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Stochastic Evolutions As Boundary Value Problems

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Abstract. Solutions to linear stochastic and quantum stochastic equations are proved to provide the interaction representations for the solutions to certain Dirac type equations with boundary conditions in pseudo Fock spaces. The latter are presented as the semi-classical limit of an appropriately dressed unitary evolutions corresponding to a boundary-value problem for general Schrödinger equations.

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1 Introduction

We are going to show that the solutions to a wide class of stochastic and quantum stochastic equations can be obtained from a positive (relativistic or non-relativistic) Hamiltonian with singular interaction as a strong limit of the input flow of quantum particles with asymptotically infinite momentum but a constant velosity. Thus the problem of stochastic approximation is reduced to a sort of quasiclassical asymptotics of a quantum mechanical boundary value problem in extra dimension.

There exists a broad literature on the stochastic limit in quantum physics in which quantum stochastics is derived from a nonsingular interaction (see monograph [1], and references therein). Here we follow a different approach recently outlined in [3]: instead of rescaling the interaction potentials we treat the singular interaction δ-potentials rigorously as the boundary conditions, and obtain the stochastic limit as an ultra-relativistic limit of a Schrödinger boundary value problem in a Hilbert space of infinite number of particles.

We start with a short Section 2 fixing some general notations that are used throughout the paper.

In Section 3, we show how the solutions to linear stochastic differential equations in Banach spaces driven by a compound Poisson process can be obtained as the interaction picture representation for the boundary- value problem for shifts in Euclidean simplices. This class of equations includes the Belavkin quantum filtering equations describing the aposterior dynamics of a quantum system under continuous non-demolition measurement of counting type [2].

Since the stochastic equations driven by a Wiener process can be obtained as the limits of equations driven by compound Poisson process, the results of Section 3 allow the representation of the solutions to linear diffusion equations in Banach spaces as the limits of certain (deterministic) boundary value problems.

After Section 4 that describes shortly the combinatorics of multiple Fock spaces, we obtain our main results in Sections 5, where we establish a connection between boundary value problem and general quantum stochastic equations (and, in particular, classical diffusion equations) directly, without a limiting procedure. To this end, we develop a theory of boundary value problems for shifts in "coloured simplices" as the restrictions of the shifts in multiple and/or pseudo Fock spaces, and then use the theory developed in [4], where it is shown that working in multiple pseudo Fock spaces allows for a representation of stochastic and quantum stochastic evolutions that preserves the number of particles (though changes their colour), and consequently reduces a study of general quantum stochastic flows to the study of Poisson driven evolutions in coloured simplices. Using pseudo Fock spaces is a characteristic feature of our method that distinguishes it from an alternative way of exploiting the connection between boundary value problems and quantum stochastics initiated in [6], (see also [7] for recent developments in this direction). Unlike [6], we systematycally consider the evolutions in general Banach spaces (and, in particular, non-unitary boundary conditions), which are important for applications to general stochatic equations, in particular those decribing the models of continuous quantum measurements.

In Section 6 we show how the boundary value problems for shifts can be obtained by a sort of semiclassical limit $h \to 0$ (which is however quite different from the usual semiclassical limit for stochastic equations [8], [10], and which generalises the ultra-relativistic limit of [3]) from evolutions described by general Schrödinger problems with a bounded below Hamiltonian.

2 Main notations

(i) *Generalities.* By H (respectively by B) we shall always denote a Hilbert (respectively a Banach) space with the norm $\| \|_H$ (respectively $\| \|_B$). In applications, B will be the Banach algebra of bounded linear operators $\mathcal{L}(H)$ in H. For a function ϕ on $\mathbb{R}$ we shall denote by $\phi(z_-)$ (respectively $\phi(z_+)$) the left (respectively the right) limit of $\phi(t)$ as $t \to z$ (when it exists, of course). For a subset $M \subset \mathbb{R}^n$ we shall denote by $\chi_M(z)$ the indicator function of M that equals 1 or 0 respectively when $z \in M$ or $z \in \mathbb{R}^n \setminus M$. We denote by $C_B(M)$ (resp. $C_B^{comp}(M)$) the space of continuous functions $M \mapsto B$ vanishing at infinity (respectively with a compact support) equipped with the usual sup-norm $\|\psi\| = \sup_z \|\psi(z)\|_B$.

(ii) *Simplicies.* We shall denote by Σ_n the infinite simplex:

$$\Sigma_n = \{z = (z_1, ..., z_n) \in \mathbb{R}^n : \quad z_1 < z_2 < ... < z_n\},$$

equipped with Lebesgue measure. Clearly this simplex can be decomposed into the union of $n+1$ cells Σ_n^k:

$$\begin{gathered} \Sigma_n^0 = \{z \in \Sigma_n : \quad z_1 \geq 0\}, \\ \Sigma_n^k = \{z \in \Sigma_n : \quad z_k \leq 0 \leq z_{k+1}\}, \quad k = 1, ..., n-1, \\ \Sigma_n^n = \{z \in \Sigma_n : \quad z_n \leq 0\}. \end{gathered} \tag{1}$$

In particular, $\Sigma_1 = \mathbb{R}$, $\Sigma_1^0 = \mathbb{R}_+$, $\Sigma_1^1 = \mathbb{R}_-$.

Vectors $z \in \Sigma_n$ are usually identified with the subsets $\zeta = \zeta(z) = \{z_1, ..., z_n\} \subset \mathbb{R}$ of the real line of cardinality $|\zeta| = n$. The representation of the points of Σ_n by the subsets of $\mathbb{R}$ (respectively by n-dimensional vectors with ordered coordinates) is more natural for defining stochastic processes (respectively, boundary value problems) we are dealing with.

(iii) *Banach valued L^p-spaces.* For a number $p \geq 1$, we denote by $L_B^p(n) = L_B^p(\Sigma_n)$ the Banach space of functions ψ: $\Sigma_n \to B$ (more precisely, equivalence classes of such functions) with the norm

$$\|\psi\| = \left(\int_{\Sigma_n} \|\psi(z)\|_B^p \, dz\right)^{1/p}.$$

In particular, $L_H^2(n)$ is the Hilbert tensor product $H \otimes L^2(\Sigma_n)$. By $L_B^p(\Sigma_n^k)$ we denote the corresponding space of functions on the cells (1). We shall also consider the locally convex topological spaces $L_B^{p,loc}(n)$ of measurable functions $\psi : \Sigma_n \mapsto H$ with the countable set of norms $\|\psi\|_N$ (N is a natural number) defined as above but with the integration over $\Sigma_n \cap \{z : |z| \leq N\}$. The notation L_B^p (respectively $L_B^{p,loc}$) is reserved for the space $L_B^p(\mathbb{R})$ (respectively $L_B^{p,loc}(\mathbb{R})$).

(iv) *Shifts and differentiation operators in $L_B^p(n)$.* By $T_n(t) = T_{n,p,B}(t)$ we shall denote the shift in $L_B^p(n)$ which takes a function $\varphi \in L_B^p(n)$ to the function $(T_n(t)\varphi)(z_1, ..., z_n) = \varphi(z_1+t, ..., z_n+t)$. This is a continuous group, whose generator (that acts on smooth functions as differentiation along the vector $(1, 1, ..., 1)$) we shall denote by $\partial_z = \partial_{z_1} + ... + \partial_{z_n}$. In particular, the operator $i\partial$ is self-adjoint in $L_H^2(n)$. We shall write shortly $T(t)$ for $T_1(t)$.

(v) *Dressing and jumps.* Let $-iE$ be a generator of a continuous group $\exp\{-itE\}$ in B, and let A be an operator in $L_B^p(n)$. For a real-valued continuous function $f(z)$ on Σ_n we define an operator in $L_B^p(n)$ by the formula

$$A_{Ef(z)}\varphi(z) = e^{iEf(z)} A e^{-iEf(z)} \varphi(z). \tag{2}$$

Clearly, if E is a self-adjoint operator in H, and if A is self-adjoint in $L_H^2(n)$, then $A_{Ef(z)}$ is also self-adjoint in $L_H^2(n)$.

For a function $A : \Sigma_n \mapsto \mathcal{L}(B)$ and a Borel subset $s \subset \Sigma_n$ we define a bounded operator A^s in $L_B^p(n)$ by the formula

$$(A^s\varphi)(z) = \chi_s(z)A(z)\varphi(z) + (1 - \chi_s(z))\varphi(z) \tag{3}$$

(more correct, but more heavy notation for A^s would be of course A^{χ_s}). Clearly, the operator A^s remains the same if s is changed on a set of Lebesgue measure zero. Clearly $T(t)A^s = A^{s-t}T(t)$ and $(A^s)_{Ef(z)} = (A_{Ef(z)})^s$, which implies, in particular, that the notation $A_{Ef(z)}^s$ is not ambiguous.

3 Stochastic equations driven by a Poisson noise as boundary value problems

Let $\sigma = \{\sigma_1, ..., \sigma_n\}$ be an arbitrary (ordered) family of operators from $\mathcal{L}(B)$ and let $-iE$ be a generator of a continuous group of linear operators in B. For a given $z \in \Sigma_n^0$, let us consider the linear multiple kick equation

$$d\eta + iE\eta\, dt = \sum_{j=1}^{n} (\sigma_j - 1)\eta\, d\chi_{(z_j,\infty)}, \quad \eta \in B, \quad t \geq 0. \tag{4}$$

As in the case of stochastic equations, this equation should be understood rigorously as the corresponding integral equation, where for any function $f(t)$ having everywhere right and left limits the integral with respect to $d\chi_{(z,\infty)}(t)$ is defined by the formula

$$\int_0^t f(\tau)\, d\chi_{(z,\infty)}(\tau) = \begin{cases} 0, & t \leq z, \\ f(z_-), & t > z. \end{cases} \tag{5}$$

If $z \in \Sigma_n^0$ is a random variable on a probability space Ω, one can rewrite equation (4) as the stochastic equation

$$d\eta + iE\eta\, dt = (\sigma_{n_t} - 1)\eta\, dn_t, \quad \eta \in B, \quad t \geq 0, \tag{6}$$

driven by the counting process $n_t(\zeta) = n_t(\zeta(z)) = |\zeta \cap [0,t)|$.

Clearly $\eta(t)$ satisfying (4) evolves according to the free equation $\frac{\partial \eta}{\partial t} = -iE\eta$ between the jump-times z_k, and at the times $t = z_k$ the wave function experiences the jump $\eta \mapsto \sigma(z_k)\eta$. This proves the following

Proposition 1 *For any $z \in \Sigma_n^0$, the operator $V(t,z)$, which gives the solution $V(t,z)\eta_0$ to the Cauchy problem for equation* (4) *with the initial function η_0, belongs to $\mathcal{L}(B)$ and has the following explicit form: for $z_k \leq t < z_{k+1}$*

$$V(t,z) = \exp\{-iE(t - z_k)\}\sigma_k \exp\{-iE(z_k - z_{k-1})\}\sigma_{k-1}...\sigma_1 \exp\{-iEz_1\}. \tag{7}$$

If B is a Hilbert space H and if all σ_j are unitary, then $V(t,z)$ is also a unitary operator.

We are going to give a representation for this operator in terms of the boundary value problem for shifts in $L_B^p(n)$, more precisely, in terms of the solutions to the equation

$$i\partial_t \varphi = (i\partial_z + E)\varphi = (i(\partial_{z_1} + \partial_{z_2} + ... + \partial_{z_n}) + E)\varphi, \quad \varphi \in L_B^p(n), \tag{8}$$

combined with the boundary conditions

$$\varphi(z_1, ..., z_{k-1}, 0_-, z_{k+1}, ..., z_n) = \sigma_k \varphi(z_1, ..., z_{k-1}, 0_+, z_{k+1}, ..., z_n), \quad k = 1, ..., n. \tag{9}$$

Let $D_\sigma = D_{\sigma_1,...,\sigma_n}(p, B)$ denote the dense subspace of functions $\varphi \in L_B^p(n)$ with the properties:

(i) for each $k = 0, ..., n$ the restriction $\varphi|_{\Sigma_n^k}$ has a continuous version such that on all lines parallel to the vector $(1, ..., 1)$ this restiction is absolutely continuous and $\partial_z \varphi|_{\Sigma_n^k} \in L_B^p(n)$.

(ii) the boundary conditions (9) are satisfied.

The differentiation operator $\partial_z^\sigma = \partial_{z_1}^{\sigma_1} + ... + \partial_{z_n}^{\sigma_n}$ acts on D_σ in the obvious way. The following result is obtained by inspection.

Proposition 2 *For an arbitrary $p \geq 1$, the operators $T^\sigma(t)$ defined by the formulas*

$$T^\sigma(t) = \sigma_n^{\{z_n \in [-t,0]\}} ... \sigma_1^{\{z_1 \in [-t,0]\}} T_n(t) = T_n(t) \sigma_n^{\{z_n \in [0,t]\}} ... \sigma_1^{\{z_1 \in [0,t]\}} \qquad (10)$$

form a continuous semigroup of operators in $L_B^p(n)$, whose generator is the closure of the operator $\partial_z^\sigma = \partial_{z_1}^{\sigma_1} + ... + \partial_{z_n}^{\sigma_n}$ defined initially on D_σ. The space D_σ is invariant under the action of the semigroup.

The semigroup T^σ gives the solution to the Cauchy problem of equation (8), (9) for vanishing E. To include a non-trivial E, let us introduce the domain

$$D_\sigma^E = D_\sigma \cap \{\varphi : E\varphi \in L_B^p(n)\},$$

which is clearly dense in $L_B^p(n)$ for any p. The operator

$$\partial_z^\sigma - iE = \partial_{z_1}^{\sigma_1} + ... + \partial_{z_n}^{\sigma_n} - iE \qquad (11)$$

acts on D_σ^E in the obvious way. By usual abuse of notations, we shall denote by the same symbol $\partial_z^\sigma - iE$ the closure of this operator. The following statement is again proved by inspection.

Proposition 3 *(i) The operator* (11) *generates a continuous semigroup*

$$U_E^\sigma(t) = (\sigma_n^{\{z_n \in [-t,0]\}})_{Ez_n} ... (\sigma_1^{\{z_1 \in [-t,0]\}})_{Ez_1} \exp\{-iEt\} T_n(t), \qquad (12)$$

(where we used notations (2), (3)*) which solves the Cauchy problem for equations* (8), (9)*. Moreover,*

$$(T_n(t))^{-1} U_E^\sigma(t) = e^{-iEt} (\sigma_n^{\{z_n \in [0,t]\}})_{Ez_n} ... (\sigma_1^{\{z_1 \in [0,t]\}})_{Ez_1}. \qquad (13)$$

(ii) The operators (12) *are invertible for all $t \geq 0$ if and only if all σ_k, $k = 1, ..., n$, are invertible. Operators* (11) *with invertible σ_k are similar. More precisely, if σ_k has a continuous inverse σ_k^{-1}, $k = 1, ..., n$, then*

$$U_E^\sigma(t) = (\sigma_n^{\{z_n<0\}})_{Ez_n} ... (\sigma_1^{\{z_1<0\}})_{Ez_1}$$

$$\times \exp\{-iEt\} T_n(t) \left((\sigma_1^{\{z_1<0\}})_{Ez_1}\right)^{-1} ... \left((\sigma_n^{\{z_n<0\}})_{Ez_n}\right)^{-1}. \qquad (14)$$

(iii) If B is a Hilbert space H and all σ_k are unitary, the operator $i\partial_z^\sigma + E$ is self-adjoint in $L_H^2(n)$.

(iv) All statements of the Proposition remain valid in the spaces $L_B^{p,loc}(n)$.

Comparing formulas (7) and (13) yields the following result.

Theorem 1 *Solution* (7) *to the Cauchy problem of equation* (4) *can be written in the form*

$$V(t,z)\eta_0 = ((T_n(t)^{-1} U_E^\sigma(t)\varphi)(z), \qquad (15)$$

where $\varphi(z) = \eta_0$ for $z \in \Sigma_n^0$ and vanishes otherwise.

Remark. The function $\varphi(z)$ which equals to a constant vector η_0 for all positive z does not belong to L^p_B, but only to $L^{p,loc}_B$, which was the main reason for introducing these spaces.

In physical language, one interpretes formula (15) by saying that the stochastic evolution $V(t,z)$ gives the solutions of equation (8), (9) in the interaction representation with respect to the "free" shift $T_n(t)$. Notice that stochastic linear equations driven by a compound Poisson noise, in particular the quantum filtering equations describing the aposterior dynamics of quantum states under continuous observations of counting type, can be reduced to equation of type (4) or (6) pathwise, because a Poisson process has almost surely a finite number of jumps on each bounded time interval. Therefore, the solutions to these stochastic equations can be obtained as the interaction representation of the solutions of problem (8), (9) with respect to the "free" shift T_n. Theorem 4 at the end of the paper shows that the model (8), (9) in its turn can be obtained as a semiclassical limit of Schrödinger evolutions with a bounded below Hamiltonian.

4 Combinatorics of the multiple Fock space

This is an auxiliary section describing the combinatorics of secondly quantised operators in multiple Fock spaces in a way that is convenient for our purposes.

By a coloured simplex of n particle having m colours we understand the set

$$CS_{n,m} = \cup_{n_1+\ldots+n_m=n}\Sigma_{n_1} \times \ldots \times \Sigma_{n_m},$$

where the (disjoint) union is taken over all partitions of the integer number n in the sum of m non-negative numbers (the order is relevant), and where it is assumed that the product is over all non-vanishing n_j. The points of $CS_{n,m}$ can be parametrised either by ordered chains of labeled variables

$$z = \{z^\alpha\} = \{z_1^{\alpha(1)} < \ldots < z_n^{\alpha(n)}\}, \tag{16}$$

with α being functions $\alpha: \{1,\ldots,n\} \mapsto \{1,\ldots,m\}$ (that label the variables in a standard simplex Σ_n), or by the families of m vector variables

$$\zeta = \{\zeta^1 = (z_1^1,\ldots,z_{n_1}^1),\ldots,\zeta^m = (z_1^m,\ldots,z_{n_m}^m)\}, \tag{17}$$

where the entries of each ζ^j are ordered: $z_1^j < \ldots < z_{n_j}^j$ (each ζ^j can be thus considered either as a vector in Σ_{n_j} or as a subset of $\mathbb{R}$ of cardinality $|\zeta^j| = n_j$), and where the subsets ζ^j are disjoint.

There is a natural projection from $CS_{n,m}$ to the standard (uncoloured) simplex Σ_n, which simply "forgets" the colour. We shall denote by $pr(z)$ (or $pr(\zeta)$) the image of the point (16) (or (17)) under this projection.

Let $l^m = C^{comp}_{\mathbb{C}^m}(\mathbb{R})$. Choosing a basis $\{e_j\}$, $j=1,\ldots,m$, in $\mathbb{C}^m$ allows one to present any function $f \in l^m$ as the sum $f = \sum f_j e_j$ with all $f_j \in l^1$. As usual, the tensor product $(\mathbb{C}^m)^{\otimes n}$ is defined as a mn-dimensional vector space with the basis $e_{\alpha(1)} \otimes \ldots \otimes e_{\alpha(n)}$ parametrised by functions $\alpha: \{1,\ldots,n\} \mapsto \{1,\ldots,m\}$. The (algebraic) symmetric tensor product $l^m_n = (l^m)^{\otimes n}_{sym}$ can be defined as the space of functions $\Sigma_n \mapsto (\mathbb{C}^m)^{\otimes n}$ generated by the monomials

$$f_1(z_1)\ldots f_n(z_n) e_{\alpha(1)} \otimes \ldots \otimes e_{\alpha(n)}, \quad f_j \in l^1. \tag{18}$$

It is convenient to get rid of tensors by transfering the index from the basis to the variables and to encode the element (18) by the function $f_1(z_1^{\alpha(1)})...f_n(z_n^{\alpha(n)})$ of n ordered labelled variables. Thus the symmetric tensor product $l_n^m = (l^m)^{\otimes n}_{sym}$ is represented as a space of functions of variables (16), or, in other words, as a space of functions on the coloured simplex $CS_{n,m}$. We shall call this representation the functional representation for the tensor product l_n^m. Similarly, the space $H \otimes l_n^m$ can be identified with a space of H-valued functions on $CS_{n,m}$. Let

$$S = \{S(k)\} = \{(S_{\mu,\nu})(k) : \quad k \in \{1, ..., n\}, \quad \mu, \nu \in \{1, ..., m\}\}, \tag{19}$$

be a family of $m \times m$-matrices with entries from $\mathcal{L}(\mathcal{H})$. Clearly, each matrix $S(k)$ defines an operator in $H \otimes l^m$ that takes the function $f(z) = \sum f_j(z)e_j$ to the function

$$(S(k)f)(z) = \sum_{\mu,\nu} S_{\mu,\nu}(k) f_\nu(z) e_\mu.$$

The tensor product $S(n) \otimes ... \otimes S(1)$ is defined as the operator in $H \otimes l_n^m$ that takes the element (18) to

$$\sum_{\mu_1,...,\mu_n} S_{\mu_n,\alpha(n)}(n)...S_{\mu_1,\alpha(1)}(1) f_1(z_1)...f_n(z_n) e_{\mu_1} \otimes ... \otimes e_{\mu_n}. \tag{20}$$

Clearly, each monomial in the sum (20) has the form

$$S_{\mu_n,\alpha(n)}(n)...S_{\mu_1,\alpha(1)}(1) f_1(z_1^{\mu_1})...f_n(z_n^{\mu_n}),$$

in the functional representation. Since the monomials (18) form the basis for the space l_n^m, we obtain the following functional representation for the operator $S(n) \otimes ... \otimes S(1)$:

Proposition 4

$$(S(n) \otimes ... \otimes S(1) f)(z_1^{\alpha(1)}, ..., z_n^{\alpha(n)}) = \sum_\beta \prod_k S_{\alpha(k)\beta(k)}(k) f(z_1^{\beta(1)}, ..., z_n^{\beta(n)}), \tag{21}$$

where the sum is taken over all functions $\beta \colon \{1, ..., n\} \mapsto \{1, ..., m\}$, and $\prod$ is the ordered product, where the index k decreases from the left to the right.

The infinite (algebraic) direct sum $\oplus_{n=0}^\infty l_n^m$ is called the (algebraic) symmetric Fock space over l^m, the space l_n^m being the n-particle subspace. In particular, if $S(j) = S$ does not depend on j, then $S^{\otimes n}$ is the restriction on the n-particle subspace of the second quantization of the operator S.

5 General stochastic evolutions as boundary value problems.

In this section, we shall study the boundary value problem in coloured simplices, when jumps may not only change the value of a function in a point, but also a colour of this point. Using the combinatorics of multiple Fock spaces from the previous section yields the key conclusion that the interaction representation for the evolutions given by the boundary value problems for shifts in coloured simplices is given by the solutions to the pure jump stochastic equations in multiple pseudo Fock spaces. This will be the key ingredient needed for the representation of general quantum stochastic evolutions as boundary value problems for shifts in pseudo Fock spaces.

First we describe cetain topologies on algebraic Fock spaces from the previous section that are parametrised by m-tuples $p = \{p_1 \leq ... \leq p_m\}$ of positive numbers, where $p_1 \geq 1$ and p_m is allowed to be $+\infty$. To be concrete, we suppose that $p_1 \leq ... \leq p_{m-1} < p_m = \infty$ (see [5] for the case $p_m \neq \infty$). Let us define a norm on the space $H \otimes l_n^m$ considered in the functional representation (see previous section), i.e. as a space of functions $\varphi: CS_{n,m} \mapsto H$:

$$\|\varphi\|_p = \sum \left(\int_{\Sigma_{n_1}} \left(\cdots \left(\max_{\Sigma_{n_m}} |\varphi(\zeta^1, ..., \zeta^m)| \right)^{p_{m-1}} \cdots \right)^{p_1/p_2} d\zeta^1 \right)^{1/p_1}, \tag{22}$$

where $\sum$ is taken over all partition $n = n_1 + ... + n_m$ of n. We shall denote by $L_H^p(CS_{n,m})$ the completion of the space $H \otimes l_n^m$ with respect to the norm (22). By $L_H^{p,loc}(CS_{n,m})$ we shall denote the corresponding locally convex space defined by the countable set of norms parametrised by the positive integers N and defined by (22) with all integrations performed not over the whole infinite simplices but over their intersections with the balls of radius N

We shall use the same notation $T_n(t) = T_{n,p,H}(t)$ as before for the shift in $L_H^p(CS_{n,m})$ or $L_H^{p,loc}(CS_{n,m})$ that shifts all variables independently of their colours. The spaces $L_H^p(CS_{n,m})$ can be considered as n-particle subspaces in the (multiple) Fock space

$$\mathcal{F}_H^p = H \oplus L_H^p(CS_{1,m}) \oplus L_H^p(CS_{2,m}) \oplus ..., \tag{23}$$

which is a Banach version of the algebraic Fock space $H \otimes (\sum_{n=0}^\infty l_n^m)$ considered in the previous section. In particular, the shifts $T^{\sigma_1,...,\sigma_n}(t)$ can be considered as the restrictions (to the n-particle subspaces) of the corresponding shifts in the Fock space $\mathcal{F}_H^p$.

Consider now the family of operators (19) under additional assumption that all $S(k)$ are block upper triangular (i.e. $S_{\mu,\nu}$ is allowed not to vanish only if either (i) $\mu \leq \nu$ or (ii) $\mu > \nu$ but $p_\mu = p_\nu$). Then these operators define a family σ of linear operators σ_k in the space $H \otimes l_n^m$ that act by the formula

$$(\sigma_k \varphi)(z_1^{\alpha(1)}, ..., z_n^{\alpha(n)})$$
$$= \sum_{\nu=\alpha(k)}^{m} S_{\alpha(k),\nu}(k) \varphi(z_1^{\alpha(1)}, ..., z_{k-1}^{\alpha(k-1)}, z_k^\nu, z_{k+1}^{\alpha(k+1)}, ..., z_n^{\alpha(n)}). \tag{24}$$

These operators may not be continuous in the spaces $L_H^p(CS_{n,m})$. However, since the matrices $S(k)$ are triangular and since for any $p_1 \leq p_2$, the standard L^{p_1} norm of any function on a compact set can be estimated by its L^{p_2} norm, the following statement holds.

Proposition 5 *The operators σ_k are continuous in $L_H^{p,loc}(CS_{n,m})$ for all p.*

Generalising the boundary value problem (8), (9), we are going to consider the equation (8) in $L_H^p(CS_{n,m})$ combined with the boundary conditions

$$\varphi(z_1^{\alpha(1)}, ..., z_{k-1}^{\alpha(k-1)}, 0_-^{\alpha(k)}, z_{k+1}^{\alpha(k+1)}, ..., z_n^{\alpha(n)})$$

$$= (\sigma_k\varphi)(z_1^{\alpha(1)}, ..., z_{k-1}^{\alpha(k-1)}, 0_+^{\alpha(k)}, z_{k+1}^{\alpha(k+1)}, ..., z_n^{\alpha(n)}), \quad k = 1, ..., n. \tag{25}$$

To deal with this problem in the same way as with the problem (8), (9), let us decompose the coloured simplex $CS_{n,m}$ into the union of $n+1$ cells $CS_{n,m}^k$ using the decomposition (1) of the underlying uncoloured simplex Σ_n:

$$CS_{n,m}^k = \{z \in CS_{n,m} : pr(z) \in \Sigma_n^k\}, \quad k = 0, ..., n,$$

and then define the subspaces $D_S = D_S(p, H)$ (respectively D_S^{loc}) of functions $\varphi(z)$ from $L_H^p(CS_{n,m})$ (respectively from $L_H^{p,loc}(CS_{n,m})$) with the properties:

(i) for each $k = 0, ..., n$ and each particion $n = n_1 + ... + n_m$ the restriction of φ on

$$CS_{n,m}^k \cap \Sigma_{n_1} \times ... \times \Sigma_{n_m}$$

has a continuous version such that it is absolutely continuous on all lines parallel to the vector $(1, 1, ..., 1)$ and such that $(\partial_z\varphi)(z) = ((\partial_{z_1} + ... + \partial_{z_n})\varphi)(z)$ belongs to $L_H^p(CS_{n,m})$ (respectively $L_H^{p,loc}(CS_{n,m})$),

(iii) the boundary conditions (25) are satisfied.

Let us use the same notation ∂_z^S for the closures of the operator ∂_z defined on the domains D_S or D_S^{loc}. We introduced the notations for coloured simplices in such a way that the main formulas of the previous section still make sense in this new framework. It remains only to assume that the use of the operator-valued functions of the variables z without a colour means that the colour is preserved. For example, the action of the operator $\exp\{-iEz_j\}$, say, is given by the formula

$$(\exp\{-iEz_j\}\varphi)(\{z_1^{\alpha(1)} < ... < z_n^{\alpha(n)}\}) = \exp\{-iEz_j^{\alpha(j)}\}\varphi(\{z_1^{\alpha(1)} < ... < z_n^{\alpha(n)}\}).$$

At last, we can define the operator $i\partial_z^S + E$ quite similarly to the case without colours. Moreover, due to Proposition 5, we get the following

Proposition 6 *Propositions 2 and 3 remain valid for spaces $L_H^{p,loc}(CS_{n,m})$ for all p under an additional assumption that all elements $S_{mm}(j)$ are identical operators in $\mathcal{L}(H)$.*

Remark. The last assumption was necessary, because jumps destroy the continuity of a function (and one can not use L^∞ spaces, because shifts are not continuous there). The assumption $S_{mm}(j) = 1$ ensures that there will be no discontinuity in the variables ζ^m.

Comparing formula (21) with the formula for the boundary value problem for shifts described in Proposition 6 (see, in particular, (10) with σ_k defined in (24)) and by straightforward generalisation of Proposition 1 one obtains the following result, which connects shifts in coloured simplices with pure jump stochastic equations and with secondly quantised operators in multiple (Banach) Fock spaces.

Theorem 2 *Let us introduce a time dependent version of the operator (21) which acts only "till the time t", i.e. the operator*

$$(S_t^{\otimes}\varphi)(z_1^{\alpha(1)}, ..., z_n^{\alpha(n)}) = (S(k(t)) \otimes ... \otimes S(1)\varphi)(z_1^{\alpha(1)}, ..., z_n^{\alpha(n)}), \tag{26}$$

where $k(t)$ is the largest k such that $pr(z)_k \leq t$. Then

$$((T_n(t))^{-1}T^{\sigma_1,\dots,\sigma_n}(t)\varphi)(z_1^{\alpha(1)},\dots,z_n^{\alpha(n)}) = (S_t^{\otimes}\varphi)(z_1^{\alpha(1)},\dots,z_n^{\alpha(n)}). \tag{27}$$

Moreover, the r.h.s. of (27) gives the solution to the Cauchy problem for a "coloured" version of the multiple-kick equation (4) with $E=0$, i.e. to the equation

$$d\varphi + iE\varphi\, dt = \sum_{j=1}^{n}(\sigma_j - 1)\varphi\, d\chi_{(pr(z)_j,\infty)}, \quad \varphi \in L_B^{p,loc}(CS_{n,m}). \tag{28}$$

with vanishing E.

As in the previous section, equation (28) can be written as a stochastic equation, if the times $pr(\zeta)$ of jumps are random variables. In fact, in terms of the counting process $n_t = |pr(\zeta)\cap[0,t)|$ equation (28) takes the form

$$d\varphi + iE\varphi\, dt = (\sigma_{n_t} - 1)\varphi\, dn_t, \quad \varphi \in L_B^{p,loc}(CS_{n,m}). \tag{29}$$

In particular, since the number of jumps of a Poisson process is almost surely finite on each finite interval of time, one can consider the process n_t in (29) to be a standard Poisson process.

Theorem 2 expresses the solutions to pure jump stochastic equations in multiple Fock spaces in terms of the boundary value problems for shifts. As was proven in [4], the general stochastic and even quantum stochastic linear equations can be obtained as the epimorphic projection of such pure jump stochastic equations. Let us recall now how this projection is constructed. To this end, one uses the Fock space (23) with $p_1 = 1$, $p_2 = \dots = p_{m-1} = 2$, $p_m = \infty$ constructed over the one-particle Banach space

$$L_H^p(CS_{1,m}) = L_H^1(\mathbb{R}) \oplus L_H^2(\mathbb{R}) \oplus \dots \oplus L^2(\mathbb{R}) \oplus C_H(\mathbb{R}). \tag{30}$$

To work in this space, it is convenient to index the variables ζ as $\zeta^-, \zeta^0, \zeta^+$ which are connected with our general notations ζ^j by the formulas: $\zeta^- = \zeta^1$, $\zeta^0 = (\zeta^{0,1},\dots,\zeta^{0,m-1})$ with $\zeta^{0,j} = \zeta^{j+1}$, and $\zeta^+ = \zeta^m$. The formula

$$((f^-,f^0,f^+)|(g^-,g^0,g^+)) = \int ((f^-,g^+)_H(z) + \sum_{j=1}^{m-1}(f_j^0,g_j^0)_H(z) + (f^+,g^-)_H(z))\, dz,$$

defines a pseudo scalar product in space (30), which is then naturally lifted to the whole Fock space $\mathcal{F}_H^p$. Then one defines the linear (pseudo) isometry operator $J : H\otimes\mathcal{F}^2 \mapsto H\otimes\mathcal{F}^{1,2,\infty}$ and its (psedo) adjoint by the formulas

$$(J(\psi))(\zeta^-,\zeta^0,\zeta^+) = \delta_\emptyset(\zeta^-)\psi(\zeta^0)1(\zeta^+), \quad J^\star\psi(\zeta) = \int \psi(\zeta^-,\zeta,\emptyset)\, d\zeta^-, \tag{31}$$

where $\delta_\emptyset(\zeta^-)$ is the indicator function of the vacuum (i.e. it equals one if ζ^- is empty and vanishes otherwise), and $1(\zeta^+)$ is the constant function which equals one for all ζ^+. The integral over ζ^- means the sum of the integrals over all finite dimensional simplices Σ_n.

Consider now the family of matrices (19) (which, for simplicitly, will be supposed not to depend explicitly on k) with the structure

$$S(k) = \begin{pmatrix} 1 & S_0^- & S_+^- \\ 0 & S_0^0 & S_+^0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{32}$$

It turns out (see [4]) that the operator $J^\star S_t^\otimes J$ solves the linear quantum stochastic equation

$$d\eta + S_+^- \eta\, dt = S_0^- \eta\, dA^+(t) + S_+^0 \eta\, dA^- + S_0^0 \eta\, d\Lambda, \tag{33}$$

where $A^\pm = A_j^{pm}$ with $j = 1, ..., m-1$ are the creation and annigilation quantum martingales respectively and Λ is the gauge process.

Therefore, the following result holds.

Theorem 3 *The solution operator for equation* (33) *is given by the formula $J^\star S_t^\otimes J$ with J, $J^\star$ defined in* (31) *and S_t defined by* (32), (26),(27).

Thus the solution to a general quantum stochastic equation is expressed in terms of the boundary value problem in a coloured (pseudo) Fock space. Due to a well known result from [9], any Lévy process can be represented in a Fock space and thus any stochastic equation driven by such a process can be written in the form of a quantum stochastic equation given above.

6 Stochastic dynamics as semi-classical limit

The aim of this section is to show that the evolutions defined by the boundary value problems for shifts can be obtained as a sort of semiclassical limit of the evolutions defined by a boundary value problem for rather general Schrödinger equations. This completes the description of stochastic evolutions as certain limits of boundary value problems for the standard (deterministic) quantum mechanical equations with physical (real and bounded below) Hamiltonians.

We first recall some notations related to pseudo-differential operators (ΨDO) with operator-valued symbols.

Recall first that if γ is a measurable function on $\mathbb{R}$ with values in linear operators in H, the pseudo-differential operator (ΨDO) $\gamma(-i\partial_z)$ in L_H^2 acts as

$$(\gamma(-i\partial_z)\varphi)(z) = \int_{-\infty}^{+\infty} e^{ikz}\gamma(k) f(k)\, dk \tag{34}$$

on the functions φ given by their Fourier transforms as

$$\varphi(z) = \int_{-\infty}^{+\infty} e^{ikz} f(k)\, dk, \quad f \in L_H^2. \tag{35}$$

The domain of the operator $\gamma(-i\partial)$ consists of the functions φ of form (35) with f from the domain of the operator of multiplication by $\gamma(k)$. The function $\gamma = \gamma(p)$ is called the symbol of the ΨDO $\gamma(-i\partial_z)$. Choosing a positive parameter h, we denote

$$\hat{\gamma} = \hat{\gamma}(h) = h^{-1}\gamma(-ih\partial_z).$$

Let $\varepsilon(p)$ be an even function on $\mathbb{R}$ with values in a set of commuting non-negative selfadjoint operators in H defined on the same dense domain $D \subset H$. Suppose also that $\varepsilon'(p)$ exist as selfadjoint operators on D for all $p \neq 0$, and for arbitary $\xi > 0$ and $v \in D$

$$||[\varepsilon(\xi + p) - \varepsilon(\xi) - p\varepsilon'(\xi)]v|| = O(|p|^2) \tag{36}$$

uniformly for p from an arbitrary compact interval. Next, let us fix a unitary operator σ in H. The operators $\hat{\varepsilon}$ and σ describe respectively the free continuous evolution and the jumps of a quantum system.

For an arbitrary selfadjoint operator E in H, which is defined on D and commutes with all $\varepsilon(p)$, and an arbitrary positive number ξ, we define the operators $\omega^{\mp}_{E,\xi} = \omega^{\mp}_{E,\xi}(h)$ in L^2_H by the formula

$$\omega^{\mp}_{E,\xi}(h) = \frac{1}{h} e^{\pm i(E+\xi/h)z}(\varepsilon(-hi\partial_z) - \varepsilon(\xi))e^{\mp i(E+\xi/h)z}. \tag{37}$$

The operators $\omega^{\mp}_{E,\xi}(h)$ are ΨDO with symbols $(\varepsilon(h(p \mp E) \mp \xi) - \varepsilon(\xi))/h$. Moreover, $\omega^{\mp}_{E,\xi}(h)$ are selfadjoint and generate the unitary evolutions

$$(\exp\{-it\omega^{\mp}_{E,\xi}(h)\}\varphi)(z) = \int_{-\infty}^{+\infty} e^{ikz} \exp\{-it[\varepsilon(h(p \mp E) \mp \xi) - \varepsilon(\xi)]/h\} f(k)\, dk \tag{38}$$

for φ given by (35).

Theorem 4 [5] *For any $\xi > 0$, and $T > 0$ the evolutions* (38) *converge strongly to the evolutions* $\exp\{-it\varepsilon'(\xi)(E \pm i\partial_z)\}$ *as $h \to 0$ uniformly for $t \in [0, T]$.*

Thus the Dirac type evolution with the unbounded generator ∂_z is obtained as a limit as $h \to 0$ of a rather general Schrödinger evolution with bounded below Hamiltonians. As an example of $\varepsilon(p)$ satisfying the assumptions of the theorem one can take the symbols $\sqrt{p^2 + m^2}$ or $p^2/2m$ of the standard relativistic or non-relativistic Schrödinger operators. We deduced this limit only for the case of a single-kick equation. The generalisations to a multi-dimensional case are straightforward.

Concluding remark. The paper is based on a lecture delivered on a conference in RIMS Kyoto, November 2000. An extended version of this paper will be published elsewhere (see [5]).

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