraic level. It consists of a C^*-algebra A and an N by N matrix $u = (u_{jk})_{j,k=1}^N$, with the elements $u_{jk} \in A$ generating a dense $*$-subalgebra $\mathcal{A}$ of A, and with the following additional structure:

1. a C^*-homomorphism $\Phi : A \to A \otimes A$, called the co-multiplication, such that

$$\Phi(u_{jk}) = \sum_{r=0}^{N} u_{jr} \otimes u_{rk} \tag{1.2}$$

2. a linear anti-multiplicative mapping $\kappa : \mathcal{A} \to \mathcal{A}$, called the co-inverse, such that $\kappa(\kappa(a^*)^*) = a$ for all elements $a \in \mathcal{A}$, and

$$\sum_{r=1}^{N} \kappa(u_{jr}) u_{rk} = \delta_{jk} I \tag{1.3}$$

$$\sum_{r=1}^{N} u_{jr} \kappa(u_{rk}) = \delta_{jk} I \tag{1.4}$$

Let us mention that later, in 1995, Woronowicz re-formulated this definition in the following way. The compact quantum group is a pair (A, Φ), consisting of a unital C^*-algebra A and a C^*-homomorphism Φ, such that:

(1)The diagram

$$A \xrightarrow{\Phi} A \otimes A \downarrow A \otimes A \xrightarrow{id\otimes\Phi} A \otimes A \otimes A \tag{1.5}$$

is commutative

(2) The sets $\{(b \otimes 1)\Phi(c) : b, c \in A\}$ and $\{(1 \otimes b)\Phi(c) : b, c \in A\}$ are both dense in $A \otimes A$.

Comparing the two definitions one may wonder, given the second definition, how to reconstruct the *-subalgebra $\mathcal{A}$ which seems essential in the first definition. The answer comes from the theory of unitary representations of compact quantum groups, and says that this *-subalgebra is generated by linear combination of matrix coefficients of the unitary representations of A.

In [SLW3] Woronowicz provided a general method for constructing compact matrix pseudogroups. The method depends on finding an N^N-element array $E = (E_{i_1,\ldots,i_N})_{i_1,\ldots,i_N=1,\ldots,N}$ of complex numbers, which is (left and right) non-degenerate. The Theorem 1.4 of [SLW3] says that if a C^*-algebra A, is generated by N^2 elements u_{jk} which satisfy:

$$\sum_{r=1}^{N} u_{jr}^* u_{rk} = \delta_{jk} I = \sum_{r=1}^{N} u_{jr} u_{rk}^* \tag{1.6}$$

$$\sum_{k_1,\ldots,k_N} u_{j_1k_1}\ldots u_{j_Nk_N} E_{k_1,\ldots,k_N} = E_{j_1,\ldots,j_N} I \tag{1.7}$$

and if the array E is non-degenerate, then (A,u) is a compact matrix quantum group, where $u = (u_{jk})_{j,k=1}^N$. If for $\mu \in (0,1]$ one defines $E_{i_1,\ldots,i_N} = (-\mu)^{i(\sigma)}$ if $\sigma(k) = i_k$ for $k = 1,\ldots,N$ is a permutation of $\{1,\ldots,N\}$ and $E_{i_1,\ldots,i_N} = 0$ otherwise, then as (A,u) one gets the quantum group $S_\mu U(N)$, called the *twisted* $SU(N)$ group. Here, for a permutation σ, $i(\sigma)$ is the number of inversions of the permutation σ, which is the number of pairs (j,k) such that $j < k$ and $i_j = \sigma(j) > \sigma(k) = i_k$. In this paper we present, for $N = 3$, this construction for another function on permutations,which gives rise to another array E.

2 Compact quantum groups associated with cycles in permutations

In this section we describe the matrix quantum groups that arise, through the general receipt of Woronowicz, by considering the function related to the number of cycles on symmetric group. We shall consider here the case of N=3.

For a sequence (i,j,k), with $\{i,j,k\} = \{1,2,3\}$, we define the function $c(i,j,k)$ as the number of cycles of the permutation $\begin{pmatrix} 1,2,3 \\ i,j,k \end{pmatrix}$. For $t > 0$ we define the array E in the following way:

$$E_{i,j,k} = \begin{cases} t^{3-c(i,j,k)} & \text{if} \quad \{i,j,k\} = \{1,2,3\} \\ 0 & \text{if} \quad \{i,j,k\} \subseteq \{1,2,3\} \quad \sharp\{i,j,k,\} \leq 2 \end{cases} \tag{2.8}$$

Then the non-zero entries of the array E are $E_{1,2,3} = 1$, $E_{1,3,2} = E_{2,1,3} = E_{3,2,1} = t$ and $E_{2,3,1} = E_{3,1,2} = t^2$.

In the sequel we shall study the Hilbert space irreducible *-representations of the C^*-algebra A generated by the elements $\{u_{jk} : j,k = 1,2,3\}$. The relations generating the algebra follow from the general theory of the unitary representations compact quantum groups. We shall skip these considerations in this exposition.

Let us say only, that the C^*-algebra A, and hence the quantum group (A,u) is generated by five elements a,b,c,d,v, which satisfy the following relations:

(1) $av = va$	(2) $cv = vc$	(3) $ac + tca = 0$
(4) $ac^* + tc^*a = 0$	(5) $cc^* = c^*c$	(6) $vv^* = v^*v = I$
(7) $aa^* + t^2cc^* = I$	(8) $a^*a + c^*c = I$	

The co-multiplication Φ in the quantum group (A, u) is given on generators by

$$\Phi(a) = a \otimes a + tc^*v^* \otimes c, \quad \Phi(c) = c \otimes a + a^*v^* \otimes c, \quad \Phi(v) = v \otimes v. \tag{2.9}$$

The co-inverse κ is defined by:

$$\kappa(a) = a^*v^*, \kappa(a^*v^*) = a, \kappa(c) = tc, \kappa(c^*v^*) = \frac{1}{t}c^*v^*, \kappa(v) = v \tag{2.10}$$

It follows from the relations $(1) - (8)$ that the elements a, c, a^*v^*, c^*v^* generate a dense *-subalgebra $\mathcal{A}$ of A. Therefore, we conclude that $G = (A, u)$ is a compact matrix quantum group, with the co-multiplication given by (2.8) and the co-inverse given by (2.9).

3 Irreducible representations of the C^*-algebra A

We shall now discuss representations of the C^*-algebra A as bounded operators on Hilbert spaces. This will follow the construction of Woronowicz and [SLW1].

Let us notice, that the elements a, c, a^*, c^* satisfy the relations defining the quantum group $SU_q(2)$ with $q = -t$. Hence, if $v = 1$, then (A, u) is equal to this quantum group. However, the group is different when the unitary is not identity.

We recall the construction from [SLW1] of the operators α, γ which satisfy the relations of $SU_q(2)$. The Hilbert space is $l_2(e_{n,k} : n \geq 0, -\infty < k < +\infty)$, and the operators are defined on the orthogonal basis as follows:

$$\alpha e_{n,k} = \sqrt{1 - q^{2n}} e_{n-1,k}, (n \geq 1), \alpha e_{0,k} = 0, \gamma e_{n,k} = q^{2n} e_{n,k+1} \tag{3.11}$$

In what follows we shall assume that $-1 < t = -q < 1$. Let H be a separable Hilbert space with a scalar product $\langle \ | \ \rangle$, and let $\pi : A \to B(H)$ be a (continuous, faithful) *-representation and let $A = \pi(a), C = \pi(c), V = \pi(v)$. Let us also assume, that there is no $\pi(A)$-invariant subspace of H. Then A, C, A^*V^*, C^*V^* satisfy the relations $1^o - 8^o$.

Since V commutes with all the other operators, and since there is no proper subspace of H, invariant for all the operators, it must be $V = \lambda I$ for some complex number $|\lambda| = 1$.

From the relations it also follows that $K_0 = kerC$ is an invariant subspace, and so is its orthogonal complement. Hence either (1) $K_0 = H$ or (2) $K_0 = \{0\}$. In the case (1) we have $C = 0$, and 7^o implies that A is then unitary. Thus in this case, since A and V commute, we have $A = \alpha I$ and $V = \lambda I$, with $|\alpha| = |\lambda| = 1$. It is evident that any pair of such α, λ defines an irreducible representation $\pi_{\alpha,\lambda}$ of $\mathcal{A}$. Therefore, we have the following:

Proposition 3.1 *Every pair α, λ of complex numbers, with $|\alpha| = |\lambda| = 1$, defines an irreducible $*$-representation $\pi_{\alpha,\lambda}$ of $\mathcal{A}$ by:*

$$\pi_{\alpha,\lambda}(A) = \alpha \cdot I, \quad \pi_{\alpha,\lambda}(V) = \lambda \cdot I, \quad \pi_{\alpha,\lambda}(C) = 0 \qquad (3.12)$$

Let us now consider the case (2) when $K_0 = \{0\}$ trivial. Then C is invertible on H. The kernel $H_0 = kerA$ of A is an invariant subspace for C, C^*, $V = \lambda \cdot I$ and V^*. Let us first observe that the kernel of A is not the whole space H. Indeed, $A = 0$ would imply $CC^* = I = t^2CC^*$, which would not be possible for an invertible C and a non-zero t with $|t| < 1$.

We are going to show that the kernel of A is non-trivial. Let us notice that, having trivial kernel, A would be invertible, as its image is an invariant subspace for A, A^*, C, C^*. The proof of $ker(A) \neq \{0\}$ follows the idea used in [C-H-M-S], in the proof of Theorem 4.4. First observe, that $P = CC^*$ is a positive operator and since $A^*A = I - P$ is also positive, we have $0 \leq P \leq I$. Hence the spectrum $Sp(P)$ of P is contained in the interval $[0,1]$. Also, zero is out of $Sp(P)$, because C is invertible. We claim that the spectrum $Sp(P)$ contains a point $\lambda < 1$. Otherwise, it would consists of 1 only, and then P would be a projection onto a subspace, on which $A^*A = 0$. Hence, the subspace would be $\{0\}$, and $P = 0$. Now, having a $\lambda \in Sp(P)$ with $0 < \lambda < 1$ it follows, that there is a sequence ξ_n of unit vectors, for which $\|P\xi_n - \lambda\xi_n\| \to 0$. This implies that $\|A\xi_n\| \to 1 - \lambda$. Hence, for $\eta_n = \frac{A\xi_n}{\|A\xi_n\|}$ one can show that $\|P\eta_n - t^{-2}\lambda\eta_n\| \to 0$, so that $t^{-2}\lambda \in Sp(P)$. It follows that $1 \in Sp(P)$ is an eigenvalue. Taking the associated eigenvector ξ with $P\xi = \xi$, one gets $A^*A\xi = (I - P)\xi = 0$, which contradicts the invertibility of A.

In what follows we shall assume that $dimH_0 \geq 1$, so that there are non-zero vectors in the kernel of A. If $x \in H_0$ and $x \neq 0$, then $C^*Cx = CC^*x = x - A^*Ax = x$, so C is unitary on H_0. This implies that $AA^*x = (1 - t^2)x$. Let us define $H_n := (A^*)^n H_0$, then

Lemma 3.2 *For all positive integers $n \neq m$ and for all $x \in H_0$, the following hold:*

1. $A(A^*)^n x = (1 - t^{2n})(A^*)^{n-1}x$,

2. $A^n(A^*)^n x = \prod_n^{k=1}(1 - t^{2k})x$,

3. $H_n \perp H_m$

Proof:

The proof in each of the three cases is inductive. We will use $AA^* = (1-t^2)I + t^2A^*A$, which easily follows from the relations on A, C, V. For the proof of (1) this relation gives the case $n = 1$. Then, for an $x \in H_0$ we have $A(A^*)^{n+1}x = AA^*(A^*)^n x = (1-t^2)x + t^2A^*A(A^*)^n x = (1-t^2)x + t^2(1-t^{2n})A^*(A^*)^{n-1}x = (1-t^{2n+2})(A^*)^n x$, from which (1) follows by induction. To proof the equality (2) we write $A^n(A^*)^n x = A^{n-1}[A(A^*)^n]x$ and then use (1) to get $A^n(A^*)^n x = (1-t^{2n})A^{n-1}(A^*)^{n-1}x$, which, through further inductive expansion, gives the desired equation.

For the proof of (3) let us assume that $n < m$ and let $k = m-n$. Then for an $y \in H_0$ we have $A^m(A^*)^n y = A^k[A^n(A^*)n]y = f_n(t) \cdot A^k y = 0$, since $k \geq 1$. Here $f_n(t)$ is the coefficient that appears in (2). Therefore, for arbitrary $x, y \in H_0$ one can compute $\langle (A^*)^m x | (A^*)^n y \rangle = \langle x | A^k[A^n(A^*)^n]y \rangle = f_n(t) \cdot \langle x | A^k y \rangle = 0$, and (3) follows. □

It is also easy to observe that both C and C^* preserve the subspaces H_n, for all positive integers n, and that on each of the subspaces C is a scalar multiple of a unitary operator.

Proposition 3.3 *For every positive integer n and for all $y \in H_n$ one has $CC^*y = t^{2n}y$*

Proof: This Proposition follows from the relation $CC^*A^* = t^2A^*CC^*$, applied n times to $CC^*y = CC^*(A^*)^n x$, with $x \in H_0$. Since C is normal, it follows that the operator D, defined on $y \in H_n$ by $Dy = (\frac{-1}{t})^n Cy$ is unitary on H_n. □

It follows that the orthogonal direct sum $\bigoplus_{n \geq 0} H_n$ is a non-trivial subspace of H, invariant for all the operators A, A^*, C, C^*, V, V hence it must be equal to the whole space H. Thus we have the following

Proposition 3.4 *The Hilbert space H has the orthogonal decomposition*

$$H = \bigoplus_{n=0}^{\infty} H_n$$

preserved by C, V, C^*, V^*, and with the action of $A : H_n \to H_{n-1}$ given by (1) of the Lemma (3.2).

We are now going to show that if $dimH_0 \geq 2$, then there is a non-trivial orthogonal decomposition of H into invariant subspaces. Thus, irreducibility would imply $dimH_0 = 1$.

Lemma 3.5 *If $dimH_0 \geq 2$ then there is a subspace $K \subset H$ invariant for all the operators A, A^*, C, C^*, V, V^*.*

Proof: Under the assumption $dimH_0 \geq 2$, for the unitary C on H_0 we have a non-trivial orthogonal decomposition $H_0 = K_0 \bigoplus K_0^\perp$, invariant for the unitary operator, and for is adjoint C^*. Then, each of the subspaces $K_n := (A^*)^n K_0$ is also invariant for both C and C^*. Moreover, the orthogonal complement of K_n in H_n is just $K_n^\perp = (A^*)^n K_0 \perp$. This can be seen with the help of the arguments that preceded the Proposition (3.4). The orthogonal sum $K = \bigoplus_{n \geq 0} K_n$ and its orthogonal complement $K^\perp = \bigoplus_{n \geq 0} K_n^\perp$ decompose H into their direct sum $H = K \bigoplus K^\perp$, which is invariant for the considered operators. □

Corollary 3.6 *If the representation π is irreducible and if the operator $A = \pi(a)$ has a non-trivial kernel H_0, then the kernel is one-dimensional. In this case, the representation π has the following form: $H = l^2$, and if $\{e_n : n \geq 0\}$ is the standard orthonormal basis, then $Ce_n = (-t)^n e_n = C^* e_n$, A is the standard unilateral shift with the adjoint A^*, and $Ve_n = \lambda e_n$ for some complex $|\lambda| = 1$.*

This way we have proved the following

Theorem 3.7 *The irreducible *-representations of the C^*-algebra $\mathcal{A}$ form the following two series:*

(1) One-dimensional characters $\pi_{\alpha,\lambda}$ with complex $\alpha, \lambda \in S^1 = \{|z| = 1\}$, given by the formulae:

$$\pi_{\alpha,\lambda}(a) = \alpha, \quad \pi_{\alpha,\lambda}(c) = 0, \quad \pi_{\alpha,\lambda}(v) = \lambda \tag{3.13}$$

(2) Infinite dimensional representations π_λ with $\lambda \in S^1$, acting on the orthonormal standard basis $\{\delta_n : \quad n \geq 0\}$ of l^2 as:

$$\begin{array}{ll} \pi_\lambda(a) = A, & A\delta_n = \sqrt{1 - t^{2n}}\delta_{n-1}, \quad A\delta_0 = 0, \\ \pi_\lambda(c) = C, & C\delta_n = (-t)^n \delta_n, \\ \pi_\lambda(v) = V, & V\delta_n = \lambda\delta_n. \end{array} \tag{3.14}$$

We end this section with description of the algebraic structure of the C^*-algebra $\mathcal{A}$. The unitary v is in the center of $\mathcal{A}$, and generates a C^*-algebra A_2 isomorphic to the algebra $C(S^1)$ of continuous functions functions on the unit circle. Also, the elements a, c generate the C^*-algebra A_1 which is isomorphic to the C^*-algebra of the quantum group $SU_{-t}(2)$. Therefore the algebra $\mathcal{A}$ is the tensor product $\mathcal{A} = A_1 \otimes A_2$.

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