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QUANTUM SEX AND MUTUAL INFORMATION

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ABSTRACT. The operational structure of quantum pairings, couplings entanglements and encodings is studied and classified for general von Neumann algebras. We show that the classical-quantum correspondences such as encodings can be treated as diagonal CP semi-classical (c-) couplings, and true entanglements are characterized by transpose-CP truly quantum (q-) couplings. The relative entropy of the diagonal compound and entangled states lead to two different types of entropies for a given quantum state on a the von Neumann entropy, which is achieved as the supremum of the information over all d-entanglements, and the dimensional entropy, which is achieved at the standard entanglement, the true quantum entanglement, coinciding with a d-entanglement only in the case of pure marginal states. The q-capacity of a quantum noiseless channel, defined as the supremum over all entanglements, is given by the logarithm of the dimensionality of the input algebra. It may double the classical capacity, achieved as the supremum over all c-couplings, or encodings, which is bounded by the logarithm of the dimensionality of a maximal Abelian subalgebra.

1. INTRODUCTION

In this paper we develop the operational approach to quantum entanglement [1], extending the notion of quantum conditional entropy and mutual information to the general von Neumann algebras with normal semifinite faithful weights. By quantum sex we call pairings, such as quantum couplings, entanglements and encoodings, of two systems $(\mathcal{A}, \mu)$ and $(\mathcal{B}, \nu)$, reffered in quantum communications as Allice and Bob, with respect to the given weights μ, ν on the von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$ respectively.

The entanglements as specifically quantum correlations, are used to study quantum information processes, in particular, quantum computations, quantum teleportation, quantum cryptography [2, 3, 4]. There have been mathematical studies of the entanglements in [5, 6, 7], in which the entangled state is defined by a state not written as a form of a convex combination $\sum_n \varrho_n \otimes \varsigma_n p(n)$ with any states ϱ_n and ς_n. However it is obvious that there exist several types of the correlated states written as 'separable' forms above. Such correlated, or classically entangled states have been also discussed in several contexts in quantum probability such as quantum measurement and filtering [8, 9], quantum compound state [10, 11] and lifting [12].

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In this paper, we study the mathematical structure of classical-quantum and quantum-quantum couplings to provide a finer classification of quantum separable and entangled states, and we discuss the informational degree of entanglement and entangled quantum mutual entropy.

The term entanglement was introduced by Schrödinger in 1935 out of the need to describe correlations of quantum states not captured by mere classical statistical correlations as the convex combinations of noncorrelated states. In this spirit the by now standard definition of entanglement is the state of a compound quantum system 'which cannot be prepared by two separated devices with only correlated classical data as their inputs' (see for example Werner, 1989. We show that the entangled states can be achieved by quantum (q-) encodings, the nonseparable couplings of states, in the same way as the separable states can be achieved by classical (c-) encodings.

The compound states, called o-coupled, are defined by orthogonal decompositions of their marginal states. This is a particular case of so called diagonal state of a compound system, the convex combination of the special product states which we call d-compound. The d-compound states are most informative among c-compound states in the sense that the maximum of mutual entropy over all c-couplings to the quantum system is achieved on the extreme d-coupled (even o-coupled) states as the von Neumann entropy $\mathsf{S}(\varsigma)$ of a given normal state ς on a simple algebra $\mathcal{A}$. Thus the maximum of mutual entropy over all classical couplings of (classical) probe systems $\mathcal{A}$ to a quantum system $\mathcal{B}$, is bounded by $\ln \operatorname{rank}\mathcal{B}$, the logarithm of the rank of the algebra $\mathcal{B}$ which is defined as the dimensionality $\dim \mathcal{H}$ of the Hilbert space $\mathcal{H}$ for irreducible representation of $\mathcal{B}$. Due to $\dim \mathcal{B} = (\operatorname{rank}\mathcal{B})^2$ for the simple $\mathcal{B}$, it is achieved on the normal tracial density operator $\sigma = (\operatorname{rank}\mathcal{B})^{-1} I$ only in the case of finite dimensional $\mathcal{B}$.

More general than o-coupled states, the d-entangled states, are defined as c-entangled states by orthogonal decomposition of only one marginal state on the probe algebra $\mathcal{A}$. In general they can give larger mutual entropy for a quantum noisy channel than the o-coupled state (which gains the same information as d-coupled extreme states in the case of a deterministic channel).

We prove that the truly entangled pure states are most informative in the sense that the maximum of mutual entropy over all entanglements to the quantum system $\mathcal{B}$ is achieved on the q-compound state, given by an extreme (standard) entanglement of the probe system $\mathcal{A} = \mathcal{B}$ with coinciding marginals, called standard for a given ς. The gained information for such extreme q-compound state defines another type of entropy, the q-entropy $\mathsf{H}(\varsigma)$ which is bigger than the von Neumann entropy $\mathsf{S}(\varsigma)$ in the case of mixed ς. The maximum of mutual entropy over all quantum couplings, including the true quantum entanglements of probe systems $\mathcal{A}$ to the system $\mathcal{B}$, is bounded by $\ln \dim\mathcal{B}$, the logarithm of the dimensionality of the von Neumann algebra $\mathcal{B}$, which is achieved on a normal tracial σ in the case of finite dimensional $\mathcal{B}$. Thus the q-entropy $\mathsf{H}(\varsigma)$, which can be called the dimensional entropy, is the true quantum entropy, in contrast to the von Neumann $\mathsf{S}(\varsigma)$, the c-entropy which is semi-classical entropy achieved as a supremum over all couplings with the classical probe systems $\mathcal{A}$. In the case of finite-dimensional $\mathcal{B}$ the q-capacity $\mathsf{C}_q = \ln \dim\mathcal{B}$ is achieved as the supremum of mutual entropy over all q-encodings, the. quantum-quantum correspondences, described by entanglements. It is strictly larger then the classical capacity $\mathsf{C}_c = \ln \operatorname{rank}\mathcal{B}$ of the identity channel, which is achieved as the supremum over usual encodings, described by the classical-quantum correspondences $\mathcal{A}^0 \to \mathcal{B}$.

In this short paper we consider the case of decomposable probe algebras $\mathcal{A}$ but simple algebra $\mathcal{B} = \mathcal{L}(\mathcal{H})$ for which the proofs are rather straightforward. More general decomposable algebra $\mathcal{B}$ including the classical discrete systems as a particular Abelian case is considered in [13], and even more general case of von Neumann algebras will be also published elsewhere.

2. Pairings, Couplings and Entanglements

Let $\mathcal{H}$ denote the Hilbert space of a quantum system, and $\mathcal{B} = \mathcal{L}(\mathcal{H})$ be the algebra of all linear bounded operators on $\mathcal{H}$. It consists of all operators $A : \mathcal{H} \to \mathcal{H}$ having the adjoints $A^\dagger$ on $\mathcal{H}$. A linear functional $\varsigma : \mathcal{B} \to \mathbf{C}$ is called a state on $\mathcal{B}$ if it is positive (i.e., $\varsigma(B) \geq 0$ for any positive operator $B = A^\dagger A$ in $\mathcal{B}$) and normalized $\varsigma(I) = 1$ for the identity operator I in $\mathcal{A}$. A normal state can be expressed as

$$\varsigma(B) = \mathrm{Tr}\varkappa^\dagger B \varkappa \equiv \langle B, \sigma \rangle, \quad B \in \mathcal{B}, \tag{1}$$

where $\varkappa$ is a linear Hilbert-Schmidt operator from $\mathcal{H}$ to (another) Hilbert space $\mathcal{G}$, $\varkappa^\dagger$ is the adjoint operator from $\mathcal{G}$ to $\mathcal{H}$. Here Tr stands for the usual trace in $\mathcal{G}$, and in the case of ambiguity it will also be denoted as $\mathrm{Tr}_\mathcal{G}$. This $\varkappa$ is called the *amplitude operator* which can always be considered on $\mathcal{G} = \mathcal{H}$ as the square root of the operator $\varkappa\varkappa^\dagger$ (it is called simply amplitude if $\mathcal{G}$ is one dimensional space $\mathbb{C}$, $\varkappa = \eta \in \mathcal{H}$ with $\varkappa^\dagger\varkappa = \|\eta\|^2 = 1$, in which case $\varkappa^\dagger$ is the functional $\eta^\dagger$ from $\mathcal{H}$ to $\mathbb{C}$).

We can always equip $\mathcal{H}$ (and will equip all auxiliary Hilbert spaces, e.g. $\mathcal{G}$) with an isometric involution $J = J^\dagger$, $J^2 = I$ having the properties of complex conjugation

$$J \sum \lambda_j \eta_j = \sum \bar{\lambda}_j J \eta_j, \quad \forall \lambda_j \in \mathbb{C}, \eta_j \in \mathcal{H},$$

and denote by $\langle B, \sigma \rangle$ the tilda-pairing $\mathrm{Tr}B\tilde{\sigma}$ of $\mathcal{B}$ with the trace class operators $\sigma \in \mathcal{T}(\mathcal{H})$ such that $\tilde{\sigma} = J\sigma^\dagger J$. We shall call $\sigma = J\varkappa\varkappa^\dagger J = \tilde{\varkappa}^\dagger\tilde{\varkappa}$ the probability density of the state (1) with respect to this pairing, and assume that the support E_σ of σ is the minimal projector $E = E^\dagger \in \mathcal{B}$ for which $\varsigma(E) = 1$, i.e. that $\overline{E_\sigma} := JE_\sigma J = E_\sigma$. The latter can also be expressed as the symmetricity property $\widetilde{E_\sigma} = E_\sigma$ with respect to the tilda operation (transposition) $\widetilde{B} = JB^\dagger J$ on $\mathcal{L}(\mathcal{H})$. One can always assume that J is the standard complex conjugation in an eigen-representation of σ such that $\bar{\sigma} = \varkappa\varkappa^\dagger = \tilde{\sigma}$ coincides with σ as the real element of the invariant maximal Abelian subalgebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ of all diagonal (and thus symmetric) operators in this basis.

The auxiliary Hilbert space $\mathcal{G}$ and the amplitude operator in (1) are not unique, however $\varkappa$ is defined uniquely up to a unitary transform $\varkappa^\dagger \mapsto U\varkappa^\dagger$ in $\mathcal{G}$, and $\mathcal{G}$ can be always taken minimal, identified with the support $\mathcal{H}_\sigma = E_\sigma\mathcal{H}$ for σ, the closure of $\sigma\mathcal{H}$ (E_σ is the minimal orthoprojector in $\mathcal{B}$ such that $\sigma E = \sigma$). In general, $\mathcal{G}$ is not one dimensional, the dimensionality $\dim \mathcal{G}$ must not be less than $\mathrm{rank}\varkappa^\dagger = \mathrm{rank}\sigma$, the dimensionality of the range $\mathcal{G}_\rho = \mathrm{ran}\varkappa^\dagger$ coinciding with the support for $\rho = \varkappa^\dagger\varkappa \simeq \tilde{\sigma}$.

Given the amplitude operator $\varkappa : \mathcal{G} \to \mathcal{H}$, one can define not only the state ς but also the normal state

$$\varrho(A) = \mathrm{Tr}\tilde{\varkappa}^\dagger A \tilde{\varkappa} \equiv \langle A, \rho \rangle, \quad A \in \mathcal{A} \tag{2}$$

on $\mathcal{A} = \mathcal{L}(\mathcal{G})$ as the marginal of the *pure compound* state

$$\omega(A \otimes B) = \mathrm{Tr}\tilde{A}\varkappa^{\dagger}B\varkappa = \mathrm{Tr}\tilde{\varkappa}^{\dagger}A\tilde{\varkappa}\widetilde{B}.$$

on the algebra $\mathcal{A} \otimes \mathcal{B}$ of all bounded operators on the Hilbert tensor product space $\mathcal{G} \otimes \mathcal{H}$.

Indeed, thus defined bilinear form with $\tilde{A} = JA^{\dagger}J$ is uniquely extended to such a state, given on $\mathcal{L}(\mathcal{G} \otimes \mathcal{H})$ by the amplitude $\psi = \varkappa'$, where $(\zeta \otimes \eta)^{\dagger} \varkappa' = \eta^{\dagger}\varkappa J\zeta$ for all $\zeta \in \mathcal{G}$, $\eta \in \mathcal{H}$.

This pure compound state ω is so called *entangled state*, unless its marginal state ς (and ϱ) is pure, corresponding to a rank one operator $\varkappa^{\dagger} = \zeta\eta^{\dagger}$, in which case $\omega = \varrho \otimes \varsigma$, given by the amplitude $v = \zeta \otimes \eta$. The amplitude operator $\varkappa$ corresponding to mixed states on $\mathcal{A}$ and $\mathcal{B}$ will be called the *entangling operator* of $\rho = \varkappa^{\dagger}\varkappa$ to $\sigma = \tilde{\varkappa}^{\dagger}\tilde{\varkappa}$.

As follows from the next theorem, any pure entangled state

$$\omega(A \otimes B) = \psi^{\dagger}(A \otimes B)\psi, \quad A \otimes B \in \mathcal{L}(\mathcal{G} \otimes \mathcal{H})$$

given by an amplitude $\psi \in \mathcal{G} \otimes \mathcal{H}$, can be achieved as described by a unique entanglement $\varkappa$ to the algebra $\mathcal{A} = \mathcal{L}(\mathcal{G})$ of the marginal state ς on $\mathcal{B} = \mathcal{L}(\mathcal{H})$.

Before to formulate this theorem in the generality which we need for further consideration, let us introduce the following notations.

Let $\mathcal{A}$ be a $*$-algebra on $\mathcal{G}$ with a normal faithful semifinite weight μ, $\mathcal{A}'$ denote the commutant $\{A' \in \mathcal{L}(\mathcal{G}) : [A', A] = 0, \forall A \in \mathcal{A}\}$ of $\mathcal{A}$, and $\left(\widetilde{\mathcal{A}}, \tilde{\mu}\right)$ denote the transposed algebra of the operators $\widetilde{A} = JA^{\dagger}J$ with $\tilde{\mu}(A) = \mu\left(\widetilde{A}\right)$ which may not coincide with $(\mathcal{A}, \mu)$ (and with $\mathcal{A}'$). We denote by $\mathcal{A}_{\mu} \subseteq \mathcal{A}$ the space of all operators $A \in \mathcal{A}$ in the form $x^{\dagger}z$, where $x, z \in \mathfrak{a}_{\mu}$, with $\mathfrak{a}_{\mu} = \left\{x \in \mathcal{A} : \mu\left(x^{\dagger}x\right) < \infty\right\}$, and by $(\mathcal{G}_{\mu}, \iota, J_{\mu})$ the standard representation $\iota : \mathcal{A} \to \mathcal{L}(\mathcal{G}_{\mu})$ given by the left multiplication $\iota(A)x = Ax$ on $\mathfrak{a}_{\mu}$, with the standard isometric involution $J_{\mu} : x \mapsto x^{\dagger}$ defining normal faithful representation $\tilde{\iota}\left(\widetilde{A}\right) = J_{\mu}\iota\left(A^{\dagger}\right)J_{\mu} \equiv \widetilde{\iota(A)}$ of the transposed algebra $\widetilde{\mathcal{A}}$ on the completion $\mathcal{G}_{\mu}$ of the left module $\mathfrak{a}_{\mu}$ with respect to the inner product $(x|z)_{\mu} = \mu\left(x^{\dagger}z\right)$. We recall that the von Neumann algebra $\mathcal{A}$ defined by $\mathcal{A}'' = \mathcal{A}$ is anti-isomorphic to $\iota(\mathcal{A})' = J_{\mu}\iota(\mathcal{A})J_{\mu}$ and thus $\widetilde{\mathcal{A}} \simeq \iota(\mathcal{A})'$, and that $\widetilde{\mathcal{A}} = \mathcal{A}_{\mu}^{*}$ as the space of all continuous functionals $\widetilde{A} : \phi \mapsto \left\langle \phi, \widetilde{A}\right\rangle_{\mu}$ with respect to the pairing

$$\left\langle x^{\dagger}z, \widetilde{A}\right\rangle_{\mu} := \left(x|\widetilde{\iota(A)}z\right)_{\mu} \equiv \left\langle A, \widetilde{x^{\dagger}z}\right\rangle_{\mu}, \quad x^{\dagger}z \in \mathcal{A}_{\mu}, .\widetilde{A} \in \widetilde{\mathcal{A}}.$$

The completion of $\mathcal{A}_{\mu}$ with respect to the $*$-norm $\left\|x^{\dagger}z\right\|_{*} = \sup\left\{\left|\left\langle A, \widetilde{x^{\dagger}z}\right\rangle_{\mu}\right| : \|A\| < 1\right\}$ and the is indentified with the predual Banach space denoted as $\mathcal{A}_{*}$ (if $\mu = \tau|\mathcal{A}$ is the usual trace $\tau = \mathrm{Tr}_{\mathcal{G}}$ on $\mathcal{A}$, then $\mathcal{A}_{\mu}$ coincides with $\mathcal{A}_{*}$ as the class $\mathcal{A}_{\tau} = \mathcal{A} \cap \mathcal{T}(\mathcal{G})$ of trace operators $\mathcal{T}(\mathcal{G}) = \left\{x^{\dagger}z : x, z \in \mathcal{S}(\mathcal{G})\right\}$, where $\mathcal{S}(\mathcal{G}) = \left\{x \in \mathcal{L}(\mathcal{G}) : \mathrm{Tr}_{\mathcal{G}}x^{\dagger}x < \infty\right\}$).

Note that $\widetilde{\mathcal{A}} \neq \mathcal{A}$, $\tilde{\mu} \neq \mu$ in the standard representation $\mathcal{H} = \mathcal{H}_{\mu}$, $J = J_{\mu}$, $\widetilde{\mathcal{A}} = \mathcal{A}'$ unles $\mathcal{A}$ is Abelian as only in this case $\mathcal{A}' = \mathcal{A}$. If $\mathcal{A}$ is not the algebra of all operators $\mathcal{L}(\mathcal{G})$, the density operator ρ for a normal state (2) is not unique even with respect to $\tau = \mathrm{Tr}_{\mathcal{G}}$. However it is uniquely defined as the bounded *probability density* $\rho = Jx^{\dagger}xJ = \bar{x}^{\dagger}\bar{x}$ with respect to the restriction $\mu = \tau|\mathcal{A}$ (i.e. as the density operator with respect to μ) describing this state as $\langle A, \rho\rangle_{\mu} = \mu\left(xAx^{\dagger}\right)$ by the additional condition $\varkappa = \bar{x} \in \widetilde{\mathcal{A}_{\mu}}$. Note that each probability density $\rho \in \widetilde{\mathcal{A}_{\mu}}$ describing the normal state $\varrho(A) = \langle A, \rho\rangle_{\mu}$ on $\mathcal{A} \ni A$ is positive and normalized as $\langle I, \rho\rangle_{\mu} = 1$, but the predual space $\widetilde{\mathcal{A}}_{*} = \widetilde{\mathcal{A}_{*}}$ as the $*$-completion of $\widetilde{\mathcal{A}_{\mu}}$ may consist of not only the bounded densities with respect to μ (however each $\rho \in \widetilde{\mathcal{A}}_{*}$ can always be approximated by the bounded $\rho_n \in \widetilde{\mathcal{A}_{\mu}}$).

In the following formulation $\mathcal{B}$ can also be more general von Neumann algebra than $\mathcal{L}(\mathcal{H})$, with a normal faithful semifinite weight $\nu : \mathcal{B}_{\nu} \mapsto \mathbb{C}$ defining the pairing $\langle B, v^{\dagger}v\rangle_{\nu} = \left(\bar{v}|\widetilde{\iota(B)}\bar{v}\right)_{\nu}$, where $v \in \widetilde{\mathfrak{b}_{\nu}}$ ($\mathcal{B}_{\nu} = \mathfrak{b}_{\nu}^{\dagger}\mathfrak{b}_{\nu}$ coincides with $\mathcal{B}_{*}$ in the case of the standard trace $\nu(B\tilde{\sigma}) = \mathrm{Tr}B\tilde{\sigma} = \langle B, \sigma\rangle_{\nu}$ when $\mathfrak{b}_{\nu}$ is the space of Hilbert-Schmidt operators $y \in \mathcal{B}$ and $\widetilde{\mathcal{B}} = \mathcal{B}$).

Theorem 2.1. *Let* $\omega : \mathcal{A} \otimes \mathcal{B} \to \mathbb{C}$ *be a normal compound state*

$$\omega(A \otimes B) = \left(\bar{v}|\iota\widetilde{(A \otimes B)}\bar{v}\right) \equiv \left\langle A \otimes B, v^{\dagger}v\right\rangle, \tag{3}$$

described by an amplitude operator $v : \mathcal{G} \otimes \mathcal{H} \to \mathcal{E} \otimes \mathcal{F}$ *on the tensor product of Hilbert spaces* $\mathcal{E}$ *and* $\mathcal{F}$, *satisfying the condition*

$$v^{\dagger}v \in \widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{B}}, \quad (\mu \otimes \nu)\left(\bar{v}^{\dagger}\bar{v}\right) = 1.$$

Here $\mu \otimes \nu$ *is the product weight the pairing of* $\mathcal{A} \otimes \mathcal{B}$ *in (3) with* $(\mathcal{A} \otimes \mathcal{B})_{\mathsf{T}} = \widetilde{(\mathcal{A} \otimes \mathcal{B})}_{*}$, *and* $\bar{v} = JvJ$. *Then this state is achieved by an entangling operator* $\varkappa : \mathcal{G} \otimes \mathcal{F} \to \mathcal{E} \otimes \mathcal{H}$ *as*

$$\left\langle A, \nu\left(\varkappa^{\dagger}(I \otimes B)\varkappa\right)\right\rangle_{\mu} = \omega(A \otimes B) = \left\langle B, \mu\left(\tilde{\varkappa}^{\dagger}(A \otimes I)\tilde{\varkappa}\right)\right\rangle_{\nu} \tag{4}$$

for all $A \in \mathcal{A}$ *and* $B \in \mathcal{B}$ *such that*

$$\nu\left(\varkappa^{\dagger}(I \otimes B)\varkappa\right) \subseteq \widetilde{\mathcal{A}}, \quad \mu\left(\tilde{\varkappa}^{\dagger}(A \otimes I)\tilde{\varkappa}\right) \subseteq \widetilde{\mathcal{B}}.$$

The operator $\varkappa$ *together with* $\tilde{\varkappa} = J\varkappa^{\dagger}J$ *is uniquely defined by* $v = U\varkappa'$, *where*

$$\left(\xi \otimes \eta'\right)^{\dagger}\varkappa'(\zeta \otimes J\eta) = (\xi \otimes \eta)^{\dagger}\varkappa(\zeta \otimes J\eta'), \quad \xi \in \mathcal{E}, \eta' \in \mathcal{F}, \zeta \in \mathcal{G}, \eta \in \mathcal{H}, \tag{5}$$

up to a unitary transformation U *of the minimal subspace space* $\mathrm{ran}v \subseteq \mathcal{E} \otimes \mathcal{F}$.

Proof. Without loss of generality we can assume that $\mathcal{E} = \mathcal{G}_{\rho}$, $\mathcal{F} = \mathcal{H}_{\sigma}$ and $v^{\dagger} = v(E_{\rho} \otimes E_{\sigma})$ as the support $(\mathcal{G} \otimes \mathcal{H})_{v^{\dagger}v} = \mathrm{ran}v^{\dagger}$ for $v^{\dagger}v$ is contained in $\mathcal{G}_{\rho} \otimes \mathcal{H}_{\sigma}$. By virtue $v^{\dagger}v \in \left(\widetilde{\mathcal{A}'} \otimes \widetilde{\mathcal{B}'}\right)'$ the range of v is invariant under the action

$$(A \otimes B)v = v(AE_{\rho} \otimes BE_{\sigma}), \quad \forall A \in \widetilde{\mathcal{A}'}, B \in \widetilde{\mathcal{B}'}$$

of the commutant $\left(\widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{B}}\right)' = \widetilde{\mathcal{A}'} \otimes \widetilde{\mathcal{B}'}$. Let us equip $\mathcal{G}$ and $\mathcal{H}$ with the involutions J leaving invariant $\mathcal{G}_{\rho} = E_{\rho}\mathcal{G}$ and $\mathcal{H}_{\sigma} = E_{\sigma}\mathcal{H}$, with $J_{\rho} = E_{\rho}J$, $J_{\sigma} = E_{\sigma}J$, and $\mathcal{E} \otimes \mathcal{F} = \mathcal{G}_{\rho} \otimes \mathcal{H}_{\sigma}$ with the induced involution $J(\zeta \otimes \eta) = J_{\rho}\zeta \otimes J_{\sigma}\eta$. It easy to check for such v and $\varkappa = v'$ defined by $v = \varkappa'$ in (5) that for any $A \in \mathcal{A}'$ and

$B \in \widetilde{\mathcal{B}}'$

$$\begin{aligned}\left(\widetilde{A}\xi \otimes \eta\right)^{\dagger} \varkappa\left(\zeta \otimes \overline{B}J\eta'\right) &= \left(\widetilde{A}\xi \otimes B\eta'\right)^{\dagger} \upsilon\left(\xi \otimes J\eta\right) = \left(\xi \otimes \eta'\right)^{\dagger} \upsilon\left(\overline{A}\xi \otimes J\widetilde{B}\eta\right) \\ &= \left(\xi \otimes \widetilde{B}\eta\right)^{\dagger} \varkappa\left(\overline{A}\zeta \otimes J\eta'\right)\end{aligned}$$

where $\overline{A} = JAJ \in \widetilde{\mathcal{A}}'$, $\overline{B} = JBJ \in \mathcal{B}'$. Hence for any $B \in \mathcal{B}$

$$\left(A \otimes B'\right) \varkappa^{\dagger} \left(I \otimes B\right) \varkappa = \varkappa^{\dagger} \left(A \otimes B'B\right) \varkappa = \varkappa^{\dagger} \left(I \otimes B\right) \varkappa \left(A \otimes B'\right),$$

where $A \in \widetilde{\mathcal{A}}'_{\rho} := \widetilde{\mathcal{A}}'E_{\rho}$, $B' \in \mathcal{B}'_{\sigma} := \mathcal{B}'E_{\sigma}$, and for any $A \in \mathcal{A}$

$$\left(A' \otimes B\right) \tilde{\varkappa}^{\dagger} \left(A \otimes I\right) \tilde{\varkappa} = \varkappa\left(A'A \otimes B\right) \varkappa^{\dagger} = \tilde{\varkappa}^{\dagger} \left(A \otimes I\right) \tilde{\varkappa}\left(A' \otimes B\right),$$

where $A' \in \mathcal{A}'$ and $B \in \widetilde{\mathcal{B}}'$. Thus for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$

$$\varkappa^{\dagger} \left(I \otimes B\right) \varkappa \in \left(\widetilde{\mathcal{A}'_{\rho}} \otimes \mathcal{B}'_{\sigma}\right)', \quad \tilde{\varkappa}^{\dagger} \left(A \otimes I\right) \tilde{\varkappa} \in \left(\mathcal{A}'_{\rho} \otimes \widetilde{\mathcal{B}'_{\sigma}}\right)'.$$

Moreover, due to $\mathcal{A}''_{\rho} = E_{\rho}\mathcal{A}E_{\rho} \equiv \mathcal{A}_{\rho}$ and $\mathcal{B}''_{\sigma} = E_{\sigma}\mathcal{B}E_{\sigma} \equiv \mathcal{B}_{\sigma}$

$$\varkappa^{\dagger} \left(I \otimes B\right) \varkappa \subseteq J_{\rho}\mathcal{A}_{\mu}J_{\rho} \otimes E_{\sigma}\mathcal{B}_{\nu}E_{\sigma} := \left(\widetilde{\mathcal{A}_{\rho}} \otimes \mathcal{B}_{\sigma}\right)_{\tilde{\mu}\otimes\nu},$$

$$\tilde{\varkappa}^{\dagger} \left(A \otimes I\right) \tilde{\varkappa} \subseteq E_{\rho}\mathcal{A}_{\mu}E_{\rho} \otimes J_{\sigma}\mathcal{B}_{\nu}J_{\sigma} := \left(\mathcal{A}_{\rho} \otimes \widetilde{\mathcal{B}_{\sigma}}\right)_{\mu\otimes\tilde{\nu}}$$

as bounded by $\|B\| \varkappa^{\dagger}\varkappa$ and by $\|A\| \tilde{\varkappa}^{\dagger}\tilde{\varkappa}$ respectively. The partial weights ν and μ on these reduced algebras are defined as

$$\nu\left(\varkappa^{\dagger}\left(I \otimes B\right)\varkappa\right) = \left\langle B, \upsilon^{\dagger}\upsilon\right\rangle_{\nu}, \; \mu\left(\tilde{\varkappa}^{\dagger}\left(A \otimes I\right)\tilde{\varkappa}\right) = \left\langle A, \upsilon^{\dagger}\upsilon\right\rangle_{\mu}, \tag{6}$$

according to $\left\langle A, \left\langle B, \upsilon^{\dagger}\upsilon\right\rangle_{\nu}\right\rangle_{\mu} = \left\langle A \otimes B, \upsilon^{\dagger}\upsilon\right\rangle = \left\langle B, \left\langle A, \upsilon^{\dagger}\upsilon\right\rangle_{\mu}\right\rangle_{\nu}$. In particular

$$\nu\left(\varkappa^{\dagger}\varkappa\right) = \tilde{\nu}\left(\upsilon^{\dagger}\upsilon\right) = \rho, \; \mu\left(\tilde{\varkappa}^{\dagger}\tilde{\varkappa}\right) = \tilde{\mu}\left(\upsilon^{\dagger}\upsilon\right) = \sigma.$$

Any other choice of υ with the minimal $\mathcal{E} \otimes \mathcal{F} \simeq \mathcal{G}_{\rho} \otimes \mathcal{H}_{\sigma}$ is unitary equivalent to $\varkappa'$. ∎

Note that the entangled state (3) is written in (4) as

$$\left\langle B, \varpi\left(A\right)\right\rangle_{\nu} = \omega\left(A \otimes B\right) = \left\langle A, \varpi^{\intercal}\left(B\right)\right\rangle_{\mu}$$

in terms of the mutually adjoint maps $\varpi : \mathcal{A} \to \mathcal{B}_{*}$ and $\varpi^{\intercal} : \mathcal{B} \to \mathcal{A}_{*}$. They are given in (6) as

$$\varpi\left(A\right) = \left\langle A, \upsilon^{\dagger}\upsilon\right\rangle_{\mu} = \widetilde{\pi^{*}\left(A\right)}, \quad \varpi^{\intercal}\left(B\right) = \left\langle B, \upsilon^{\dagger}\upsilon\right\rangle_{\nu} = \widetilde{\pi\left(B\right)}, \tag{7}$$

where the linear map $\pi : \mathcal{B} \to \mathcal{A}_{\mu}$ and the adjoint $\pi^{*} : \mathcal{A} \to \mathcal{B}_{\nu}$ are defined as partial weights

$$\pi\left(B^{\dagger}\right) = J\left\langle B, \upsilon^{\dagger}\upsilon\right\rangle_{\nu} J, \quad \pi^{*}\left(A^{\dagger}\right) = J\left\langle A, \upsilon^{\dagger}\upsilon\right\rangle_{\mu} J.$$

The linear normal map ϖ in (6) is written in the Kraus-Steinspring form [17] and thus is completely positive (CP) but not unital, normalized to the density operators $\sigma = \omega\left(I\right)$ with respect to the weight ν.

A linear map $\pi : \mathcal{B} \to \mathcal{A}_{*}$ is called *tilda-positive* if $\pi^{\sim}\left(B\right) := J\pi\left(B\right)^{\dagger} J$ is positive for any positive (and thus Hermitian) operator $B \geq 0$ in the sense of non-negative definiteness of B. It is called *tilda-completely positive* (TCP) if the operator-matrix $\pi^{\sim}\left(\mathbf{B}\right) = J\pi\left(\mathbf{B}\right)^{\dagger} J$ is positive for every positive operator-matrix $\mathbf{B} = \left[B_{ik}\right] = \mathbf{B}^{*}$, where $\mathbf{A}^\dagger = \left[A^\dagger_{ik}\right]$, $\mathbf{B}^* = \left[B^\dagger_{ki}\right]$ (and thus $\mathbf{A}^\dagger = [A_{ki}]$ for $\mathbf{A} = [A_{ik}] \geq 0$, and $\mathbf{B}^* = \mathbf{B}$ for $\mathbf{B} \geq 0$). Obviously every tilda-positive and tilda-completely positive π is positive as positive is $\tilde{A} = JA^\dagger J$ for every positive A, but it is not necessarily completely positive unless $\tilde{A} = A$ for all $A \in \mathcal{A}$, in which case $\mathcal{A}$ is Abelian (or the Abelian is $\mathcal{B}$).

The map π defined in (8) as a TCP $\dagger$-map, $\pi\left(B^\dagger\right) = \pi(B)^\dagger$, is obviously transpose-CP in the sense of positivity of $\pi(\mathbf{B})^\dagger = [\pi(B_{ki})] = \pi\left(\mathbf{B}^\dagger\right)$ for any $\mathbf{B} \geq 0$, but it is in general not CP. Because every transpose-CP map can be represented as tilda-CP, there might be a positive-definite matrix $\mathbf{B}$ for which $\pi(\mathbf{B})$ is not positive. Note that the adjoint map $\pi^* = \bar{\pi}^\intercal$ is also TCP, as well as the maps $\tilde{\pi} = \bar{\pi}$ and $\pi^\intercal = \bar{\pi}^*$, where $\bar{\pi}(B) = J\pi\left(\overline{B}\right)J$, obtained from (6) as partial tracings

$$\bar{\pi}(B) = \nu\left(\varkappa^\dagger\left(I \otimes \widetilde{B}\right)\varkappa\right), \quad \pi^\intercal(A) = \mu\left(\tilde{\varkappa}^\dagger\left(\widetilde{A} \otimes I\right)\tilde{\varkappa}\right). \tag{8}$$

In these terms of the compound state (4) is written as

$$\langle A|\pi(B)\rangle_\mu = \omega\left(A^\dagger \otimes B\right) = \langle \pi^*(A)|B\rangle_\nu,$$

where $\langle x|y\rangle = \langle y, \overline{x}\rangle$ defines an inner product which coincides in the case of traces with the GNS product $(x|y)$.

In the following definition the predual space $\mathcal{B}_\intercal = \widetilde{\mathcal{B}}_*$ (as well as $\mathcal{A}_\intercal = \widetilde{\mathcal{A}}_*$) is identified by the pairing $\langle B, \sigma\rangle_\nu = \varsigma(B)$ with the space of generalized density operators σ which are thus uniquely defined as selfadjoint operators (could be unbounded) in $\mathcal{H}$. Note that $\mathcal{B}_\intercal = \mathcal{B}_\nu$ if $\mathcal{B} = \widetilde{\mathcal{B}}$ and $\nu = \mathrm{Tr}_\mathcal{H} = \tilde{\nu}$.

Definition 2.1. *A TCP map $\pi : \mathcal{B} \to \mathcal{A}_*$ (or $\mathcal{B} \to \mathcal{A}_\mu \subseteq \mathcal{A}_*$) normalized as $\mu(\pi(I)) = 1$ and having an adjoint with $\pi^*(\mathcal{A}) \subseteq \mathcal{B}_*$ ($\pi^*(\mathcal{A}) \subseteq \mathcal{B}_\nu$) is called normal coupling (bounded coupling) of the state $\varsigma = \mu \circ \pi$ on $\mathcal{B}$ to the state $\varrho = \nu \circ \pi^*$ on $\mathcal{A}$. The CP map $\varpi : \mathcal{A} \to \mathcal{B}_\intercal$ (or $\mathcal{A} \to \widetilde{\mathcal{B}_\nu} \subseteq \mathcal{B}_\intercal$) normalized to the probability density $\sigma = \varpi(I)$ of ς with $\varpi^\intercal(I) \in \mathcal{B}_*$ ($\varpi^\intercal(I) \in \widetilde{\mathcal{A}_\mu}$) will be called normal entanglement (bounded entanglement) of the system $(\mathcal{A}, \varrho)$ with the probability density $\rho = \varpi^\intercal(I)$ to $(\mathcal{B}, \varsigma)$. The coupling π (entanglement ϖ) is called truly quantum if it is not CP (not TCP). The self-adjoint entanglement $\varpi_q = \varpi^*_q$ on $(\mathcal{A}, \varrho) = \left(\widetilde{\mathcal{B}}, \tilde{\varsigma}\right)$ (or symmetric coupling $\pi_q = \pi^\intercal_q$ into $\mathcal{A}_* = \mathcal{B}_\intercal$) is called standard for the system $(\mathcal{B}, \varsigma)$ if it is given by*

$$\varpi_q(A) = \sigma^{1/2} A \sigma^{1/2}, \quad \pi_q(B) = \sigma^{1/2}\widetilde{B}\sigma^{1/2}. \tag{9}$$

Note that the standard entanglement is true as soon as the reduced algebra $\mathcal{B}_\sigma = E_\sigma \mathcal{B} E_\sigma$ on the support $\mathcal{H}_\sigma = E_\sigma \mathcal{H}$ of the state ς is not Abelian, i.e. is not one-dimensional in the case $\mathcal{B} = \mathcal{L}(\mathcal{H})$, corresponding to a pure normal ς on $\mathcal{B} = \mathcal{L}(\mathcal{H})$. Indeed, π^q restricted to $\mathcal{B}_\sigma$ is the composition of the nondegenerated multiplication $\mathcal{B}_\sigma \ni B \mapsto \tilde{\sigma}^{1/2} B\, \tilde{\sigma}^{1/2}$ (which is CP) and the transposition $\widetilde{B} = JB^\dagger J$ on $\mathcal{B}_\sigma$ (which is TCP but not CP if $\dim \mathcal{H}_\sigma > 1$).

The standard entanglement in the purely quantum case $\mathcal{B} = \mathcal{B}(\mathcal{H}) = \widetilde{\mathcal{B}}$, $\nu = \mathrm{Tr} = \tilde{\nu}$ corresponds to the pure *standard compound state*

$$\mathrm{Tr} A \sigma^{1/2} \widetilde{B} \sigma^{1/2} = \omega_q(A \otimes B) = \mathrm{Tr} B \tilde{\sigma}^{1/2} \widetilde{A} \tilde{\sigma}^{1/2} \tag{10}$$

on the algebra $\mathcal{B} \otimes \mathcal{B}$. It is given by the amplitude $v' \simeq |\sigma^{1/2}) \equiv \psi$, with $|\sigma^{1/2})^\dagger = \varkappa' \equiv (\sigma^{1/2}|$ defined in (5) as $\varkappa'(\zeta \otimes J\eta) = \eta^\dagger \varkappa \zeta$ for $\varkappa = \sigma^{1/2}$.

Any entanglement on $\mathcal{A} = \mathcal{L}(\mathcal{G})$, $\mu = \mathrm{Tr}$ corresponding to a pure compound state is true if $\mathrm{rank}\rho = \mathrm{rank}\sigma$ is not one. If the space $\mathcal{G}$ is also minimal, $\mathcal{G} = \mathcal{G}_\rho$, $\pi^\intercal$ is unitary equivalent to the standard one π_q. Indeed, $\varpi(A) = \tilde{\varkappa}^\dagger A \tilde{\varkappa}$ can be decomposed as

$$\varpi(A) = \sigma^{1/2} U^\dagger A U \sigma^{1/2} = \varpi_q\left(U^\dagger A U\right),$$

where $U : \sigma^{1/2}\eta \mapsto \tilde{\varkappa}\eta$ is a unitary operator from $\mathcal{H}_\sigma$ onto the support $\mathcal{G}_\rho$ of $\rho = U\sigma U^\dagger$ with nonabelian $\mathcal{A}_\rho = \mathcal{L}(\mathcal{G}_\rho)$ and $\mathcal{B}_\sigma = U^\dagger \mathcal{A}_\rho U = \mathcal{L}(\mathcal{H}_\sigma)$.

Note that the compound state (4) with $\tilde{\varkappa} = \sigma^{1/2}$ corresponding to the standard $\varpi = \varpi_q$ can always be extended to a vector state on $\widetilde{\mathcal{B}} \vee \mathcal{B}$ in the standard representation $(\mathcal{H}_\nu, \iota, J_\nu)$ of $\mathcal{B} \equiv \iota(\mathcal{B})$ when $\widetilde{\mathcal{B}} = J_\nu \mathcal{B} J_\nu = \mathcal{B}'$, but it cannot be extended to a normal state on $\widetilde{\mathcal{B}} \otimes \mathcal{B}$ in the case of nonatomic $\mathcal{B}$. If $\mathcal{B}$ is a factor, this state is pure, given in the standard representation $\widetilde{\mathcal{B}} \vee \mathcal{B} = \mathcal{L}(\mathcal{H}_\nu)$ by the unit vector $y = \hat{\sigma}^{1/2} \in \mathcal{H}_\nu$; however it is not normal on $\widetilde{\mathcal{B}} \otimes \mathcal{B}$ unless $\mathcal{B}$ is type I: $\mathcal{B} \simeq \mathcal{L}(\mathcal{H})$.

3. C-, D- and O-Couplings and Encodings

The compound states play the role of joint input-output probability measures in classical information channels, and can be pure in quantum case even if the marginal states are mixed. The pure compound states achieved by an entanglement of mixed input and output states exhibit new, non-classical type of correlations which are responsible for the EPR type paradoxes in the interpretation of quantum theory. However mixed, so called *separable* states on $\mathcal{A} \otimes \mathcal{B}$, the convex product combinations

$$\omega_c(A \otimes B) = \sum_n \varrho_n(A)\varsigma_n(B)p(n),$$

which we refer as the *c-compound states*, do not exhibit such paradoxical behavior. Here $p(n) > 0$, $\sum p_n = 1$, is a probability distribution, and $\varrho_n : \mathcal{A} \to \mathbb{C}$, $\varsigma_n : \mathcal{B} \to \mathbb{C}$ are usually normal states defined by the product densities $\rho_n \otimes \sigma_n \in \mathcal{A}_\intercal \otimes \mathcal{B}_\intercal$ of $\omega_n = \varrho_n \otimes \sigma_n$. Such compound states are achieved by *c-couplings* $\pi_c : \mathcal{B} \to \mathcal{A}_*$ given by $\pi_c^{-} = \varpi_c^\intercal$, where

$$\varpi_c(A) = \sum_n \varrho_n(A)\sigma_n p(n), \quad \varpi_c^\intercal(B) = \sum_n \varsigma_n(B)\rho_n p(n),$$

Here $\rho_n \in \mathcal{A}_*$ and $\sigma_n \in \mathcal{B}_*$ are the probability densities for ϱ_n and ς_n with respect to given weights μ and ν on $\mathcal{A}$ and $\mathcal{B}$. Note that the *c-entanglement* ϖ_c, being the convex combinations of the primitive CP-TCP maps $\varpi_n(A) = \varrho_n(A)\sigma_n \in \mathcal{B}_\intercal$, is not truly quantum.

The separable states of the particular form

$$\omega_d(A \otimes B) = \sum_n \langle n|A|n\rangle \varsigma(n, B), \tag{11}$$

where $\varrho_n(A) = \langle n|A|n\rangle$ are pure states on $\mathcal{A} = \mathcal{L}(\mathcal{G}) = \widetilde{\mathcal{A}}$ given by an ortho-normal system $\{|n\rangle\} \subset \mathcal{G}$, and $\varsigma(n, B) = \langle B, \sigma(n)\rangle_\nu$ with $\sigma(n) = \sigma_n p(n)$, are usually considered as the proper candidates for the input-output states in the communication channels involving the classical-quantum (c-q) encodings. Such separable state was introduced by Ohya [10, 22] using a Schatten decomposition $\rho = \sum |n\rangle\langle n| p(n)$ of the input density operator $\rho \in \mathcal{T}(\mathcal{G})$ into the orthogonal one-dimensional projectors $\rho_n = |n\rangle\langle n|$. Here we note that such state is the mixture of the classical-quantum correspondences $n \mapsto |n\rangle\langle n| \otimes \sigma_n$ which can be described as the composition of quantum channeling $|n\rangle\langle n| \mapsto \sigma_n$ and the errorless *encodings* $n \mapsto |n\rangle\langle n|$ in the sense that they can be inverted by the measurements $|n\rangle\langle n| \mapsto n$ as input *decodings*. We shall call such separable states *d-compound* as they are achieved by the diagonal couplings $\pi_d = \varpi_d^{\mathsf{T}}$ (*d-couplings*) to the subalgebra $\mathcal{A}_d \subseteq \mathcal{A}$ of the diagonal operators $A = \sum a(n)\,|n\rangle\langle n|$, where

$$\varpi_d(A) = \sum_n \langle n|A|n\rangle \sigma(n)\,,\ \varpi_d^{\mathsf{T}}(B) = \sum_n \varsigma(n, B)\,|n\rangle\langle n|. \tag{12}$$

for the respect to the standard transposition $\langle n|\widetilde{A}|m\rangle = \langle m|A|n\rangle$ in the eigenbasis of ρ.

Actually Ohya obtained the compound states ω_d as the result of composition

$$\omega_d(A \otimes B) = \omega_o(A \otimes \Lambda(B))$$

of quantum channels as normal unital CP maps $\Lambda : \mathcal{B} \to \mathcal{A}$ and the special, *o-compound* states

$$\omega_o(A \otimes B) = \sum_n \langle n|A|n\rangle p(n)\,\langle n|B|n\rangle \tag{13}$$

corresponding to the orthogonal decompositions

$$\varpi_o(A) = \sum_n \langle n|A|n\rangle p(n)\,|n\rangle\langle n| = \varpi_o^{\mathsf{T}}(A) \tag{14}$$

such that $\varsigma_n(B) = \langle n|\Lambda(B)\,|n\rangle$, $\sigma_n = \Lambda^{\mathsf{T}}(|n\rangle\langle n|)$, where $\langle B, \Lambda^{\mathsf{T}}(\rho)\rangle_\nu = \mathrm{Tr}_{\mathcal{G}}\Lambda(B)\,\tilde{\rho}$.

Assuming that $\langle A, \rho\rangle = \mathrm{Tr}_{\mathcal{G}} A\tilde{\rho}$, we can extend this construction to any discretely-decomposable algebra $\mathcal{A} = \widetilde{\mathcal{A}}$ on the Hilbert sum $\mathcal{G} = \oplus \mathcal{G}_i$ with invariant components $\mathcal{G}_i$ under the standard complex conjugation J in the eigen-basis of the density operator $\tilde{\rho} = J\rho J = \rho$. In particular, the von Neumann algebra $\mathcal{A}$ might be Abelian, as it is in the case $\widetilde{A} = A$ for all $A \in \mathcal{A}$, e.g. when $\mathcal{A} = \widetilde{\mathcal{A}}$ is the diagonal algebra of pointwise multiplications $Ag = ag = \widetilde{A}g$ by the bounded functions $n \mapsto a(n) \in \mathbb{C}$ on the functional Hilbert space $\mathcal{G} = \ell^2 \ni g$ with the standard complex conjugation $Jg = \bar{g}$. In this case the densities $\rho \in \mathcal{A}_*$ are given by the summable functions $p \in \ell^1$ with respect to the standard trace $\mu(\rho) = \sum p(n)$, and any compound state has the separable form with $\varrho_n(A) = a(n)$ corresponding to the Kronecker δ-densities $\rho_n \simeq \delta_n$. The normal states on the $\mathcal{A} \simeq \ell^\infty$ are described by the probability densities $p(n) \geq 0$, $\sum p(n) = 1$ with respect to the standard pairing

$$\langle A, \rho\rangle_\mu = \sum a(n)\,p(n)\,, \quad p \in \ell^1, a \in \ell^\infty$$

of $\mathcal{A}_\mu = \mathcal{A}_*$ with the commutative algebra $\mathcal{A}$. Every normal compound state ω on $\mathcal{A} \otimes \mathcal{B}$ is defined by

$$\omega_c(A \otimes B) = \sum_n a(n)\,\langle B, \sigma(n)\rangle_\nu\,,$$

where $\sigma(n) = \sigma_n p(n)$ is the function with positive values $\sigma(n) \in \mathcal{B}_{\mathsf{T}}$ normalized to the probability density $p(n) = \langle I, \sigma(n)\rangle_\nu$. Thus all normal compound states on $\ell^\infty \otimes \mathcal{B}$ are achieved by *c-couplings* $\pi_c = \varpi_c^{\mathsf{T}} : \mathcal{B} \to \ell^1$ with $\pi_c^{\mathsf{T}} = \varpi_c$ given by convex combinations of the primitive CP-TCP maps $\varpi_n(a) = a(n)\,\sigma_n \in \mathcal{B}_*$,

$$\varpi_c(A) = \sum_n a(n)\,\sigma(n)\,,\ \varpi_c^{\mathsf{T}}(B) = \sum_n \varsigma(n, B)\,\delta_n,$$

where $\varsigma(n,B) = \langle B,\sigma(n)\rangle_{\nu}$.

Note that any d-coupling can be regarded as such quantum-classical c-coupling which is achieved by the identification $a(n) = \langle n|A|n\rangle$ of the reduced diagonal algebra $\mathcal{A}^0 = \{\sum |n\rangle a(n)\langle n| : A \in \mathcal{A}\}$ and $\ell^\infty \ni a$. This simply follows from the commutativity of the density operators $\rho = \sum |n\rangle\langle n| p(n)$ for the induced states $\varrho(A) = \omega_d(A \otimes I)$ identified with $p \in \ell^1$.

In the case $\mathcal{A} = \mathcal{L}(\mathcal{G})$ and pure elementary states ω_n described by probability amplitudes $v_n = \chi_n \otimes \psi_n$, where $\tilde{\chi}_n \equiv |\chi_n\rangle \in \mathcal{G}$, $\tilde{\psi}_n \equiv |\psi_n\rangle \in \mathcal{H}$, we have density operators $\rho_n = \chi_n^\dagger \chi_n$ and $\sigma_n = \psi_n^\dagger \psi_n$ of rank one. The total compound amplitude is obviously $v = \sum |n\rangle v(n)$, where $v(n) = \chi_n \otimes \psi_n p(n)^{1/2}$ are the amplitude operators $\mathcal{G} \otimes \mathcal{H} \to \ell^2$ satisfying the orthogonality relations

$$v(n)^\dagger v(m) = \rho_n \otimes \sigma_n p(n)\,\delta_n^m$$

corresponding to the decomposition $v^\dagger v = \sum \rho_n \otimes \sigma_n p(n)$. The "entangling" operator for the separable state $\varkappa$ can be chosen as either as $\varkappa = \sum |n\rangle \varkappa(n)$ or as $\varkappa = \sum \varkappa(n)\langle n|$ or even as $\varkappa = \sum |n\rangle \varkappa(n)\langle n|$ with $\varkappa(n) = \chi_n \otimes \tilde{\psi}(n)$, where $\tilde{\psi}_n(n) = \tilde{\psi}_n p(n)^{1/2}$. In particular, d-entangling operator $\varkappa$ corresponding to d-encodings (12) is diagonal, $\varkappa = \sum |n\rangle \tilde{\psi}(n)\langle n|$ on $\mathcal{G} = \ell^2$, corresponding to the orthogonal $\tilde{\chi}_n = |n\rangle$. Thus, we have proved the Theorem 2 below in the case of pure states ς_n and ϱ_n. But before formulating this theorem in a natural generality let us introduce the following notations.

The general c-compound states on $\mathcal{A} \otimes \mathcal{B}$ are defined as integral convex combinations

$$\omega(A \otimes B) = \int \varrho_x(A)\,\varsigma_x(B)\,p(\mathrm{d}x)$$

given by a probability distribution p on the product-states $\varrho_x \otimes \varsigma_x$. Such compound states are achieved by convex combinations of the primitive CP-TCP maps $\pi_x = \varpi_x^\intercal$ with $\varpi_x(A) = \varrho_x(A)\,\sigma_x$:

$$\varpi_c(A) = \int \varrho_x(A)\,\sigma_x p(\mathrm{d}x)\,,\quad \varpi_c^\intercal(B) = \int \varsigma_x(B)\,\rho_x p(\mathrm{d}x)\,. \tag{15}$$

This is always the case when the von Neumann algebra $\mathcal{A}$ is Abelian, and thus can be identified with the diagonal algebra of multiplications $(Ag)(x) = a(x)\,g(x)$ by the functions $a \in L_\mu^\infty$ on the functional Hilbert space $\mathcal{G} = L_\mu^2$ with respect to a (not necessarily finite) measure μ on X. It defines trace μ on $\mathcal{A}_\mu \simeq L_\mu^1 \cap L_\mu^\infty$ as the integral $\mu(\rho) = \int p(x)\,\mu(\mathrm{d}x)$ for the bounded multiplication densities $(\rho g)(x) = p(x)\,g(x)$. The normal states on $\mathcal{A}$ are given by the probability densities $p \in L_\mu^1$ with respect to the standard pairing

$$\langle A,\rho\rangle_\mu = \int a(x)\,p(x)\,\mu(\mathrm{d}x)\,,\quad p \in L_\mu^1, a \in L_\mu^\infty.$$

of $\mathcal{A}_* = \mathcal{A}_\intercal \simeq L_\mu^1$ and $\mathcal{A} = \widetilde{\mathcal{A}} \simeq L_\mu^\infty$ corresponding to the trivial transposition $\tilde{a} = a$. Any normal compound state ω on $\mathcal{A} \otimes \mathcal{B} \simeq L_\mu^\infty(X \to \mathcal{B})$ is the c-compound state, defined on the diagonal algebra $\mathcal{A}$ by

$$\omega_d(A \otimes B) = \int a(x)\,\varsigma(x,B)\,\mu(\mathrm{d}x)\,, \tag{16}$$

where $\varsigma(x,B) = \langle B,\sigma(x)\rangle_\nu$ is absolutely integrable function with density operator values $\sigma(x) = \sigma_x p(x)$ normalized to the probability density $p(x) = \langle I,\sigma(x)\rangle_\nu =$

$\varsigma(x, I)$. It corresponds to d-couplings $\pi_d = \varpi_d^{\intercal} = \pi_d^{-}$ with $\pi_d^{\intercal} = \varpi_d$ decomposing into $\varpi(x, A) = a(x)\sigma(x)$:

$$\varpi_d(A) = \int a(x)\sigma(x)\mu(\mathrm{d}x), \ \varpi_d^{\intercal}(B) = \int \varsigma(x, B)\delta_x\mu(\mathrm{d}x), \tag{17}$$

where δ_x is the (generalized) density operator of the Dirac state $\varrho_x(A) = \langle A, \delta_x\rangle_\mu = a(x)$ on the diagonal algebra $\mathcal{A}$.

Theorem 3.1. *Let $\omega_c : \mathcal{A} \otimes \mathcal{B} \to \mathbb{C}$ be a normal c-compound state given as*

$$\omega_c(A \otimes B) = \int \mu_x\left(\tilde{\chi}_\cdot^{\dagger} A \tilde{\chi}_\cdot\right) \nu_x\left(\tilde{\psi}_\cdot^{\dagger} B \tilde{\psi}_\cdot\right) p(\mathrm{d}x), \tag{18}$$

where $\chi_x : \mathcal{G} \to \mathcal{E}_x$, $\psi_x : \mathcal{H} \to \mathcal{F}_x$ are linear operators having bounded transpose $\tilde{\chi}_\cdot = J\chi_\cdot^{\dagger}J$. $\tilde{\psi}_\cdot = J\psi_\cdot^{\dagger}J$. on Hilbert spaces $\mathcal{E}_\cdot = \int^{\oplus} \mathcal{E}_x p(\mathrm{d}x)$, $\mathcal{F}_\cdot = \int^{\oplus} \mathcal{F}_x p(\mathrm{d}x)$ with respect to pointwise involution $J_\cdot = J_\cdot^{\dagger}$. We also assume that

$$\chi_x^{\dagger}\chi_x \in \widetilde{\mathcal{A}}, \psi_x^{\dagger}\psi_x \in \widetilde{\mathcal{B}}, \quad \mu_x\left(\tilde{\chi}_\cdot^{\dagger}\tilde{\chi}_\cdot\right) = 1 = \nu_x\left(\tilde{\psi}_\cdot^{\dagger}\tilde{\psi}_\cdot\right)$$

with respect to the weights

$$\mu_x\left(\tilde{\chi}_\cdot^{\dagger}\tilde{\chi}_\cdot\right) = \langle I, \chi_x^{\dagger}\chi_x\rangle_\mu, \quad \nu_x\left(\tilde{\psi}_\cdot^{\dagger}\tilde{\psi}_\cdot\right) = \left\langle I, \psi_x^{\dagger}\psi_x\right\rangle_\nu. \tag{19}$$

Then this state is achieved by decomposable entangling operator $\varkappa = \int^{\oplus} \chi_x \otimes \psi_x p(\mathrm{d}x)$ defining c-entanglement (15) with

$$\varrho_x(A) = \mu_x\left(\tilde{\chi}_\cdot^{\dagger} A \tilde{\chi}_\cdot\right), \ \varsigma_x(B) = \nu_x\left(\tilde{\psi}_\cdot^{\dagger} A \tilde{\psi}_\cdot\right), \tag{20}$$

corresponding to the probability densities $\rho_x = \chi_x^{\dagger}\chi_x$, $\sigma_x = \psi_x^{\dagger}\psi_x$. In particular, every d-compound state (16) corresponding to $p(\mathrm{d}x) = p(x)\mu(\mathrm{d}x)$ with the Abelian algebra $\mathcal{A}$ can be achieved by the orthogonal sum of entangling operators $\varkappa_x = \delta_x \otimes \tilde{\psi}_x$ defining d-entanglement (17) with

$$\sigma(x) = \psi_x^{\dagger}\psi_x p(x), \quad \varsigma(x, B) = \nu_x\left(\tilde{\psi}_\cdot^{\dagger} A \tilde{\psi}_\cdot\right) p(x).$$

Proof. The amplitude operator $\upsilon = \int^{\otimes} \upsilon_x p(\mathrm{d}x)$ corresponding to c-compound state (18) is defined on as the orthogonal sum of $\upsilon_x = \chi_x \otimes \psi_x$ on $\mathcal{G} \otimes \mathcal{H}$ into $\int^{\oplus} \mathcal{E}_x \otimes \mathcal{F}_x p(\mathrm{d}x)$. Without loss of generality we can assume that $\mathcal{E}_x = \mathcal{G}_\rho$, $\mathcal{F}_x = \mathcal{H}_\sigma$ and $\upsilon_x^{\dagger} = \upsilon_x(E_\rho \otimes E_\sigma)$ because the support $(\mathcal{G} \otimes \mathcal{H})_{\upsilon_x^{\dagger}\upsilon_x} = \mathrm{ran}\upsilon_x^{\dagger}$ for

$$\upsilon_x^{\dagger}\upsilon_x = \chi_x^{\dagger}\chi_x \otimes \psi_x^{\dagger}\psi_x = \rho_x \otimes \sigma_x$$

is in $\mathcal{G}_\rho \otimes \mathcal{H}_\sigma$. Due to $\chi_x^{\dagger}\chi_x \in \widetilde{\mathcal{A}'}$, $\psi_x^{\dagger}\psi_x \in \widetilde{\mathcal{B}}'$ for almost all x, the operators χ_x and ψ_x commute with $A \in \widetilde{\mathcal{A}'}$ and $B \in \widetilde{\mathcal{B}}'$ respectively, and $\tilde{\psi}_x$ commutes with $B \in \mathcal{B}'$ for almost all x. Thus,

$$\tilde{\chi}_x^{\dagger} A \tilde{\chi}_x \in \mathcal{A}, \quad \tilde{\psi}_x B \tilde{\psi}_x \in \mathcal{B}$$

which defines the weights (19) on $L_p^{\infty} \otimes \mathcal{A}$ and $L_p^{\infty} \otimes \mathcal{B}$ for almost all x. The rest of the proof is the repetition of the proof of the Theorem 1 for each x with the addition that $\varkappa_x$ is the product $\upsilon_x' = \chi_x \otimes \tilde{\psi}_x$ for each x. The total entangling operator $\varkappa : \mathcal{G} \otimes \mathcal{F}_\cdot \to \mathcal{E}_\cdot \otimes \mathcal{H}$ acts componentwise as $\varkappa_x(\zeta \otimes \eta_\cdot) = \chi_x\zeta \otimes \tilde{\psi}_x\eta_x$.

In the case of d-compound state (16) one should take $\mathcal{G} = L^2_\mu$, $\mathcal{E}_x = \mathbb{C}$, and $\chi_x g = g(x)$. Thus the entangling operator in this case is given as

$$\varkappa (g \otimes \eta.) = \int^{\otimes} g(x)\,\tilde{\psi}_x \eta_x \mu(\mathrm{d}x), \quad \forall g \in L^2_\mu, \eta. = \int^{\oplus} \eta_x \mu(\mathrm{d}x) \in \mathcal{F}..$$

∎

Note that c-entanglements ϖ_c in (15) are both CP and TCP and thus are not true quantum. The map $\varpi_c : \mathcal{A} \to \mathcal{B}_\intercal$ with and Abelian algebra $\mathcal{A}$ in (17) is described by a $\mathcal{B}_\intercal$-valued measure $\sigma(\mathrm{d}x) = \sigma(x)\,\mu(\mathrm{d}x)$ normalized to the input probability measure as $p(\mathrm{d}x) = \langle I, \sigma(\mathrm{d}x)\rangle_\nu$. This gives the concise form for the description of random classical-quantum state correspondences $x \mapsto \sigma_x$ with the given probability measure p, called *encodings* of $\sigma = \int \sigma(\mathrm{d}x)$.

Definition 3.1. *Let both algebras $\mathcal{A}$ and $\mathcal{B}$ be non-Abelian. The map $\varpi : \mathcal{A} \to \mathcal{B}_\intercal$ is called* c-encoding *of $(\mathcal{B}, \varsigma)$ if it is a convex combination of the primitive maps $\sigma_n \varrho_n$ given by the probability densities $\sigma_n \in \mathcal{B}_\intercal$ and normal states $\varrho_n : \mathcal{A} \to \mathbb{C}$. It is called* d-encoding *if it has the diagonalizing form (12) on $\mathcal{A}$, and it is called* o-encoding *if all density operators σ_n are mutually orthogonal: $\sigma_m \sigma_n = 0$ for all $m \neq n$ as in (14) The entanglement which is described by non-separable CP map $\varpi : \mathcal{A} \to \mathcal{B}_\intercal$ will be called* q-encoding.

Note that due to the commutativity of the operators $A \otimes I$ with $I \otimes B$ on $\mathcal{G} \otimes \mathcal{H}$, one can treat the encodings as nondemolition measurements [9] in $\mathcal{A}$ with respect to $\mathcal{B}$. The corresponding compound state is the state prepared for such measurements on the input $\mathcal{G}$. It coincides with the mixture of the states, corresponding to those after the measurement without reading the message sent. The set of all d-encodings for a Schatten decomposition of the input state ρ on $\mathcal{A}$ is obviously convex with the extreme points given by the pure output states ς_n on $\mathcal{B}$, corresponding to the not necessarily orthogonal (not Schatten) decompositions $\sigma = \sum \sigma(n)$ into the one-dimensional density operators $\sigma(n) = p(n)\,\sigma_n$.

The Schatten decompositions $\sigma = \sum_n q(n)\,\sigma_n$ correspond to o-encodings, the extreme d-encodings $\sigma_n = \eta_n \eta_n^\dagger$, $p(n) = q(n)$ characterized by the orthogonality $\sigma_m \sigma_n = 0$, $m \neq n$. For each Schatten decomposition of σ they form a convex subset of d-encodings with mixed commuting σ_n .

4. Quantum Entropy via Entanglements

As we have seen in the previous section, the encodings $\varpi : \mathcal{A} \to \mathcal{B}_\intercal$, which are usually described as in (17) with a discrete Abelian $\mathcal{A}$, correspond to the case (12) when the general entanglement (7) is *d-encoding*, with the diagonal coupling $\pi = \varpi^\intercal$ in the eigen-representation of a discrete probability density ρ on non-Abelian $\mathcal{A}$. The true quantum entanglements with non-Abelian $\mathcal{A}$ cannot be achieved by d-, and more general, c-encodings even in the case of discrete $\mathcal{A}$. The nonseparable, true entangled states ω called in [22] *q-compound* states, can be achieved by *q-encodings*, the quantum-quantum nonseparable correspondences (6) which are not diagonal. in the eigen-representation of ρ.

As we shall prove in this section, the self-dual standard true entanglement $\varpi_q = \varpi_q^\intercal$ to the probe system $(\mathcal{A}^0, \varrho_0) = \left(\widetilde{\mathcal{B}}, \tilde{\varsigma}\right)$, which is defined in (9), is the most informative for a quantum system $(\mathcal{B}, \varsigma)$ in the sense that it achieves the maximal mutual information in the coupled system $(\mathcal{A} \otimes \mathcal{B}, \omega)$ when $\omega = \omega_q$ is given in (10).

Let us consider entangled mutual information and quantum entropies of states by means of the above three types of compound states. To define the quantum mutual entropy, we need to apply a quantum version of the relative entropy to compound state on the algebra $\mathcal{M} = \mathcal{A} \otimes \mathcal{B}$, called also the information divergency of the state ω with respect to a reference state φ on $\mathcal{M}$. The relative entropy was defined in [20, 21, 24] even for most general von Neumann algebra $\mathcal{M}$, but for our purposes we need the following its explicit description.

Let $\mathcal{M}$ be a semi-finite algebra with normal states ω and φ having the density operator $\upsilon^\dagger \upsilon$ and $\phi \in \widetilde{\mathcal{M}}$ with respect to the pairing

$$\langle M, \upsilon^\dagger \upsilon \rangle = \left(\bar{\upsilon} | \widetilde{\iota(M)} \bar{\upsilon} \right), \quad M \in \mathcal{M}, \upsilon^\dagger \upsilon \in \widetilde{\mathcal{M}}$$

given by a normal faithful weight τ on the transposed algebra $\widetilde{\mathcal{M}} = J\mathcal{M}J$ (not necessary decomposable as $\tau = \tilde{\mu} \otimes \tilde{\nu}$ in (3) in the case of $\mathcal{M} = \mathcal{A} \otimes \mathcal{B}$). Then the *relative entropy* $\mathsf{R}(\omega; \varphi)$ of the state ω with respect to φ is given by the formula

$$\mathsf{R}(\omega : \varphi) = \tau\left(\upsilon\left(\ln \upsilon^\dagger \upsilon - \ln \phi\right) \upsilon^\dagger\right) = \tau\left(\omega\left(\ln \omega - \ln \phi\right)\right). \tag{21}$$

(For the notational simplicity here and below we identify the state ω with its density operator $\upsilon^\dagger \upsilon$). It has a positive value $\mathsf{R}(\omega : \varphi) \in [0, \infty]$ if the states are equally normalized, say (as usually) $\tau(\omega) = 1 = \tau(\phi)$, and it can be finite only if the state ω is absolutely continuous with respect to the reference state φ, i.e. iff $\omega(E) = 0$ for the maximal null-orthoprojector $E \in \mathcal{M}$, $E\phi = 0$. Note that this definition depends on the choice of the semi-finite weight τ, and it can be extended also to the arbitrary normal ω and φ with unbounded self-adjoint density operators $\upsilon^\dagger \upsilon$ and ϕ.

The most important property of the information divergence R is its monotonicity property [20, 25], i.e. nonincrease of the divergency $\mathsf{R}(\omega_0 : \varphi_0)$ after the application of the pre-dual of a normal completely positive unital map $\mathrm{K} : \mathcal{M} \to \mathcal{M}^0$ to the states ω_0 and φ_0 on a von Neumann algebra $\mathcal{M}^0$:

$$\omega = \omega_0 \mathrm{K}, \varphi = \varphi_0 \mathrm{K} \Rightarrow \mathsf{R}(\omega : \varphi) \leq \mathsf{R}(\omega_0 : \varphi_0). \tag{22}$$

The *mutual information* $\mathsf{I}(\pi) = \mathsf{I}(\pi^*)$ in a compound state ω achieved by a coupling $\pi : \mathcal{B} \to \mathcal{A}_*$, or by $\pi^* : \mathcal{A} \to \mathcal{B}_*$ with the marginals

$$\varrho(A) = \omega(A \otimes I) = \langle A, \rho \rangle_\mu, \ \varsigma(B) = \omega(I \otimes B) = \langle B, \sigma \rangle_\nu$$

is defined as the relative entropy

$$\mathsf{I}(\pi) = \tau\left(\omega\left(\ln \omega - \ln(\rho \otimes I) - \ln(I \otimes \sigma)\right)\right) = \mathsf{R}(\omega : \varrho \otimes \varsigma). \tag{23}$$

of the state ω on $\mathcal{M} = \mathcal{A} \otimes \mathcal{B}$ with respect to the product state $\varphi = \varrho \otimes \varsigma$ for $\tau = \tilde{\mu} \otimes \tilde{\nu}$. This quantity, generalizing the classical mutual information corresponding to the case of Abelian $\mathcal{A}$, $\mathcal{B}$, describes an information gain in a quantum system $(\mathcal{A}, \varrho)$ via the entanglement $\varpi^\intercal = \pi^{\tilde{}}$, or in $(\mathcal{B}, \varsigma)$ via an entanglement $\varpi : \mathcal{A} \to \mathcal{B}_\intercal$. It is naturally treated as a measure of the strength of the generalized entanglement having zero value only for completely disentangled states $\omega = \varrho \otimes \varsigma$.

Proposition 4.1. *Let $\left(\mathcal{A}^0, \mu_0\right)$ be quantum system with a normal faithful semifinite weight, and $\pi_0 : \mathcal{A}^0 \to \mathcal{B}_*$ be a normal coupling of the state $\varrho_0 = \nu \circ \pi_0$ on $\mathcal{A}^0$ to $\varsigma = \mu \circ \pi$, defining an entanglement $\varpi = \pi^{*\tilde{}}$ of $(\mathcal{A}, \varrho)$ to $(\mathcal{B}, \varsigma)$ by the composition $\pi^* = \pi_0 \mathrm{K}$ with a normal completely positive unital map $\mathrm{K} : \mathcal{A} \to \mathcal{A}^0$. Then $\mathsf{I}(\pi) \leq \mathsf{I}\left(\pi^0\right)$, where $\pi^0 = \pi_0^*$. In particular, for each normal c-coupling given by (15) such as $\pi^{\tilde{}} = \varpi_{\mathrm{c}}^\intercal$ there exists a not less informative d-coupling $\pi^0 = \varpi_0^\intercal$*

with Abelian $\mathcal{A}^0$ corresponding to the encoding $\varpi_0 = \pi_0^{\bar{}}$ of $(\mathcal{B}, \varsigma)$, and the standard q-coupling $\pi^0 = \pi_q$, $\pi_q(B) = \sigma^{1/2}\tilde{B}\sigma^{1/2}$ to $\varrho_0 = \tilde{\varsigma}$ on $\mathcal{A}^0 = \widetilde{\mathcal{B}}$ is the maximal one in this sense.

Proof. The first follows from the monotonicity property (22) applied to the ampliation $\mathrm{K}(A \otimes B) = \mathrm{K}(A) \otimes B$ of the CP map K from $\mathcal{A} \to \mathcal{A}^0$ to $\mathcal{A} \otimes \mathcal{B} \to \mathcal{A}^0 \otimes \mathcal{B}$. The compound state $\omega_0(\mathrm{K} \otimes \mathrm{I})$ (I denotes the identity map $\mathcal{B} \to \mathcal{B}$) is achieved by the entanglement $\varpi = \varpi_0\mathrm{K}$, and $\varphi_0(\mathrm{K} \otimes \mathrm{I}) = \varrho \otimes \varsigma$, $\varrho = \varrho_0\mathrm{K}$ corresponding to $\varphi_0 = \varrho_0 \otimes \varsigma$. It corresponds to the coupling $\pi = \mathrm{K}^*\pi_0$ which is defined by $\mathrm{K}^* : \mathcal{A}_*^0 \to \mathcal{A}_*$ as $\mathrm{K}^*\tilde{\rho}_0 = J(\mathrm{K}^\intercal\rho_0)^\dagger J$, where

$$\langle A, \mathrm{K}^\intercal\rho_0\rangle_\mu = \langle \mathrm{K}A, \rho_0\rangle_{\mu_0}, \quad \forall A \in \mathcal{A}, \rho_0 \in \mathcal{A}_\intercal^0.$$

This monotonicity property proves, in particular, that for any separable compound state (18) on $\mathcal{A} \otimes \mathcal{B}$, which is prepared by the c-entanglement $\pi_c^{\bar{}} = \varpi_c^\intercal$, there exists a d-entanglement $\pi_0 = \varpi_0^\intercal = \pi_0^{\bar{}}$ with $(\mathcal{A}^0, \varrho_0)$ having the same, or even larger information gain (23). One can take even a classical system $(\mathcal{A}^0, \varrho_0)$, say the diagonal sublalgebra $\mathcal{A}^0 \simeq L_p^\infty$ on $\mathcal{G}_0 = L_p^2$ with the state ϱ_0, induced by the measure $\mu = p$, and consider the classical-quantum correspondence (encoding)

$$\varpi_0(A^0) = \int a(x)\sigma_x p(\mathrm{d}x), \quad A^0 = \int^\oplus a(x)p(\mathrm{d}x), a \in L_p^\infty$$

assigning the states $\varsigma_x(B) = \langle B, \sigma_x\rangle_\nu$ to the letters x with the probabilities $p(\mathrm{d}x)$. In this case the state ϱ is described by the density $\rho = I$ the multiplication by identity function in L_p^2, ω is multiplication by σ. in $L_p^2 \otimes \mathcal{H}$, and the mutual information (23) is given as

$$\text{(24)} \qquad \mathsf{I}(\pi^0) = \int \tilde{\nu}_x(\sigma_x(\ln\sigma_x - \ln\sigma))p(\mathrm{d}x) = \mathsf{S}(\sigma) - \int \mathsf{S}(\sigma_x)p(\mathrm{d}x),$$

where $\mathsf{S}(\sigma) = -\tilde{\nu}(\sigma\ln\sigma)$. The achieved information gain $\mathsf{I}(\pi^0)$ is larger than $\mathsf{I}(\pi)$ corresponding to $\omega = \int \rho_x \otimes \sigma_x p(\mathrm{d}x)$ because the c-entanglement ϖ_c in (15) is represented as the composition $\varpi_0\mathrm{K}$ of the encoding $\varpi_0 : \mathcal{A}^0 \to \mathcal{B}_\intercal$ with the CP map

$$\mathrm{K}(A) = \int^\oplus \varrho_x(A)p(\mathrm{d}x), \quad A \in \mathcal{A}$$

given by $a(x) = \varrho_x(A)$ for each $A \in \mathcal{A}$. Hence

$$\pi^*(A) = \widetilde{\varpi(A)} = \widetilde{\varpi_0\mathrm{K}A} = \pi_0(\mathrm{K}A), \quad \forall A \in \mathcal{A}$$

where $\pi_0 = \varpi_0^{\bar{}}$, and thus $\mathsf{I}(\pi^0) \geq \mathsf{I}(\mathrm{K}^*\pi^0) = \mathsf{I}(\pi)$, where $\pi^0 = \pi_0^* = \varpi_0^\intercal$.

The inequality (22) can also be applied to the standard entanglement corresponding to the compound state (10) on $\widetilde{\mathcal{B}} \otimes \mathcal{B}$. Indeed, any normal entanglement $\varpi(A) = \mu(\tilde{\varkappa}^\dagger(A \otimes I)\tilde{\varkappa})$ on $\mathcal{A}$ into $\mathcal{B}_\intercal$ as a CP map $\mathcal{A} \to \widetilde{\mathcal{B}_\nu}$ can be decomposed as

$$\mu(\tilde{\varkappa}^\dagger(A \otimes I)\tilde{\varkappa}) = \sigma^{1/2}\mu(X^\dagger(A \otimes I)X)\sigma^{1/2} = \varpi_0(\mathrm{K}A),$$

where $\mathrm{K}A = \mu(X^\dagger(A \otimes I)X)$ is a normal unital CP map $\mathcal{A} \to \widetilde{\mathcal{B}}$. It is uniquely given by an operator $X : \mathcal{E} \otimes \mathcal{H} \to \mathcal{G} \otimes \mathcal{F}$ with $\mathcal{E} = \mathcal{G}_\rho$, $\mathcal{H} = \mathcal{F}_\sigma$ satisfying the condition $X(I \otimes \sigma)^{1/2} = \tilde{\varkappa}$, and thus $X \in \mathcal{A} \otimes \mathcal{B}'$ due to the commutativity of $\tilde{\varkappa}$ with $\mathcal{A}' \otimes \mathcal{B}$ and σ with $\mathcal{B}$. Moreover, the partial weight μ of $X^\dagger X$ is well-defined by

$\mu(\varkappa^\dagger\varkappa) = \sigma$ as $\mu(X^\dagger X) = I$. Thus $\varpi = \varpi_q \mathrm{K}$ and $\pi = \mathrm{K}^* \pi_q$, where K is a normal unital CP map $\mathcal{A} \to \widetilde{\mathcal{B}}$, and $\mathrm{K}^* : \mathcal{B}_\mathsf{T} = \widetilde{\mathcal{B}}_* \to \mathcal{A}_*$. Hence the standard entanglement (coupling) (9) corresponds to the maximal mutual information, $\mathsf{I}(\pi_q) \geq \mathsf{I}(\mathrm{K}^*\pi_q) = \mathsf{I}(\pi)$. ∎

Note that the mutual information (23) is written as

$$\mathsf{I}(\pi) = \mathsf{S}(\rho) + \mathsf{S}(\sigma) - \mathsf{S}(\omega/\varphi),$$

where $\varphi = \mu \otimes \nu$, $\mathsf{S}(\rho) = \mathsf{S}(\varrho/\mu)$, $\mathsf{S}(\sigma) = \mathsf{S}(\varsigma/\nu)$ and

$$\mathsf{S}(\omega/\varphi) = -\tilde{\varphi}\left(v\left(\ln v^\dagger v\right) v^\dagger\right) \equiv -\tilde{\varphi}\left(v^\dagger v \ln v^\dagger v\right) \tag{25}$$

denotes the *entropy of the density operator* $v^\dagger v \in \widetilde{\mathcal{M}}$ of the state ω with respect to the weight φ on $\mathcal{M}$. Note that the entropy $\mathsf{S}(\omega/\varphi)$, coinciding with $-\mathsf{R}(\omega : \varphi)$ (cf. with (21 in the case $\tau = \tilde{\varphi}$), is not in general positive, and may not even be bounded from below as a function of ω. However in the case of irreducible $\mathcal{M}$ it can always be made positive by the choice of the standard trace $\tau = \mathrm{Tr}$ on $\mathcal{M}$, in which case it is called the von Neumann entropy of the state ω $(= v^\dagger v)$, denoted simply as $\mathsf{S}(\omega)$:

$$\mathsf{S}(\omega/\tau) = -\mathrm{Tr}\omega \ln \omega \equiv \mathsf{S}(\omega). \tag{26}$$

In the following we shall assume that $\mathcal{B}$ is a discrete decomposition of the irreducible $\mathcal{B}_i = \mathcal{L}(\mathcal{H}_i) = \widetilde{\mathcal{B}}_i$ with the trace $\nu = \mathrm{Tr}_\mathcal{H} = \tilde{\nu}$ induced on $\mathcal{B}_* = \mathcal{B}_\tau$. The entropy $\mathsf{S}(\sigma) = \mathsf{S}(\varsigma/\nu)$ of the density operator σ for the normal state ς on $\mathcal{B}$ can be found in this case as the maximal information $\mathsf{S}(\varsigma) = \sup \mathsf{I}(\pi_c)$ achieved via all c-encodings $\varpi : \mathcal{A} \mapsto \mathcal{B}_\tau$ of the system $(\mathcal{B}, \varsigma)$ such that, $\varpi(I) = \sigma$ $\varpi^\mathsf{T} = \pi_c^{\tilde{}}$. Indeed, as follows from the proposition above, is sufficient to find the maximum of $\mathsf{I}(\pi)$ over all d-couplings $\pi^0 = \varpi^\mathsf{T}$ mapping $\mathcal{B}$ into Abelian $\mathcal{A}$ with fixed $\varpi(I) = \sigma$, i.e. to find maximum of (14) under the condition $\int \sigma_x p(\mathrm{d}x) = \sigma$. Due to positivity of the *d-conditional entropy*

$$\mathsf{S}(\pi_d) = -\int \mathrm{Tr}\left(\sigma_x \ln \sigma_x\right) p(\mathrm{d}x) = \int \mathsf{S}(\sigma_x)\, p(\mathrm{d}x) \tag{27}$$

the information $\mathsf{I}(\pi^0) = \mathsf{I}(\pi_d)$ has the maximum $\mathsf{S}(\sigma)$ which is achieved on an extreme d-coupling π_d^0 when almost all $\mathsf{S}(\sigma_x)$ are zero, i.e. when almost all σ_x are one-dimensional projectors $\sigma_x^0 = P_x$ corresponding to pure states ς_x. One can take for example, the maximal Abelian subalgebra $\mathcal{A}^0 \subseteq \mathcal{B}$ generated by $P_n = |n\rangle\langle n| \in \mathcal{B}$ for a Schatten decomposition $\sigma = \sum_n |n\rangle\langle n| p(n)$ of $\sigma \in \mathcal{B}_\tau$. The maximal value $\ln \operatorname{rank} \mathcal{B}$ of the von Neumann entropy is defined by the dimensionality $\operatorname{rank} \mathcal{B} = \dim \mathcal{A}^0$ of the maximal Abelian subalgebra of the decomposable algebra $\mathcal{B}$, i.e. by $\dim \mathcal{H}$.

However, if π is not c-coupling, the difference $\mathsf{S}(\pi) = \mathsf{S}(\sigma) - \mathsf{I}(\pi)$ can achieve the negative value, and cannot serve as a measure of conditional entropy in such case.

Definition 4.1. *The supremum of the mutual information*

$$\mathsf{H}(\varsigma) = \sup\{\mathsf{I}(\pi) : \mu \circ \pi = \varsigma\} = \mathsf{I}(\pi_q), \tag{28}$$

which is achieved on $\mathcal{A} = \widetilde{\mathcal{B}}$ for a fixed state $\varsigma(B) = \mathrm{Tr}_\mathcal{H} B\sigma$ by the standard q-coupling $\pi_q(B) = \sigma^{1/2}\widetilde{B}\sigma^{1/2}$, is called q-entropy *of the state ς. The maximum*

$$\mathsf{S}(\varsigma) = \sup\{\mathsf{I}(\pi_c) : \mu \circ \pi_c = \varsigma\} = \mathsf{I}(\pi_d^0)$$

over all c-couplings π_c corresponding to c-encodings (15), which is achieved on an extreme d-coupling π_d^0, is called c-entropy *of the state ς. The differences*

$$\mathsf{H}(\pi) = \mathsf{H}(\varsigma) - \mathsf{I}(\pi), \quad \mathsf{S}(\pi) = \mathsf{S}(\varsigma) - \mathsf{I}(\pi)$$

are called respectively, the q-conditional entropy *on $\mathcal{B}$ with respect to $\mathcal{A}$ and the* (degree of) disentanglement *for the coupling $\pi : \mathcal{B} \to \mathcal{A}$. A compound state is said to be* essentially entangled *if $\mathsf{S}(\pi) < 0$, and $\mathsf{S}(\pi) \geq 0$ for a c-coupling $\pi = \pi_c$ is called* c-conditional entropy *on $\mathcal{B}$ with respect to $\mathcal{A}$.*

Obviously, $\mathsf{H}(\varsigma)$ and $\mathsf{S}(\varsigma)$ are both positive, do not depend unlike $\mathsf{S}(\sigma) = \mathsf{S}(\varsigma/\nu)$ on the choice of the faithful weight ν on $\mathcal{B}$, and $\mathsf{H}(\varsigma) \geq \mathsf{S}(\varsigma)$. The same is true for the conditional entropies $\mathsf{H}(\pi)$ and $\mathsf{S}(\pi)$, where $\mathsf{S}(\pi)$ has always a positive value

$$\mathsf{S}(\pi) \geq \mathsf{S}(\pi^0) \geq 0$$

in the case of a c-coupling $\pi = \pi_c$ due to $\pi_c^* = \pi_d^* \mathrm{K}$ for a normal unital CP map $\mathrm{K} : \mathcal{A} \to \mathcal{A}^0$, where $\pi^0 = \pi_d$ is a d-coupling with Abelian $\mathcal{A}^0$. But the disentanglement $\mathsf{S}(\pi)$ can also achieve the negative value

$$\inf\{\mathsf{S}(\pi) : \mu \circ \pi = \varsigma\} = \mathsf{S}(\varsigma) - \mathsf{H}(\varsigma) = -\sum_i \varkappa(i)\,\mathsf{S}(\sigma_i) \tag{29}$$

as the following theorem states in the case of the discrete $\mathcal{B}$. Here the $\sigma_i \in \mathcal{L}(\mathcal{H}_i)$ are the density operators of the normalized factor-states $\varsigma_i = \varkappa(i)^{-1}\varsigma|\mathcal{L}(\mathcal{H}_i)$ with $\varkappa(i) = \varsigma(I^i)$, where I^i are the orthoprojectors onto $\mathcal{H}_i$. Note that $\mathsf{H}(\varsigma) = \mathsf{S}(\varsigma)$ if the algebra $\mathcal{B}$ is completely decomposable, i.e. Abelian. In this case the maximal value $\ln \operatorname{rank}\mathcal{B}$ of $\mathsf{S}(\varsigma)$ can be written as $\ln \dim \mathcal{B}$. The disentanglement $\mathsf{S}(\pi)$ is always positive in this case, and $\mathsf{S}(\pi) = \mathsf{H}(\pi)$ as in the case of Abelian $\mathcal{A}$.

Theorem 4.2. *Let $\mathcal{B}$ be a discrete decomposable algebra on $\mathcal{H} = \oplus_i \mathcal{H}_i$, with a normal state given by the density operator $\sigma = \oplus \sigma(i)$ with respect to the trace $\mu = \mathrm{Tr}_{\mathcal{H}}$ on $\mathcal{B}$, and $\mathcal{C} \subseteq \mathcal{B}$ be its center with the state $\varkappa = \varsigma|\mathcal{C}$ induced by the probability distribution $\varkappa(i) = \mathrm{Tr}\,\sigma(i)$. Then the c-entropy $\mathsf{S}(\varsigma)$ is given as the von Neumann entropy (26) of the density operator σ, and the q-entropy (28) is given by the formula*

$$\mathsf{H}(\varsigma) = \sum_i \left(\varkappa(i)\ln\varkappa(i) - 2\mathrm{Tr}_{\mathcal{H}_i}\sigma(i)\ln\sigma(i)\right). \tag{30}$$

This can be written as $\mathsf{H}(\varsigma) = \mathsf{H}_{\mathcal{B}|\mathcal{C}}(\varsigma) + \mathsf{H}_{\mathcal{C}}(\varsigma)$, where $\mathsf{H}_{\mathcal{C}}(\varsigma) = -\sum_i \varkappa(i)\ln\varkappa(i)$, and

$$\mathsf{H}_{\mathcal{B}|\mathcal{C}}(\varsigma) = -2\sum_i \varkappa(i)\,\mathrm{Tr}_{\mathcal{H}_i}\sigma_i \ln\sigma_i = 2\mathsf{S}_{\mathcal{B}|\mathcal{C}}(\varsigma),$$

with $\sigma_i = \sigma(i)/\varkappa(i)$. $\mathsf{H}(\varsigma)$ is finite iff $\mathsf{S}(\varsigma) < \infty$, and if $\mathcal{B}$ is finite-dimensional, it is bounded, with the maximal value $\mathsf{H}(\varsigma^\circ) = \ln\dim\mathcal{B}$ which is achieved for $\sigma^\circ = \oplus\sigma_i^\circ\varkappa^\circ(i)$

$$\sigma_i^\circ = (\dim\mathcal{H}_i)^{-1} I^i, \quad \varkappa^\circ(i) = \dim\mathcal{B}(i)/\dim\mathcal{B},$$

where $\dim\mathcal{B}(i) = (\dim\mathcal{H}_i)^2$, $\dim\mathcal{B} = \sum_i \dim\mathcal{B}(i)$.

Proof. We have already proven that $\mathsf{S}(\varsigma) = \mathsf{S}(\sigma)$, where

$$\mathsf{S}(\sigma) = -\sum \mathrm{Tr}_{\mathcal{H}_i}\sigma(i)\ln\sigma(i) = \mathsf{S}_{\mathcal{C}}(\varsigma) + \mathsf{S}_{\mathcal{B}|\mathcal{C}}(\varsigma),$$

with $\mathsf{S}_{\mathcal{C}}(\varsigma) = \mathsf{H}_{\mathcal{C}}(\varsigma)$, $\mathsf{S}_{\mathcal{B}|\mathcal{C}}(\varsigma) = \sum \varkappa(i)\,\mathsf{S}(\sigma_i) = \frac{1}{2}\mathsf{H}_{\mathcal{B}|\mathcal{C}}(\varsigma)$.

The q-entropy $\mathsf{H}(\varsigma)$ is the supremum (28) of the mutual information (23) which is achieved on the standard entanglement, corresponding to the density operator $\omega = \oplus\omega(i,k)$ with $\omega(i,k) = \varkappa(i)\,|\sigma_i^{1/2})(\sigma_i^{1/2}|\delta_k^i$ of the standard compound state (10) with $\widetilde{\mathcal{B}} = \mathcal{B}$, $\rho = \sigma$. Thus $\mathsf{H}(\varsigma) = \mathsf{I}(\pi_q)$, where

$$\begin{aligned}\mathsf{I}(\pi_q) &= \mathrm{Tr}\omega\left(\ln\omega - \ln(\sigma\otimes I) - \ln(I\otimes\sigma)\right) = \mathsf{S}(\omega) - 2\mathsf{S}(\sigma)\\ &= \sum_i \varkappa(i)\ln\varkappa(i) - 2\mathrm{Tr}\,\sigma\ln\sigma = -\sum_i \varkappa(i)\left(\ln\varkappa(i) + 2\mathrm{Tr}_{\mathcal{H}_i}\sigma_i\ln\sigma_i\right).\end{aligned}$$

Here we used that $\mathrm{Tr}\;\omega\ln\omega = \sum_i \varkappa(i)\ln\varkappa(i)$ due to

$$\omega\ln\omega = \oplus_{i,k}\omega(i,k)\ln\omega(i,k) = \oplus_i\varkappa(i)\,|\sigma_i^{1/2})(\sigma_i^{1/2}|\ln\varkappa(i),$$

and that $\mathrm{Tr}\;\sigma\ln\sigma = \sum_i \varkappa(i)\left(\ln\varkappa(i) - \mathsf{S}_{\mathcal{B}_i}(\varsigma_i)\right)$ due to

$$\sigma\ln\sigma = \oplus_i\sigma(i)\ln\sigma(i) = \oplus_i\varkappa(i)\,\sigma_i\left(\ln\varkappa(i) + \ln\sigma_i\right)$$

for the orthogonal decomposition $\sigma = \oplus_i\varkappa(i)\,\sigma_i$, where $\varkappa(i) = \mathrm{Tr}\,\sigma(i)$.

Thus $\mathsf{H}(\varsigma) = \mathsf{H}_{\mathcal{B}|\mathcal{C}}(\varsigma) + \mathsf{H}_{\mathcal{C}}(\varsigma) = 2\mathsf{S}_{\mathcal{B}|\mathcal{C}}(\varsigma) + \mathsf{S}_{\mathcal{C}}(\varsigma) \le 2\mathsf{S}(\varsigma)$, and it is bounded by

$$\begin{aligned}\mathsf{C}_{\mathcal{B}} &= \sup_{\varkappa}\sum_i \varkappa(i)\left(2\sup_{\varsigma_i}\mathsf{S}_{\mathcal{B}(i)}(\varsigma_i) - \ln\varkappa(i)\right)\\ &= -\inf_{\varkappa}\sum_i \varkappa(i)\left(\ln\varkappa(i) - 2\ln\dim\mathcal{H}_i\right) = \ln\dim\mathcal{B}.\end{aligned}$$

Here we used the fact that the supremum of von Neumann entropies $\mathsf{S}(\sigma_i)$ for the simple algebras $\mathcal{B}(i) = \mathcal{L}(\mathcal{H}_i)$ with $\dim\mathcal{B}(i) = (\dim\mathcal{H}_i)^2 < \infty$ is achieved on the tracial density operators $\sigma_i = (\dim\mathcal{H}_i)^{-1}I^i \equiv \sigma_i^\circ$, and the infimum of the relative entropy

$$\mathsf{R}(\varkappa : \varkappa^\circ) = \sum_i \varkappa(i)\left(\ln\varkappa(i) - \ln\varkappa^\circ(i)\right),$$

where $\varkappa^\circ(i) = \dim\mathcal{B}(i)\,/\dim\mathcal{B}$, is zero, achieved at $\varkappa = \varkappa^\circ$. ∎

5. Quantum Channel and Entropic Capacities

Let $\mathcal{H}_1$ be a Hilbert space describing a quantum input system and $\mathcal{H}$ describe its output Hilbert space. A quantum channel is an affine operation sending each input state defined on $\mathcal{H}_1$ to an output state defined on $\mathcal{H}$ such that the mixtures of states are preserved. A deterministic quantum channel is given by a linear isometry $U\colon\mathcal{H}_1 \to \mathcal{H}$ with $U^\dagger U = I^1$ (I^1 is the identify operator in $\mathcal{H}_1$) such that each input state vector $\eta_1 \in \mathcal{H}_1$, $\|\eta_1\| = 1$, is transmitted into an output state vector $\eta = U\eta_1 \in \mathcal{H}$, $\|\eta\| = 1$. The orthogonal sums $\varsigma_1 = \oplus\varsigma_1(n)$ of pure input states $\varsigma_1(B,n) = \eta_1(n)^\dagger B\eta_1(n)$ are sent into the orthogonal sums $\varsigma = \oplus\varsigma(n)$ of pure states on $\mathcal{B} = \mathcal{L}(\mathcal{H})$ corresponding to the orthogonal state vectors $\eta(n) = U\eta_1(n)$.

A noisy quantum channel sends pure input states ς_1 on an algebra $\mathcal{B}^1 \subseteq \mathcal{L}(\mathcal{H}_1)$ into mixed ones $\varsigma = \varsigma_1\Lambda$ given by the composition with a normal completely positive unital map $\Lambda : \mathcal{B} \to \mathcal{B}^1$. We shall assume that $\mathcal{B}^1$ (as well as $\mathcal{B}$) is equipped with a normal faithful semifinite weight ν_1 defining the pairing $\langle B, u^\dagger u\rangle_1 = \nu_1\left(\tilde{u}^\dagger B\tilde{u}\right)$

of $\mathcal{B}^1$ and $\mathcal{B}_{\intercal}^1 = \widetilde{\mathcal{B}_*^1}$. Then the input-output state transformations are described by the transposed map $\Lambda^\intercal : \mathcal{B}_\intercal^1 \to \mathcal{B}_\intercal$

$$\langle B, \Lambda^\intercal(\sigma_1)\rangle = \langle \Lambda(B), \sigma_1\rangle_1, \quad B \in \mathcal{B}, \sigma_1 \in \mathcal{B}_\intercal^1$$

defining the output density operators $\sigma = \Lambda^\intercal(\sigma_1)$ for any input normal state $\varsigma_1(B) = \langle B, \sigma_1\rangle_1$. Without loss of generality the input algebra $\mathcal{B}^1$ can be assumed being the smallest decomposable algebra generated by the range $\Lambda(\mathcal{B})$ of the channel map Λ ($\mathcal{B}^1$ is Abelian if $\Lambda(\mathcal{B})$ consists of only commuting operators on $\mathcal{H}_1$)

The input generalized entanglements $\varpi^1 : \mathcal{A} \to \mathcal{B}_\intercal^1$, including encodings of the state ς_1 with the density $\sigma_1 = \varpi^1(I)$, will be defined by the couplings $\kappa^* : \mathcal{B}^1 \to \mathcal{A}_*$ as $\varpi^1 = \kappa^{\tilde{}}$. Here $\kappa : \mathcal{A} \to \mathcal{B}_*^1$ is a normal TCP map defining the state $\varrho = \nu_1 \circ \kappa$ of a probe system $(\mathcal{A}, \mu)$ which is entangled to $(\mathcal{B}^1, \varsigma_1)$ by $\kappa^{\tilde{}}(A) = J\kappa(A^\dagger)J$, and the adjoint map κ^* is defined as usually by

$$\langle A|\kappa^*(B)\rangle_\mu = \omega_1(A^\dagger \otimes B) = \langle \kappa(A)|B\rangle_1, \quad \forall A \in \mathcal{A}, B \in \mathcal{B}_1,$$

where ω_1 is the corresponding compound state on $\mathcal{A} \otimes \mathcal{B}^1$.

These (generalized) entanglements describe the quantum-quantum correspondences (q-, c-, or o-encodings) of the probe systems $(\mathcal{A}, \varrho)$ with the density operators $\rho = \kappa^\intercal(I^1)$, to the input $(\mathcal{B}^1, \varsigma_1)$ of the channel Λ. In particular, the most informative standard input entanglement $\varpi_q^1 : \widetilde{\mathcal{B}^1} \to \mathcal{B}_\intercal^1$ is the entanglement of the transposed input system $(\mathcal{A}^0, \varrho_0) = \left(\widetilde{\mathcal{B}^1}, \widetilde{\varsigma_1}\right)$ corresponding to the TCP map $\kappa_q(A) = J\sigma_1^{1/2}A^\dagger\sigma_1^{1/2}J$. In the case of discrete decomposable $\mathcal{A}^0 = \widetilde{\mathcal{B}^1} = \mathcal{B}^1$ with the density operator $\sigma_1 = \oplus_i\sigma_1(i)$ this extreme input q-encoding defines the following density operator

$$\omega_q = (\mathrm{I} \otimes \Lambda^\intercal)(\omega_{q1}), \quad \omega_{q1} = \oplus_i|\sigma_1(i)^{1/2})(\sigma_1(i)^{1/2}| \tag{31}$$

of the input-output compound state $\omega_{q1}\Lambda$ on $\mathcal{A}^0 \otimes \mathcal{B} = \mathcal{B}^1 \otimes \mathcal{B}$.

The other extreme case of the generalized input entanglements, the pure c-encodings corresponding to (12), are less informative then the pure d-encodings $\varpi_d^1 = \kappa_d^{\tilde{}}$ given by the decompositions $\kappa_d^* = \sum|n\rangle\langle n|\varsigma_1(n)$ with pure states $\varsigma_1(B, n) = \eta(n)^\dagger B\eta(n)$ on $\mathcal{B}_1$. They define the density operators

$$\omega_d = (\mathrm{I} \otimes \Lambda^\intercal)(\omega_{d1}), \quad \omega_{d1} = \sum_n |n\rangle\langle n| \otimes \eta_1(n)\eta_1(n)^\dagger, \tag{32}$$

of the $\mathcal{B}^1 \otimes \mathcal{B}$-compound state $\omega_{d1}\Lambda = \omega_{d1} \circ (\mathrm{I} \otimes \Lambda)$. They are known as the Ohya compound states $\omega_o = \omega_{o1}\Lambda$ [10] in the case

$$\sigma_1(n) = \eta_1^o(n)\eta_1^o(n)^\dagger, \quad \eta_1^o(n)^\dagger\eta_1^o(m) = p_1(n)\delta_n^m,$$

of orthogonality of the density operators $\sigma_1(n)$ normalized to the eigen-values $p_1(n)$ of σ_1. The o-compound states are achieved by pure o-encodings $\varpi_d^1 = \kappa_o^{\tilde{}}$ described by the couplings $\kappa_o = \sum|n\rangle\langle n|\varsigma_1^o(n)$ with ς_1^o corresponding to η_1^o. The input-output density operator

$$\omega_o = (\mathrm{I} \otimes \Lambda^\intercal)\omega_{o1}, \quad \omega_{o1} = \sum_n |n\rangle\langle n| \otimes \eta_1^o(n)\eta_1^o(n)^\dagger \tag{33}$$

of the Ohya compound state ω_o is achieved by the coupling $\lambda = \kappa^* \Lambda$ of the output $(\mathcal{B}, \varsigma)$ to the extreme probe system $\left(\mathcal{A}^0, \varrho_0\right) = \left(\mathcal{B}^1, \varsigma_1\right)$ as the composition of κ^* and the channel Λ.

If $\mathrm{K} : \mathcal{A} \to \mathcal{A}^0$ is a normal completely positive unital map

$$\mathrm{K}(A) = \mathrm{Tr}_{\mathcal{F}_-} \widetilde{X} A \widetilde{X}^\dagger, \quad A \in \mathcal{A},$$

where X is a bounded operator $\mathcal{F}_- \otimes \mathcal{G}_0 \to \mathcal{G}$ with $\mathrm{Tr}_{\mathcal{F}_-} X^\dagger X = I^0$, the compositions $\kappa = \kappa_0 \mathrm{K}$, $\pi = \Lambda^* \kappa$ describe the entanglements of the probe system $(\mathcal{A}, \varrho)$ to the channel input $\left(\mathcal{B}^1, \varsigma_1\right)$ and the output $(\mathcal{B}, \varsigma)$ via this channel respectively. The state $\varrho = \varrho_0 \mathrm{K}$ is given by

$$\mathrm{K}^\intercal \left(\rho_0\right) = X \left(I^- \otimes \rho_0\right) X^\dagger \in \mathcal{A}_*$$

for each density operator $\rho_0 \in \mathcal{A}^0_*$, where I^- is the identity operator in $\mathcal{F}_-$. The resulting entanglement $\pi = \lambda^* \mathrm{K}$ defines the compound state $\omega = \omega_{01} \circ (\mathrm{K} \otimes \Lambda)$ on $\mathcal{A} \otimes \mathcal{B}$ with

$$\omega_{01}\left(A^0 \otimes B^1\right) = \mathrm{Tr}\ \tilde{A}^0 \kappa_0^* \left(B^1\right) = \mathrm{Tr}\ \tilde{v}_{01}^\dagger \left(A^0 \otimes B^1\right) \tilde{v}_{01}$$

on $\mathcal{A}^0 \otimes \mathcal{B}^1$. Here $v_{01} : \mathcal{G}_0 \otimes \mathcal{H}_1 \to \mathcal{F}_{01}$ is the amplitude operator uniquely defined by the input compound density operator $\omega_{01} \in \mathcal{A}^0_\intercal \otimes \mathcal{B}^1_\intercal$ up to a unitary operator U^0 on $\mathcal{F}_{01}$. The effect of the input entanglement κ and the output channel Λ can be written in terms of the amplitude operator of the state ω as

$$v = (X \otimes Y) \left(I^- \otimes v_{01} \otimes I^+\right) U$$

up to a unitary operator U in $\mathcal{F} = \mathcal{F}_- \otimes \mathcal{F}_{01} \otimes \mathcal{F}_+$. Thus the density operator of the input-output compound state ω is given by $\omega_{01} (\mathrm{K} \otimes \Lambda)$ with the density

$$(\mathrm{K} \otimes \Lambda)^* \left(\omega_{01}\right) = (X \otimes Y)\, \omega_{01}\, (X \otimes Y)^\dagger, \tag{34}$$

where $\omega_{01} = v_{01} v_{01}^\dagger$.

Let $\mathcal{K}_q^1$ be the set of all normal TCP maps $\kappa : \mathcal{A} \to \mathcal{B}^1_*$ with any probe algebra $\mathcal{A}$ normalized as $\mathrm{Tr}\kappa(I) = 1$, and $\mathcal{K}_q(\varsigma_1)$ be the subset of all $\kappa \in \mathcal{K}_q^1$ with $\kappa(I) = \varsigma_1$. Each $\kappa \in \mathcal{K}_q^1$ can be decomposed as $\kappa_q \mathrm{K}$, where $\kappa_q : \mathcal{A}^0 \to \mathcal{B}^1$ defines the standard input entanglement $\varpi_q^1 = \kappa_q^{\tilde{}}$, and K is a normal unital CP map $\mathcal{A} \to \widetilde{\mathcal{B}^1}$.

Further let $\mathcal{K}_c^1$ be the set of all CP-TCP maps κ described by the combinations

$$\kappa(A) = \sum_n \varrho_n(A)\, \sigma_1(n) \tag{35}$$

of the primitive maps $A \mapsto \varrho_n(A)\, \sigma_1(n)$, and $\mathcal{K}_d^1$ be the subset of the diagonalizing entanglements κ, i.e. the decompositions

$$\kappa(A) = \sum_n \langle n|A|n\rangle \sigma_1(n). \tag{36}$$

As in the first case $\mathcal{K}_c(\varsigma_1)$ and $\mathcal{K}_d(\varsigma_1)$ denote the subsets corresponding to a fixed $\kappa(I) = \varsigma_1$. Each $\mathcal{K}_c(\varsigma_1)$ can be represented as the composition $\kappa = \kappa_d \mathrm{K}$, where κ_d normalized to ς_1 describes a pure d-encoding $\varpi_d^1 = \kappa_d^{\tilde{}}$ of $\left(\mathcal{B}^1, \varsigma_1\right)$ for a proper choice of the CP map $\mathrm{K} : \mathcal{A} \to \mathcal{B}^1$.

Furthermore let $\mathcal{K}_o^1$ (and $\mathcal{K}_o(\varsigma_1)$) be the subset of all decompositions (36) with orthogonal $\sigma_1(n)$ (and fixed $\sum_n \sigma_1(n) = \sigma_1$):

$$\sigma_1(m)\, \sigma_1(n) = 0,\ m \neq n.$$

Each $\kappa \in \mathcal{K}_o(\varsigma_1)$ can also be represented as $\kappa = \kappa_o \mathrm{K}$, with κ_o describing the pure o-encoding $\varpi_o^1 = \kappa_o^{\cdot}$ of $(\mathcal{B}^1, \varsigma_1) = (\mathcal{A}^0, \varrho_0)$.

Now, let us maximize the entangled mutual entropy for a given quantum channel Λ (and a fixed input state ς_1 on the decomposable $\mathcal{B}^1 = \widetilde{\mathcal{B}^1}$) by means of the above four types entanglements κ. The mutual information (23) was defined in the previous section by the density operators of the corresponding compound state ω on $\mathcal{A} \otimes \mathcal{B}$ and the product-state $\varphi = \varrho \otimes \varsigma$ of the marginals ϱ, ς for ω. In each case

$$\omega = \omega_{01}(\mathrm{K} \otimes \Lambda), \quad \varphi = \varphi_{01}(\mathrm{K} \otimes \Lambda),$$

where K is a CP map $\mathcal{A} \to \mathcal{A}^0 = \mathcal{B}^1$, ω_{01} is one of the corresponding extreme compound states ω_{q1}, $\omega_{c1} = \omega_{d1}$, ω_{o1} on $\mathcal{B}^1 \otimes \mathcal{B}^1$, and $\varphi_{01} = \rho_0 \otimes \varsigma_1$. The density operator $\omega = (\mathrm{K} \otimes \Lambda)^{\intercal}(\omega_{01})$ is written in (34), and $\phi = \rho \otimes \sigma$ can be written as

$$\phi = \kappa^{\intercal}(I) \otimes \lambda^{\intercal}(I),$$

where $\lambda^{\intercal} = \Lambda^{\intercal}\pi_1^0$. This proves the following proposition.

Proposition 5.1. *The entangled mutual informations achieve the following maximal values*

$$\sup_{\kappa \in \mathcal{K}_q(\varsigma_1)} \mathsf{I}(\kappa^*\Lambda) = \mathsf{I}_q(\varsigma_1, \Lambda) := \mathsf{I}(\kappa_q^*\Lambda), \tag{37}$$

$$\mathsf{I}_c(\varsigma_1, \Lambda) := \sup_{\kappa \in \mathcal{K}_c(\varsigma_1)} \mathsf{I}(\kappa^*\Lambda) = \sup_{\kappa_d} \mathsf{I}(\kappa_d^*\Lambda) \equiv \mathsf{I}_d(\varsigma_1, \Lambda),$$

$$\sup_{\kappa \in \mathcal{K}_o(\varsigma_1)} \mathsf{I}(\kappa^*\Lambda) = \mathsf{I}_o(\varsigma_1, \Lambda) := \sup_{\kappa_o} \mathsf{I}(\kappa_o^*\Lambda), \tag{38}$$

where $\kappa_\cdot$ are the corresponding extremal input couplings $\mathcal{A}^0 \to \mathcal{B}_*^1$ with $\mu \circ \kappa_\cdot^* = \varsigma_1$. They are ordered as

$$\mathsf{I}_q(\varsigma_1, \Lambda) \geq \mathsf{I}_c(\varsigma_1, \Lambda) = \mathsf{I}_d(\varsigma_1, \Lambda) \geq \mathsf{I}_o(\varsigma_1, \Lambda). \tag{39}$$

In the following definition the maximal informations $\mathsf{I}_c(\varsigma_1, \Lambda) = \mathsf{I}_d(\varsigma_1, \Lambda)$ is simply denoted as $\mathsf{I}_1(\varsigma_1, \Lambda)$.

Definition 5.1. *The suprema*

$$\mathsf{C}_q(\Lambda) = \sup_{\kappa \in \mathcal{K}_q^1} \mathsf{I}(\kappa^*\Lambda) = \sup_{\varsigma_1} \mathsf{I}_q(\varsigma_1, \Lambda),$$

$$\sup_{\kappa \in \mathcal{K}_d^1} \mathsf{I}(\kappa^*\Lambda) = \mathsf{C}_1(\Lambda) := \sup_{\varsigma_1} \mathsf{I}_1(\varsigma_1, \Lambda), \tag{40}$$

$$\mathsf{C}_o(\Lambda) = \sup_{\kappa \in \mathcal{K}_o^1} \mathsf{I}(\kappa^*\Lambda) = \sup_{\varsigma_1} \mathsf{I}_o(\varsigma_1, \Lambda),$$

are called the q-, c- or d-, and o-capacities respectively for the quantum channel defined by a normal unital CP map $\Lambda : \mathcal{B} \to \mathcal{B}^1$.

Obviously, the capacities (40) satisfy the inequalities

$$\mathsf{C}_o(\Lambda) \leq \mathsf{C}_1(\Lambda) \leq \mathsf{C}_q(\Lambda).$$

Theorem 5.2. *Let $\Lambda(B) = U^{\dagger}BU$ be a unital CP map $\mathcal{B} \to \mathcal{B}^1$ describing a quantum deterministic channel. Then*

$$\mathsf{I}_1(\varsigma_1, \Lambda) = \mathsf{I}_o(\varsigma_1, \Lambda) = \mathsf{S}(\varsigma_1), \quad \mathsf{I}_q(\varsigma_1, \Lambda) = \mathsf{S}_q(\varsigma_1),$$

where $\mathsf{S}_q(\varsigma_1) = \mathsf{H}(\varsigma_1)$, and thus in this case

$$\mathsf{C}_1(\Lambda) = \mathsf{C}_o(\Lambda) = \ln \operatorname{rank} \mathcal{B}^1, \quad \mathsf{C}_q(\Lambda) = \ln \dim \mathcal{B}^1.$$

Proof. It was proved in the previous section for the case of the identity channel $\Lambda = \mathrm{I}$, and thus it is also valid for any isomorphism $\Lambda : B \mapsto U^{\dagger}BU$ describing the state transformations $\Lambda^{\intercal} : \sigma \mapsto Y\sigma Y^{\dagger}$ by a unitary operator $U = \overline{Y}$. In the case of non-unitary Y we can use the identity

$$\operatorname{Tr} Y(\sigma_1 \otimes I^+) Y^{\dagger} \ln Y(\sigma_1 \otimes I^+) Y^{\dagger} = \operatorname{Tr} S(\sigma_1 \otimes I^+) \ln S(\sigma_1 \otimes I^+),$$

where $S = Y^{\dagger}Y$. Due to this $\mathsf{S}(\varsigma_1\Lambda) = -\operatorname{Tr} S(\sigma_1 \otimes I^+) \ln S(\sigma_1 \otimes I^+)$, and $\mathsf{S}(\omega_{01}(\mathrm{K} \otimes \Lambda)) =$

$$-\operatorname{Tr} (R \otimes S)(I^- \otimes \omega_{01} \otimes I^+) \ln (R \otimes S)(I^- \otimes \omega_{01} \otimes I^+),$$

where $R = X^{\dagger}X$. Thus $\mathsf{S}(\varsigma_1\Lambda) = \mathsf{S}(\varsigma_1)$, $\mathsf{S}(\omega_{01}(\mathrm{K} \otimes \Lambda)) = \mathsf{S}(\omega_{01}(\mathrm{K} \otimes \mathrm{I}))$ if $Y^{\dagger}Y = I$, and

$$\begin{aligned} \mathsf{I}((\pi_1\Lambda)) &= \mathsf{S}(\varrho_0\mathrm{K}) + \mathsf{S}(\varsigma_1) - \mathsf{S}(\omega_{01}(\mathrm{K} \otimes \mathrm{I})) \\ &\leq \mathsf{S}(\varrho_0) + \mathsf{S}(\varsigma_1) - \mathsf{S}(\omega_{01}) = \mathsf{I}(\omega_{01}) \end{aligned}$$

for $\kappa = \kappa_0\mathrm{K}$ with any normal unital CP map $\mathrm{K} : \mathcal{A} \to \mathcal{A}^0$ and a compound state ω_{01} on $\mathcal{A}^0 \otimes \mathcal{B}^1$. The supremum (37), which is achieved at the standard entanglement, corresponding to $\omega_{01} = \omega_{q1}$, coincides with q-entropy $\mathsf{H}(\varsigma_1)$, and the supremum (??), coinciding with $\mathsf{S}(\varsigma_1)$, is achieved for a pure o-entanglement, corresponding to $\omega_{01} = \omega_{o1}$ given by any Schatten decomposition for σ_1. Moreover, the entropy $\mathsf{H}(\varsigma_1)$ is also achieved by any pure d-entanglement, corresponding to $\omega_{01} = \omega_{d1}$ given by any extreme decomposition for σ_1, and thus is the maximal mutual information $\mathsf{I}_1(\varsigma_1, \Lambda)$ in the case of deterministic Λ. Thus the capacity $\mathsf{C}_1(\Lambda)$ of the deterministic channel is given by the maximum $\mathsf{C}_o = \ln \dim \mathcal{H}_1$ of the von Neumann entropy S, and the q-capacity $\mathsf{C}_q(\Lambda)$ is equal $\mathsf{C}_{\mathcal{B}^1} = \ln \dim \mathcal{B}^1$. ∎

In the general case, d-entanglements can be more informative than o-entanglements as can be shown by an example of a quantum noisy channel for which

$$\mathsf{I}_1(\varsigma_1, \Lambda) > \mathsf{I}_o(\varsigma_1, \Lambda), \quad \mathsf{C}_1(\Lambda) > \mathsf{C}_o(\Lambda).$$

The last equalities of the above theorem will be related to the work on entropy by Voiculescu [26].

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