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A construction of interacting Fock spaces derived from quantum decomposition and its applications

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1. Introduction

Random walks on discrete graphs have been discussed in algebraic probability theory as they provide us a rich resource of many new ideas. Recently we introduce the idea of *quantum decomposition* in the category of discrete groups [12]. Let G be a discrete group with a length function $|\cdot|: G\rightarrow \mathbf{N}\cup\{0\}$. Suppose that an element $a\in G$ has the property

$$|ag|=|g|\pm 1, \quad \forall g\in G.$$

Then we naturally come to a 'decomposition'

$$a=a^{+}+a^{-}$$

where $a^{+}g=ag$ if $|ag|=|g|+1$ (resp. $a^{-}g=ag$ if $|ag|=|g|-1$) and $a^{+}g=0$ if $|ag|\neq|g|+1$ (resp. $a^{-}g=0$ if $|ag|\neq|g|-1$). The idea traces back to the Hudson-Parthasarathy theory [14] on quantum analysis of stochastic evolutions. Our approach of the quantum decomposition has advantage in quantum analysis of set partition statistics. We construct a sequence of one-mode interacting Fock spaces associated with a filtration of the group. Then we directly observe an intrinsic Fock space structure associated with discrete Laplacians on large Cayley graphs. We show stochastic convergence of such a sequence of interacting Fock spaces and obtain a fully quantum version of the de Moivre-Laplace theorem for the discrete Laplacians (Theorem 4.1).

In this paper we present various applications of quantum decomposition to the limit theorems for the discrete Laplacians on Cayley graphs. In Section 5.2, we see that the Haagerup states give rise to a transform on set partition statistics through their coherent expression. Hence we explain the reason for the appearance of the free Poisson law in the context of the central limit theorem under the Haagerup states and reproduce the result in [10]. We observe the asymptotic behavior of products of free elements in Section 5.3 and the anti-commutation of free elements in Section 5.4. The idea of quantum decomposition is applied to more general discrete graphs. As an instance, we illustrate a result on the Hamming graph in Section 6.

2. Quantum decomposition in a discrete group

Let G be a discrete group equipped with a length function $|\cdot|: G\rightarrow \mathbf{N}_{0}:=\mathbf{N}\cup\{0\}$ and a set of generators $\Sigma:=\{g_{\alpha}\mid\alpha\in \mathbf{Z}^{\times}\}$. Throughout this paper we adopt the convention $\{g_{-\alpha}=g_{\alpha}^{-1}\}$ and assume $g_{-\alpha}\neq g_{\alpha}$ for simplicity. We say that Σ is *compatible* with respect to the length function $|\cdot|$ if

(A1) $|g_{\alpha}|=1$ for any $g_{\alpha}\in\Sigma$,
(A2) $|g_{\alpha}\cdot g|=|g|\pm 1$ for any $g_{\alpha}\in\Sigma$ and $g\in G$.

Then we introduce a 'quantum' decomposition of g_{α}:

(2.1) $$g_{\alpha}=g_{\alpha}^{+}+g_{\alpha}^{-}.$$

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We take the left regular representation $\pi : G \to l^2(G)$. Let $\pi(g_\alpha^\pm)$ be bounded operators defined by

$$\text{(2.2)} \qquad \pi(g_\alpha^+)\delta_g = \begin{cases} \delta_{g_\alpha g}, & \text{if } |g_\alpha g| = |g|+1, \\ 0, & \text{otherwise}, \end{cases} \qquad \pi(g_\alpha^-)\delta_g = \begin{cases} \delta_{g_\alpha g}, & \text{if } |g_\alpha g| = |g|-1, \\ 0, & \text{otherwise}. \end{cases}$$

Here δ_g denotes the characteristic function of a singlet $\{g\}$. Then we see that $\pi(g_\alpha) = \pi(g_\alpha^+) + \pi(g_\alpha^-)$, $\pi(g_\alpha^+)^* = \pi(g_{-\alpha}^-)$ and $\|\pi(g_\alpha^\pm)f\|_{l^2} \le \|f\|_{l^2}$ for $f \in l^2(G)$.

A family of subsets $\Sigma^{(N)} := \{g_\alpha \mid |\alpha| = 1, \dots, N\} \subset \Sigma$ gives a filtration of G: $G^{(1)} \subset G^{(2)} \subset \cdots \subset G$, where $G^{(N)}$ is the subgroup generated by $\Sigma^{(N)}$. For each $g \in G^{(N)}$ put

$$\text{(2.3)} \qquad \begin{aligned} \omega_+^{(N)}(g) &:= \{(g_\alpha, x) \in \Sigma^{(N)} \times G^{(N)} \mid \pi(g_\alpha^+)\delta_x = \delta_g\}, \\ \omega_-^{(N)}(g) &:= \{(g_\alpha, x) \in \Sigma^{(N)} \times G^{(N)} \mid \pi(g_\alpha^-)\delta_x = \delta_g\}. \end{aligned}$$

We note that $\pi(g_\alpha^\pm)\delta_x = \delta_g$ implies $|g| = |x| \pm 1$ and $\#\omega_+^{(N)}(g) + \#\omega_-^{(N)}(g) = 2N$ by assumption (A2). With these notations we assume

(A3) for each $n \in \mathbf{N}$, there exist $\omega_n \in \mathbf{N}$ and $C_n > 0$ such that

$$\#\{g \in G^{(N)} \mid |g| = n \text{ and } \#\omega_+^{(N)}(g) \ne \omega_n\} \le C_n (2N)^{n-1},$$

(A4) for each $n \in \mathbf{N}$,

$$\sup_{N \in \mathbf{N}} \sup_{g \in G^{(N)}, |g|=n} \#\omega_+^{(N)}(g) =: W_n < \infty,$$

and $\limsup_{n\to\infty} W_n^{1/n} < \infty$.

(A5) if $|g| = n$ and $g \in G^{(N)}$, there exists an n-tuple $(g_{\alpha_1}, \dots, g_{\alpha_n}) \in (\Sigma^{(N)})^n$ such that

$$\delta_g = \pi(g_{\alpha_1}^+) \cdots \pi(g_{\alpha_n}^+)\delta_e.$$

Canonical examples are given in section 5.

3. Asymptotic interacting Fock space associated with a discrete group

For each subgroup $G^{(N)}$, we construct a one-mode interacting Fock space [2] as follows. The vacuum vector is defined by $|0\rangle := \delta_e$. A vector $v \in l^2(G^{(N)})$ is called *n-homogeneous* if it has a form of

$$\text{(3.1)} \qquad v = \sum_{g \in G^{(N)}, |g|=n} v(g)\pi(g)|0\rangle, \qquad v(g) \in \mathbf{C}.$$

We define a sequence of homogeneous vectors $|n\rangle^{(N)} \in l^2(G)$ for each n and N, by

$$|n\rangle^{(N)} := \left(\frac{1}{\sqrt{2N}}\right)^n \sum_{g \in G^{(N)},\ |g|=n} \pi(g)|0\rangle.$$

Then the operators

$$\begin{aligned} a_N^+ &:= \pi(g_{-N}^+) + \cdots + \pi(g_{-1}^+) + \pi(g_1^+) + \cdots + \pi(g_N^+) \\ a_N^- &:= \pi(g_{-N}^-) + \cdots + \pi(g_{-1}^-) + \pi(g_1^-) + \cdots + \pi(g_N^-) \end{aligned}$$

behave like a creation and an annihilation, respectively.

Lemma 3.1. *For any $n \geq 0$,*

$$\frac{a_N^+}{\sqrt{2N}}|n\rangle^{(N)} = \omega_{n+1}|n+1\rangle^{(N)} + v_{n+1}(\frac{1}{\sqrt{N}}),$$

$$\frac{a_N^-}{\sqrt{2N}}|n+1\rangle^{(N)} = |n\rangle^{(N)} + v_n(\frac{1}{N}), \quad \text{and} \quad \frac{a_N^-}{\sqrt{2N}}|0\rangle = 0,$$

where $v_n(N^m)$ is an n-homogeneous vector such that $\|v_n(N^m)\|_{l^2} = O(N^m)$.

Proof. By definition we have

$$\frac{a_N^+}{\sqrt{2N}}|n\rangle^{(N)} = \left(\frac{1}{\sqrt{2N}}\right)^{n+1} \sum_{|\alpha|=1,\dots,N} \pi(g_\alpha^+) \sum_{x\in G^{(N)},\ |x|=n} \pi(x)|0\rangle. \tag{3.2}$$

By virtue of assumption (A5), the right hand side becomes

$$\begin{aligned} &= \left(\frac{1}{\sqrt{2N}}\right)^{n+1} \sum_{g\in G^{(N)},\ |g|=n+1} \sum_{(g_\alpha,x)\in\omega_+^{(N)}(g)} \pi(g_\alpha^+)\pi(x)|0\rangle \\ &= \left(\frac{1}{\sqrt{2N}}\right)^{n+1} \sum_{g\in G^{(N)},\ |g|=n+1} \#\omega_+^{(N)}(g)\pi(g)|0\rangle, \\ &= \omega_{n+1}|n+1\rangle^{(N)} + v \end{aligned} \tag{3.3}$$

where v is an $(n+1)$-homogeneous vector

$$v = \left(\frac{1}{\sqrt{2N}}\right)^{n+1} \sum_{g\in G^{(N)},\ |g|=n+1} (\#\omega_+^{(N)}(g) - \omega_{n+1})\pi(g)|0\rangle.$$

It follows from assumptions (A3) and (A4) that $\|v\|^2 = O(1/N)$. The second relation is obtained similarly. The last one is proved by definition. □

Lemma 3.2. *For each $n \geq 0$ and $N \geq 1$ it holds that*

$$\langle n|n\rangle^{(N)} = \frac{1}{(\omega_n)!} + O(\frac{1}{N}),$$

where $(\omega_n)! = \omega_1\omega_2\cdots\omega_n$.

Proof. Note that $(2N)^n\langle n|n\rangle^{(N)} = \#\{g \in G^{(N)} \mid |g| = n\}$. Then, the relation (3.2)(3.3) and assumptions (A3)(A4) imply that

$$2N\#\{g \in G^{(N)} \mid |g| = n\} = \omega_{n+1}\#\{g \in G^{(N)} \mid |g| = n+1\} + O(N^n).$$

Hence the assertion follows. □

Definition 3.3. Let $\Gamma_n^{(N)}$ be a one-dimensional complex vector space equipped with a pre-scalar product

$$(z|w)_n^{(N)} := \bar{z}w\langle\Phi(n)|\Phi(n)\rangle^{(N)}, \qquad z, w \in \mathbf{C},$$

where $|\Phi(n)\rangle^{(N)}$ stands for a *number vector* defined by

$$|\Phi(n)\rangle^{(N)} := (\omega_n)!|n\rangle^{(N)}.$$

The completion of the orthogonal sum $\oplus_n\{\Gamma_n^{(N)}, (\cdot|\cdot)_n^{(N)}\}$ is denoted by $\Gamma(G^{(N)})$

By Lemmas 3.1 and 3.2, we have for any $n \geq 0$,

$$\frac{a_N^+}{\sqrt{2N}}|\Phi(n)\rangle^{(N)} = |\Phi(n+1)\rangle^{(N)} + v_n(\frac{1}{\sqrt{N}}),$$

$$\frac{a_N^-}{\sqrt{2N}}|\Phi(n+1)\rangle^{(N)} = \omega_{n+1}|\Phi(n)\rangle^{(N)} + v_n(\frac{1}{N}),$$

$$\frac{a_N^-}{\sqrt{2N}}|\Phi(0)\rangle^{(N)} = 0$$

and $\langle\Phi(n)|\Phi(n)\rangle^{(N)} = (\omega_n)! + O(1/N)$. With each $G^{(N)}$ we thereby associate a *one-mode interacting Fock space* $\Gamma(G^{(N)})$ with parameters $\{\lambda_0 := 1, \lambda_n := (\omega_n)!\}$. (See [2] for definitions.)

Definition 3.4. For $\lambda \in \mathbf{C}$ we define

$$|\mathcal{E}(\lambda)\rangle^{(N)} := \sum_{n=0}^{\infty} \lambda^n |n\rangle^{(N)} = \sum_{n=0}^{\infty} \frac{\lambda^n}{(\omega_n)!}|\Phi(n)\rangle^{(N)},$$

whenever $\langle\mathcal{E}(\lambda)|\mathcal{E}(\lambda)\rangle = \sum_{n=0}^{\infty} |\lambda|^{2n}/(\omega_n)!$ converges. We call $|\mathcal{E}(\lambda)\rangle^{(N)}$ a *coherent vector.*

It is shown that a coherent vector is an eigenvector of a_N^- in an asymptotic sense, in fact,

Lemma 3.5. *For* $\lambda \in \mathbf{C}$ *in a neighborhood of* 0 *we have*

$$\frac{a_N^-}{\sqrt{2N}}|\mathcal{E}(\lambda)\rangle^{(N)} = \lambda|\mathcal{E}(\lambda)\rangle^{(N)} + v(\frac{1}{N}).$$

Proof. Assumption (A4) yields the estimate:

$$\left\| \frac{a_N^-}{\sqrt{2N}}|\mathcal{E}(\lambda)\rangle^{(N)} - \lambda|\mathcal{E}(\lambda)\rangle^{(N)} \right\| = O(\frac{1}{N}).$$

□

Lemma 3.6. *Let* $v \in l^2(G^{(N)})$ *be an n-homogeneous vector. Then,* $a_N^+ v$ *and* $a_N^- v$ *are respectively* $(n+1)$*- and* $(n-1)$*-homogeneous vectors with the norm estimates:*

$$\|a_N^+ v\|_{l^2}^2 \leq 2NW_n\|v\|_{l^2}^2,$$

$$\|a_N^- v\|_{l^2}^2 \leq 2NW_n\|v\|_{l^2}^2.$$

Proof. Suppose that v is given as in (3.1). Then,

$$\begin{aligned}
\|a_N^+ v\|^2 &= \sum_{g \in G^{(N)}, |g|=n+1} \left\| \Big(\sum_{(g_\alpha, x) \in \omega_+^{(N)}(g)} v(x)\ \pi(g_\alpha^+)\pi(x) \Big) |0\rangle \right\|^2 \\
&= \sum_{g \in G^{(N)}, |g|=n+1} \Big| \sum_{(g_\alpha, x) \in \omega_+^{(N)}(g)} v(x) \Big|^2 \leq \sum_{g \in G^{(N)}, |g|=n+1} \#\omega_+^{(N)}(g) \ \Big(\sum_{(g_\alpha, x) \in \omega_+^{(N)}(g)} |v(x)|^2 \Big) \\
&\leq W_n \sum_{x \in G^{(N)}, |x|=n} \#\omega_-^{(N)}(x)|v(x)|^2 \leq 2NW_n\|v\|^2,
\end{aligned}$$

where $\#\omega_-^{(N)}(x) \leq 2N$ is taken into account. Assumption (A3) implies the inequality for a_N^-

4. Interacting Fock space in the limit

We construct an interacting Fock space for the limit of a_N^+ and a_N^- as $N \to \infty$. Let $\Gamma(G)$ denote the one-mode interacting Fock space with parameters $\{\lambda_0 := 1, \lambda_n := (\omega_n)!\}$. By definition $\Gamma(G)$ is the completion of the orthogonal sum of one-dimensional space $\Gamma_n := \mathbf{C}|\Phi(n)\rangle$ equipped with a pre-scalar product

$$(z|w)_n := \bar{z}w\langle\Phi(n)|\Phi(n)\rangle, \qquad z, w \in \mathbf{C},$$

where $|\Phi(n)\rangle$ stands for a number vector with norm $\sqrt{\lambda_n} = \sqrt{(\omega_n)!}$. The creation a^+ and annihilation a^- are uniquely determined by

$$a^+|\Phi(n)\rangle = |\Phi(n+1)\rangle, \qquad a^-|\Phi(n+1)\rangle = \omega_{n+1}|\Phi(n)\rangle, \qquad a^-|\Phi(0)\rangle = 0.$$

For $\lambda \in \mathbf{C}$, we define a coherent vector

$$|\mathcal{E}(\lambda)\rangle := \sum_{n=0}^{\infty} \frac{\lambda^n}{(\omega_n)!}|\Phi(n)\rangle \tag{4.1}$$

whenever $\langle\mathcal{E}(\lambda)|\mathcal{E}(\lambda)\rangle = \sum_{n=0}^{\infty} |\lambda|^{2n}/(\omega_n)!$ converges. By definition, $a^-|\mathcal{E}(\lambda)\rangle = \lambda|\mathcal{E}(\lambda)\rangle$ holds.

For $u = \sum u_n|\Phi(n)\rangle \in \Gamma(G)$, we put

$$u^{(N)} := \sum u_n|\Phi(n)\rangle^{(N)} \in \Gamma(G^{(N)}). \tag{4.2}$$

Theorem 4.1. *Let $m \geq 1$ and $\epsilon_1, \ldots, \epsilon_m \in \{\pm\}$. Then, for any $u \in \Gamma(G)$ and $n \in \mathbf{N}_0$ we have*

$$\lim_{N\to\infty} \langle u^{(N)}| \left(\frac{a_N^{\epsilon_1}}{\sqrt{2N}}\right) \cdots \left(\frac{a_N^{\epsilon_m}}{\sqrt{2N}}\right) |\Phi(n)\rangle^{(N)}_{\Gamma(G^{(N)})} = \langle u|a^{\epsilon_1}\cdots a^{\epsilon_m}|\Phi(n)\rangle_{\Gamma(G)}.$$

Proof. By definition, there exists a constant $\omega_n(\epsilon_1, \ldots, \epsilon_m)$ depending on $n \in \mathbf{N}$ and the indices of the product such that

$$a^{\epsilon_1}\cdots a^{\epsilon_m}|\Phi(n)\rangle = \omega_n(\epsilon_1, \ldots, \epsilon_m)|\Phi(l)\rangle,$$

where $l = n + \#\{i \mid \epsilon_i = +\} - \#\{i \mid \epsilon_i = -\}$. Then, combining Lemmas 3.1, 3.2 and 3.6, we see that

$$\left(\frac{a_N^{\epsilon_1}}{\sqrt{2N}}\right) \cdots \left(\frac{a_N^{\epsilon_m}}{\sqrt{2N}}\right) |\Phi(n)\rangle^{(N)} = \omega_n(\epsilon_1, \ldots, \epsilon_m)|\Phi(l)\rangle^{(N)} + |v\rangle^{(N)},$$

where $|v\rangle^{(N)}$ is an l-homogeneous vector whose norm is of order $1/\sqrt{N}$. Then the assertion follows. $\square$

5. Applications to limit theorems in algebraic probability theory

5.1. Central limit theorems on the vacuum state. Since Theorem 4.1 is nothing but a general form of limit theorems in algebraic probability theory, it particularly leads to the algebraic central limit theorem with respect to the vacuum state $\langle 0|\cdot|0\rangle$:

$$\lim_{N\to\infty} \langle 0|\left(\frac{a_N^+ + a_N^-}{\sqrt{2N}}\right)^m |0\rangle^{(N)}_{\Gamma(G^{(N)})} = \langle\Phi(0)|(a^+ + a^-)^m|\Phi(0)\rangle_{\Gamma(G)}. \tag{5.1}$$

Let A (resp. F) be a free abelian group (resp. a free group) generated by $\Sigma := \{g_\alpha\}$, where $g_\alpha^{-1} = g_{-\alpha}$ according to our convention. Each element $g \in A$ (resp. $g \in F$) has a canonical expression in terms of Σ:

$$g = g_{\alpha_1}^{\epsilon_1} \cdots g_{\alpha_m}^{\epsilon_m},$$

$$0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m, \ \epsilon_i \in \mathbf{Z}$$
$$(\text{ resp. } \quad |\alpha_1| \neq |\alpha_2| \neq \cdots \neq |\alpha_m|, \ \epsilon_i \in \mathbf{Z} \ ),$$

and a reduced length function is defined by

$$|g| = |\epsilon_1| + \cdots + |\epsilon_m|.$$

Conditions (A1)(A2)(A5) are obvious. As for (A3)(A4), we employ some combinatorial arguments. In fact, we see for $g \in A$ (resp. $g \in F$) with $|g| = n$,

$$\#\omega_+^{(N)}(g) \leq n, \ \#\{g \in A \mid |g| = n, \#\omega_+^{(N)}(g) < n\} \leq (2N)^{n-1},$$
$$2N - n \leq \#\omega_-^{(N)}(g) \leq 2N - 1,$$
$$(\text{ resp. } \#\omega_+^{(N)}(g) = 1, \ \#\omega_-^{(N)}(g) = 2N - 1).$$

Hence we have $\omega_n = n$ (resp. $\omega_n = 1$), which implies that the interacting Fock space in the limit is the one-mode boson (resp. free) Fock space. In fact, by (5.1) the limit distribution $d\mu$ of the field operator $(a_N^+ + a_N^-)/\sqrt{2N}$ under the vacuum state is determined by its $2m$-th moments, which is well known:

$$\langle 0|(a^+ + a^-)^{2m}|0\rangle_{\Gamma(G)} = \begin{cases} \dfrac{(2m)!}{2^m m!}, & \text{for } G = A, \\ \dfrac{(2m)!}{(m+1)!m!}, & \text{for } G = F. \end{cases}$$

In other words,

$$d\mu(x) = \begin{cases} \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2} dx, & \text{for } G = A, \\ \dfrac{1}{2\pi} \chi_{[-2,2]} \sqrt{4 - x^2} dx, & \text{for } G = F. \end{cases}$$

5.2. **Limit theorems for Haagerup states.** It is well known ([4][6]) that the Haagerup function $\varphi_\gamma(g) = \gamma^{|g|}$ $(0 \leq \gamma < 1)$ on a finitely generated free group $F^{(N)}$ is positive definite. In the previous paper [10], the limit distribution of the field operator $(a_N^+ + a_N^-)/\sqrt{2N}$ under the Haagerup state was obtained by way of Fourier transform of the Bessel function. The result was thought as an analogy of a central limit theorem for a vacuum state, though the limit distribution had peculiar properties. In fact, the limit distribution coincides with the free Poisson law [22] up to translation. It is noticeable that limit behaviors of the field operators under a vacuum state are completely described by pair-partition statistics [3], while the Poisson laws are induced from all set partition statistics [14][19][17]. The next result shows that the Haagerup states give rise to a transform of set partition statistics through a coherent expression of the Haagerup functions.

Let us first consider a general interacting Fock space Γ with parameters $\{\lambda_n = (\omega_n)!\}$. For $n \geq 0$, let $\sigma_n : \mathbf{C}[\omega_1, \omega_2, \dots] \to \mathbf{C}[\omega_1, \omega_2, \dots]$ be a shift operator on the polynomial ring in ω_i's defined by $\sigma_n(\omega_i) = \omega_{i+n}$. As $\langle \Phi(p)|a^{\epsilon_1} \cdots a^{\epsilon_m}|\Phi(q)\rangle_\Gamma \in \mathbf{C}[\omega_1, \omega_2, \dots]$, $\sigma_n\big(\langle \Phi(p)|a^{\epsilon_1} \cdots a^{\epsilon_m}|\Phi(q)\rangle_\Gamma\big)$ is well defined.

Proposition 5.1. *For an interacting Fock space $(\Gamma, \{\lambda_n\})$ and a coherent vector $|\mathcal{E}(\lambda)\rangle_\Gamma$ given by (4.1), we have a recurrence decomposition*

$$\begin{aligned}\langle\mathcal{E}(\lambda)|(a^+ + a^-)^m|\Phi(0)\rangle_\Gamma &= \langle\Phi(0)|(a^+ + a^-)^m|\Phi(0)\rangle_\Gamma \\ &+\sum_{n=1}^{\infty}\sum_{l=0}^{\lfloor(m-1)/2\rfloor}\sigma_n\Big(\langle\Phi(0)|(a^+ + a^-)^{2l}|\Phi(0)\rangle_\Gamma\Big) \\ &\cdot\frac{1}{\omega_n}\langle\Phi(n)|\lambda a^+a^-|\Phi(n)\rangle_\Gamma\cdot\frac{\lambda^{n-1}}{(\omega_{n-1})!}\langle\Phi(n-1)|(a^+ + a^-)^{m-2l-1}|\Phi(0)\rangle_\Gamma.\end{aligned}$$

Proof. By definition, the left hand side is a linear combination of $\langle\Phi(n)|(a^+ + a^-)^m|\Phi(0)\rangle$ where $n \geq 0$. The term for $n = 0$ appears in the first term of the right hand side. Suppose $n \geq 1$. If $\langle\Phi(n)|a^{\epsilon_1}\cdots a^{\epsilon_m}|\Phi(0)\rangle \neq 0$, there exists a unique l satisfying

$$a^{\epsilon_{2l+2}}\cdots a^{\epsilon_m}|\Phi(0)\rangle \in \mathbf{C}|\Phi(n-1)\rangle, \qquad a^{\epsilon_{2l+1}} = a^+, \tag{5.2}$$

$$a^{\epsilon_k}\cdots a^{\epsilon_{2l}}|\Phi(n)\rangle \in \oplus_{p=n}^{\infty}\mathbf{C}|\Phi(p)\rangle \quad \text{for any} \quad 1 \leq k \leq 2l. \tag{5.3}$$

It follows from (5.3) that

$$\langle\Phi(n)|a^{\epsilon_1}\cdots a^{\epsilon_{2l}}|\Phi(n)\rangle = \sigma_n\Big(\langle\Phi(0)|a^{\epsilon_1}\cdots a^{\epsilon_{2l}}|\Phi(0)\rangle\Big)\langle\Phi(n)|\Phi(n)\rangle.$$

Note that $\langle\Phi(n)|a^{\epsilon_{2l+1}}|\Phi(n-1)\rangle = \langle\Phi(n)|a^+a^-|\Phi(n)\rangle/\omega_n$. Then we have a decomposition

$$\begin{aligned}&(\omega_n)!\langle\Phi(n)|a^{\epsilon_1}\cdots a^{\epsilon_m}|\Phi(0)\rangle \\ &\quad= \sigma_n\Big(\langle\Phi(0)|a^{\epsilon_1}\cdots a^{\epsilon_{2l}}|\Phi(0)\rangle\Big)\cdot\langle\Phi(n)|a^+a^-|\Phi(n)\rangle\cdot\langle\Phi(n-1)|a^{\epsilon_{2l+2}}\cdots a^{\epsilon_m}|\Phi(0)\rangle,\end{aligned}$$

which leads to

$$\begin{aligned}(\omega_n)!\langle\Phi(n)|(a^+ + a^-)^m|\Phi(0)\rangle &= \sum_{l=0}^{\lfloor(m-1)/2\rfloor}\sigma_n\Big(\langle\Phi(0)|(a^+ + a^-)^{2l}|\Phi(0)\rangle\Big) \\ &\qquad\cdot\langle\Phi(n)|a^+a^-|\Phi(n)\rangle\cdot\langle\Phi(n-1)|(a^+ + a^-)^{m-2l-1}|\Phi(0)\rangle.\end{aligned}$$

Thus the assertion follows. □

Let us return to the special case of $\Gamma(F)$.

Lemma 5.2. *For any $\lambda \in \mathbf{C}$ with $|\lambda| < 1$,*

$$\varphi_{\frac{\lambda}{\sqrt{2N}}}(g) = \langle\mathcal{E}(\lambda)|\pi(g)|0\rangle^{(N)}_{\Gamma(F^{(N)})}$$

holds for any $g \in F^{(N)}$.

Proof. If $|g| = n$, by definition we have

$$\varphi_{\frac{\lambda}{\sqrt{2N}}}(g) = \left(\frac{\lambda}{\sqrt{2N}}\right)^n = \sum_{k=0}^{\infty}\lambda^k\langle k|\pi(g)|0\rangle^{(N)} = \langle\mathcal{E}(\lambda)|\pi(g)|0\rangle^{(N)}.$$

The assumption $|\lambda| < 1$ guarantees that $|\mathcal{E}(\lambda)\rangle \in \Gamma(G^{(N)})$ is defined. □

It follows from Theorem 4.1 that the moments of the field operator in the limit are given by

$$\lim_{N\to\infty}\varphi_{\frac{\lambda}{\sqrt{2N}}}\left(\left(\frac{a_N^+ + a_N^-}{\sqrt{2N}}\right)^m\right) = \langle\mathcal{E}(\lambda)|(a^+ + a^-)^m|\Phi(0)\rangle_{\Gamma(F)}.$$

Since the parameters λ_n's are identically equal to 1, we have $\sigma_n\Big(\langle\Phi(0)|(a^+ + a^-)^{2l}|\Phi(0)\rangle\Big) = \langle\Phi(0)|(a^+ + a^-)^{2l}|\Phi(0)\rangle$. Denoting by P_0 the vacuum projection, we rewrite the factor

$$\langle\Phi(n)|a^+a^-|\Phi(n)\rangle = 1 = \langle\Phi(0)|P_0|\Phi(0)\rangle.$$

Then it follows from Proposition 5.1 that

$$\begin{aligned}(5.4)\quad \langle\mathcal{E}(\lambda)|(a^+ + a^-)^m|\Phi(0)\rangle &= \langle\Phi(0)|(a^+ + a^-)^m|\Phi(0)\rangle \\ &+ \sum_{l=0}^{\lfloor(m-1)/2\rfloor} \langle\Phi(0)|(a^+ + a^-)^{2l}|\Phi(0)\rangle \\ &\qquad \cdot \langle\Phi(0)|\lambda P_0|\Phi(0)\rangle \cdot \langle\mathcal{E}(\lambda)|(a^+ + a^-)^{m-2l-1}|\Phi(0)\rangle,\end{aligned}$$

while an expansion of $(a^+ + a^- + \lambda P_0)^m$ gives

$$\begin{aligned}\langle\Phi(0)|(a^+ + a^- + \lambda P_0)^m|\Phi(0)\rangle &= \langle\Phi(0)|(a^+ + a^-)^m|\Phi(0)\rangle \\ &+ \sum_{l=0}^{\lfloor(m-1)/2\rfloor} \langle\Phi(0)|(a^+ + a^-)^{2l}|\Phi(0)\rangle \\ &\qquad \cdot \langle\Phi(0)|\lambda P_0|\Phi(0)\rangle \cdot \langle\Phi(0)|(a^+ + a^- + \lambda P_0)^{m-2l-1}|\Phi(0)\rangle.\end{aligned}$$

By induction we then obtain the following

Corollary 5.3. *In the free Fock space $\{\Gamma(F), \{\lambda_n \equiv 1\}\}$, the identity*

$$\langle\mathcal{E}(\lambda)|(a^+ + a^-)^m|\Phi(0)\rangle_{\Gamma(F)} = \langle\Phi(0)|(a^+ + a^- + \lambda P_0)^m|\Phi(0)\rangle_{\Gamma(F)}$$

holds, where P_0 is the vacuum projection.

Remarks.

(1) Since in the free Fock space, the vacuum projection $P_0 = I - a^+a^-$, the random variable $a^+ + a^- + \lambda P_0$ is nothing but the free Poisson one [19], that is,

$$a^+ + a^- + \lambda P_0 = \lambda + \frac{1}{\lambda} - \left(\sqrt{\lambda}a^+ - \frac{1}{\sqrt{\lambda}}\right)\left(\sqrt{\lambda}a^- - \frac{1}{\sqrt{\lambda}}\right).$$

(2) Corollary 5.3 describes a *Gaussian-Poisson transform* in terms of coherent vectors, while Oravecz [16] recently introduced a Gaussian-Poisson transform called the *$\mathcal{F}$-transform* for an arbitrary symmetric measure from the viewpoint of orthogonal polynomials.

(3) With the recurrence formula (5.4) one obtains a functional identity for the moment generating functions. Put $F(t) = 1 + \sum_{n=1}^{\infty} F_m t^m$ with $F_m = \langle\mathcal{E}(\lambda)|(a^+ + a^-)^m|\Phi(0)\rangle$ and $C(t) = (1 - \sqrt{1-4t^2})/2t^2$, which is the moment generating function of the normalized semi-circle law. Then we have the identity $F(t) = C(t) + \lambda t C(t) F(t)$, from which the limit distribution is computed easily:

$$d\mu(x) = \max\left\{0, \frac{1}{2}\left(1 - \frac{1}{\lambda^2}\right)\right\}\delta_{\lambda+\frac{1}{\lambda}}(x) + \frac{1}{2\pi}\chi_{[-2,2]}\frac{\sqrt{4-x^2}}{\lambda^2+1-\lambda x}dx.$$

This coincides with the result in [10] up to translation in the case of $\lambda \in [0,1]$.

5.3. **Multiplication of free elements.** Corollary 5.3 implies that under the Haagerup state, $S_N/\sqrt{2N} = \pi(g_1) + \pi(g_1^{-1}) + \cdots + \pi(g_N) + \pi(g_N^{-1})$ is decomposed into a creation a^+, an annihilation a^- and a^+a^- in the limit. In this subsection we present a central limit theorem associated with a multiplication of free elements, and again we obtain the same decomposition as Corollary 5.3.

Let F be a free product of $\mathbf{Z}/2\mathbf{Z}$, $F = *_{i=1}^{\infty}\mathbf{Z}/2\mathbf{Z}\sigma_i$, where σ_i's are generator of order 2. Note that σ_i's are free from each other with respect to the vacuum state $\langle 0| \cdot |0\rangle$ in the sense of Voiculescu [22]. Let us consider a set of products of free elements $\Sigma_2 := \{w_{ij} := \sigma_i\sigma_j\ (i \neq j)\}$, which are not free from each other, and a subgroup $F_2 \subset F$ generated by Σ_2. We use a new length function $|\cdot|_2 : F_2 \longrightarrow \mathbf{N}_0$ with respect to Σ_2 different from the one introduced in Section 5.1. Since any product $w_{i_1j_1}\cdots w_{i_mj_m}$ has a reduced expression in terms of σ_i's,

$$w_{i_1j_1}\cdots w_{i_mj_m} = \sigma_{k_1}\cdots\sigma_{k_l},$$

where $k_1 \neq k_2 \neq \cdots \neq k_l$, its length is defined by

$$|w_{i_1j_1}\cdots w_{i_mj_m}|_2 := l/2. \tag{5.5}$$

Then we see for any $g = \sigma_{k_1}\cdots\sigma_{k_m} \in F_2$ and w_{ij}, the following three cases occur:

$$|w_{ij}g|_2 = \begin{cases} |g|_2 + 1, & \text{if } j \neq k_1, \\ |g|_2 - 1, & \text{if } (i,j) = (k_2, k_1), \\ |g|_2, & \text{if } j = k_1 \text{ and } i \neq k_2. \end{cases} \tag{5.6}$$

It follows from $|w_{ij}|_2 = 1$ and (5.6) that $|g|_2 \in \mathbf{N}_0$ for any $g \in F_2$.

The existence of the third case is a noticeable difference from our discussions before. Concerning the new case, we extend our argument. The condition (A2) in Section 2 is replaced to

(A2)' $\quad |g_\alpha \cdot g| = |g|$ or $|g| \pm 1$ for any $g_\alpha \in \Sigma$ and $g \in G$.

In accordance with (A2)' we have a 'quantum' decomposition of g_α:

$$g_\alpha = g_\alpha^+ + g_\alpha^- + g_\alpha^\circ, \tag{5.7}$$

where $g_\alpha^\pm$ are defined by (2.1) and g_α° is given by

$$\pi(g_\alpha^\circ)\delta_g = \begin{cases} \delta_{g_\alpha g}, & \text{if } |g_\alpha g| = |g|, \\ 0, & \text{otherwise.} \end{cases}$$

Let us return to the example (F_2, Σ_2). We put $\Sigma_2^{(N)} := \{w_{ij} \mid 1 \leq i \neq j \leq N\}$. $F_2^{(N)}$ denotes the subgroup generated by $\Sigma_2^{(N)}$. For $g \in F_2^{(N)}$ we define $\omega_\pm^{(N)}(g)$ by (2.3) where we take $G^{(N)}$ for $F_2^{(N)}$, and $\omega_\circ^{(N)}(g)$ by

$$\omega_\circ^{(N)}(g) := \{(w_{ij}, x) \in \Sigma_2^{(N)} \times F_2^{(N)} \mid \pi(g_\alpha^\circ)\delta_x = \delta_g\}. \tag{5.8}$$

By definition, we have $\omega_+^{(N)}(e) = \omega_\circ^{(N)}(e) = \emptyset$ and $\omega_-^{(N)}(e) = \Sigma_2^{(N)}$ for the unit $e \in F_2$. It is easy to see that for $g \in F_2^{(N)}$ with $|g|_2 = n \geq 1$ and for large N,

$$\#\omega_+^{(N)}(g) = 1,\ \#\omega_-^{(N)}(g) = (N-1)^2 \text{ and } \#\omega_\circ^{(N)}(g) = N - 2.$$

Also note that $\#\omega_+^{(N)}(g) + \#\omega_-^{(N)}(g) + \#\omega_\circ^{(N)}(g) = \#\Sigma_2^{(N)} = N(N-1)$. According to Section 3, we define a number vector

$$|\Phi(n)\rangle^{(N)} := \left(\frac{1}{\sqrt{\#\Sigma_2^{(N)}}}\right)^n \sum_{g\in F^{(N)}(2), |g|_2=n} \pi(g)|0\rangle,$$

and define canonical operators corresponding to the decomposition (5.7),

$$a_N^\epsilon := \frac{1}{\sqrt{\#\Sigma_2^{(N)}}} \sum_{1\le i\ne j\le N} \pi(w_{ij}^\epsilon),$$

where $\epsilon = \pm, \circ$. Then by similar arguments in Section 3, we have

$$\begin{aligned}
a_N^+|\Phi(n)\rangle^{(N)} &= |\Phi(n+1)\rangle^{(N)} + v_{n+1}(\frac{1}{N}),\\
a_N^-|\Phi(n+1)\rangle^{(N)} &= |\Phi(n)\rangle^{(N)} + v_n(\frac{1}{N^2}),\\
a_N^\circ|\Phi(n)\rangle^{(N)} &= |\Phi(n)\rangle^{(N)} + v_n(\frac{1}{N}), \qquad \text{for } n\ge 1,\\
a_N^-|0\rangle^{(N)} &= a_N^\circ|0\rangle^{(N)} = 0,
\end{aligned} \tag{5.9}$$

where v_n is a homogeneous vector defined in Section 3. We also construct an asymptotic one-mode Fock space $\Gamma(F_2^{(N)})$ by way of similar manner in Section 3.

Let $\Gamma(F)$ be the one-mode free Fock space, a^+, a^- and $a^\circ := P_0$ be the creation, the annihilation and the vacuum projection respectively, used in the previous subsection. It follows from (5.9) that we have a 'quantum' central limit theorem associated with $(F_2^{(N)}, \Sigma_2^{(N)})$ as follows.

Theorem 5.4. *Let $m\ge 1$ and $\epsilon_1,\ldots,\epsilon_m \in \{\pm,\circ\}$. Then for any $u\in\Gamma(F)$ and $n\in\mathbf{N}_0$, we have*

$$\lim_{N\to\infty} \langle u^{(N)}| \left(\frac{a_N^{\epsilon_1}}{\sqrt{\#\Sigma_2^{(N)}}}\right)\cdots\left(\frac{a_N^{\epsilon_m}}{\sqrt{\#\Sigma_2^{(N)}}}\right)|\Phi(n)\rangle^{(N)}_{\Gamma(F_2^{(N)})} = \langle u|a^{\epsilon_1}\cdots a^{\epsilon_m}|\Phi(n)\rangle_{\Gamma(F)},$$

where we use the notation (4.2).

Particularly we obtain a classical central limit theorem.

Corollary 5.5.

$$\lim_{N\to\infty}\langle 0|\Big(\frac{1}{\sqrt{N(N-1)}}\sum_{1\le i\ne j\le N}\pi(w_{ij})\Big)^m|0\rangle^{(N)}_{\Gamma(F_2^{(N)})} = \langle 0|(a^+ + a^- + a^\circ)^m|0\rangle_{\Gamma(F)}.$$

5.4. **Anti-commutation of semi-circular elements.** Throughout this paragraph, we use the notations in Section 5.3. By taking subsets of $\Sigma_2^{(N)}$, we obtain various kinds of 'quantum' decomposition. See [11] for detail, where combinatorial moments are calculated by way of path counting on graphs. Here we observe one of the decompositions, which leads us an analysis of anti-commutations of semi-circular elements.

Let us take a subset $\Sigma_{2,\gamma}^{(N)} \subset \Sigma_2^{(N)}$ for $0<\gamma<1$,

$$\Sigma_{2,\gamma}^{(N)} := \{w_{ij}\in\Sigma_2^{(N)} \mid 1\le i\le \max\{1,\gamma N\} < j\le N \text{ or } 1\le j\le\max\{1,\gamma N\}<i\le N\}.$$

$F_{2,\gamma}^{(N)}$ denote the subgroup of $F_2^{(N)}$ generated by $\Sigma_{2,\gamma}^{(N)}$. Since (5.7) holds, we again obtain a decomposition

$$w_{ij} = w_{ij}^{+} + w_{ij}^{-} + w_{ij}^{\circ}. \tag{5.10}$$

According to the decomposition, we define operators

$$a_N^{\epsilon} := \frac{1}{\sqrt{v}} \sum_{w_{ij} \in F_{2,\gamma}^{(N)}} \pi(w_{ij}^{\epsilon}), \tag{5.11}$$

where $\epsilon = \pm, \circ$ and $v := \#\Sigma_{2,\gamma}^{(N)} \sim 2\gamma(1-\gamma)N^2$. The purpose of this paragraph is to establish a limit theorem on the operators (5.11) for a constant γ. General cases are discussed in [11].

It is easy to see that for any $g \in F_{2,\gamma}^{(N)}$, we have

$$2(\gamma N - 1)((1-\gamma)N - 1) \le \#\omega_{-}^{(N)}(g) \le 2\gamma(1-\gamma)N^2. \tag{5.12}$$

However, to determine $\#\omega_{+}^{(N)}(g)$ and $\#\omega_{\circ}^{(N)}(g)$, we need a delicate argument as follows. Let $I := [1, \gamma N] \cap \mathbf{N}$ and $J := (\gamma N, N] \cap \mathbf{N}$. For each $n > 0$, let us decompose the set $\mathcal{V}_n$ of elements $g \in F_{2,\gamma}^{(N)}$ with $|g|_2 = n$ into 2^{2n} disjoint subsets,

$$\mathcal{V}_n := \{g \in F_{2,\gamma}^{(N)} \mid |g|_2 = n\} = \coprod_{\eta_1, \dots, \eta_{2n} \in \{I, J\}} \Delta(\eta_1, \dots, \eta_{2n})$$

where $\Delta(\eta_1, \dots, \eta_{2n})$'s are given by

$$\Delta(\eta_1, \dots, \eta_{2n}) := \{\sigma_{i_1} \cdots \sigma_{i_{2n}} \in \mathcal{V}_n \mid i_1 \in \eta_1, \dots, i_{2n} \in \eta_{2n}\}.$$

With the notations above, we see

$$\begin{aligned}
&\#\omega_{+}^{(N)}(g) = 1, \text{ for } g \in \Delta(I, J, \eta_3, \dots, \eta_n) \cup \Delta(J, I, \eta_3, \dots, \eta_{2n}), \\
&\#\omega_{+}^{(N)}(g) = 0, \text{ for } g \in \Delta(I, I, \eta_3, \dots, \eta_n) \cup \Delta(J, J, \eta_3, \dots, \eta_{2n}), \\
&(1-\gamma)N - 1 \le \#\omega_{\circ}^{(N)}(g) \le (1-\gamma)N, \text{ for } g \in \Delta(I, \eta_2, \dots, \eta_{2n}), \\
&\gamma N - 1 \le \#\omega_{\circ}^{(N)}(g) \le \gamma N, \text{ for } g \in \Delta(J, \eta_2, \dots, \eta_{2n}).
\end{aligned} \tag{5.13}$$

According to the decomposition of $\mathcal{V}_n$, the number vector

$$|\Phi(n)\rangle^{(N)} := \left(\frac{1}{\sqrt{v}}\right)^n \sum_{g \in \mathcal{V}_n} \pi(g)|0\rangle$$

has an orthogonal decomposition into 2^{2n} homogeneous vectors

$$|\Phi(n)\rangle^{(N)} = \sum_{\eta_1, \dots, \eta_{2n} \in \{I, J\}} |\eta_1, \dots, \eta_{2n}\rangle^{(N)},$$

where $|\eta_1,\ldots,\eta_{2n}\rangle^{(N)} := \sum_{g\in\Delta(\eta_1,\ldots,\eta_{2n})}\pi(g)|0\rangle/\sqrt{v}$. Then we define a Fock space $\Gamma(F_{2,\gamma}^{(N)}) := \oplus_{\eta_1,\ldots,\eta_{2n}\in\{I,J\}}|\eta_1,\ldots,\eta_{2n}\rangle^{(N)}$. It follows from (5.12) and (5.13), we have

$$
\begin{aligned}
&a_N^+(|\eta_1,\ldots,\eta_{2n}\rangle^{(N)}) = |I,J,\eta_1,\ldots,\eta_{2n}\rangle^{(N)} + |J,I,\eta_1,\ldots,\eta_{2n}\rangle^{(N)} + v_n\left(\frac{1}{\sqrt{v}}\right),\\
&a_N^-|I,J,\eta_3,\ldots,\eta_{2n}\rangle^{(N)} = a_N^-|J,I,\eta_3,\ldots,\eta_{2n}\rangle^{(N)} = \frac{1}{2}|\eta_3,\ldots,\eta_{2n}\rangle^{(N)} + v_n\left(\frac{1}{v}\right),\\
&a_N^-|I,I,\eta_3,\ldots,\eta_{2n}\rangle^{(N)} = a_N^-|J,J,\eta_3,\ldots,\eta_{2n}\rangle^{(N)} = 0,\\
&a_N^\circ|I,J,\eta_3,\ldots,\eta_{2n}\rangle^{(N)} = \sqrt{\frac{\gamma}{2(1-\gamma)}}|J,J,\eta_3,\ldots,\eta_{2n}\rangle^{(N)} + v_n\left(\frac{1}{\sqrt{v}}\right),\\
&a_N^\circ|I,I,\eta_3,\ldots,\eta_{2n}\rangle^{(N)} = \sqrt{\frac{\gamma}{2(1-\gamma)}}|J,I,\eta_3,\ldots,\eta_{2n}\rangle^{(N)} + v_n\left(\frac{1}{\sqrt{v}}\right),\\
&a_N^\circ|J,I,\eta_3,\ldots,\eta_{2n}\rangle^{(N)} = \sqrt{\frac{1-\gamma}{2\gamma}}|I,I,\eta_3,\ldots,\eta_{2n}\rangle^{(N)} + v_n\left(\frac{1}{\sqrt{v}}\right),\\
&a_N^\circ|J,J,\eta_3,\ldots,\eta_{2n}\rangle^{(N)} = \sqrt{\frac{1-\gamma}{2\gamma}}|I,J,\eta_3,\ldots,\eta_{2n}\rangle^{(N)} + v_n\left(\frac{1}{\sqrt{v}}\right).
\end{aligned}
\tag{5.14}
$$

In view of the relations (5.14), we prepare a two-mode free Fock space, $\Gamma^2 := \oplus_{n=0}^\infty\Gamma^2(n)$, where

$$\Gamma^2(n) := \mathbf{C}\text{-linear span}\{|\eta_1,\ldots,\eta_{2n}\rangle \mid \eta_i\in\{I,J\}\},$$

equipped with a canonical inner product $\langle\eta_1,\ldots,\eta_{2n}|\eta_1',\ldots,\eta_{2n}'\rangle = \delta_{\eta_1,\eta_1'}\cdots\delta_{\eta_{2n},\eta_{2n}'}$, and operators a^+, a^- and a° specified by

$$
\begin{aligned}
&a^+(|\eta_1,\ldots,\eta_{2n}\rangle) = |I,J,\eta_1,\ldots,\eta_{2n}\rangle + |J,I,\eta_1,\ldots,\eta_{2n}\rangle,\\
&a^-|I,J,\eta_3,\ldots,\eta_{2n}\rangle = a^-|J,I,\eta_3,\ldots,\eta_{2n}\rangle = \frac{1}{2}|\eta_3,\ldots,\eta_{2n}\rangle,\\
&a^-|I,I,\eta_3,\ldots,\eta_{2n}\rangle = a^-|J,J,\eta_3,\ldots,\eta_{2n}\rangle = 0,\\
&a^\circ|I,J,\eta_3,\ldots,\eta_{2n}\rangle = \sqrt{\frac{\gamma}{2(1-\gamma)}}|J,J,\eta_3,\ldots,\eta_{2n}\rangle,\\
&a^\circ|I,I,\eta_3,\ldots,\eta_{2n}\rangle = \sqrt{\frac{\gamma}{2(1-\gamma)}}|J,I,\eta_3,\ldots,\eta_{2n}\rangle,\\
&a^\circ|J,I,\eta_3,\ldots,\eta_{2n}\rangle = \sqrt{\frac{1-\gamma}{2\gamma}}|I,I,\eta_3,\ldots,\eta_{2n}\rangle,\\
&a^\circ|J,J,\eta_3,\ldots,\eta_{2n}\rangle = \sqrt{\frac{1-\gamma}{2\gamma}}|I,J,\eta_3,\ldots,\eta_{2n}\rangle.
\end{aligned}
\tag{5.15}
$$

Summing up the arguments above, we have established a central limit theorem associated with the decomposition (5.10) as follows.

Theorem 5.6. *Let $m \geq 1$ and $\epsilon_1,\ldots,\epsilon_m \in \{\pm,\circ\}$. Then for any constant $0<\gamma<1$, $u\in\Gamma^2$ and $\eta_1,\ldots,\eta_n\in\{I,J\}$, we have*

$$\lim_{N\to\infty}\langle u^{(N)}|\left(\frac{a_N^{\epsilon_1}}{\sqrt{v}}\right)\cdots\left(\frac{a_N^{\epsilon_m}}{\sqrt{v}}\right)|\eta_1,\ldots,\eta_n\rangle^{(N)}_{\Gamma(F_{2,\gamma}^{(N)})} = \langle u|a^{\epsilon_1}\cdots a^{\epsilon_m}|\eta_1,\ldots,\eta_n\rangle_{\Gamma^2},$$

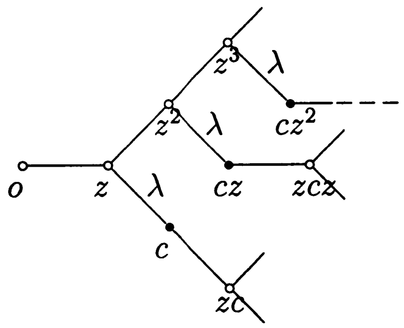
where we use the notation (4.2). The operators a^+, a^- and a° are specified by (5.15).

In particular, combinatorial moments with respect to the vacuum state are given by

$$.16) \qquad \lim_{N\to\infty} \langle 0| \Big(\frac{1}{\sqrt{v}} \sum_{w_{ij}\in\Sigma_{2,\gamma}^{(N)}} \pi(w_{ij}) \Big)^m |0\rangle^{(N)}_{\Gamma(F_{2,\gamma}^{(N)})} = \langle 0|(a^+ + a^- + a^\circ)^m|0\rangle_{\Gamma^2}.$$

Remarks.

(1) By a combinatorial argument, we see that the combinatorial moments (5.16) are given as the number of closed walks on a induced subgraph of a weighted binary tree. (The weights $\lambda = 1/\sqrt{2}$ are given in the figure below.) Indeed m-th combinatorial moment



F_m is given by the number of m-step walks which leave o and return to o. Let f_m be the number of m-step walks which leave z and return to z without arriving at o. By the self-similarity of the graph, we see for $m \geq 2$,

$$\begin{aligned} f_m &= \sum_{k=0}^{m-2} (f_k + \frac{F_k}{\sqrt{2}}) f_{m-k-2}, \\ F_m &= \sum_{k=0}^{m-2} f_k F_{m-k-2}, \end{aligned}$$

where $f_0 = F_0 = 1$. Then the moment generating function $F(t) = \sum_m F_m t^m$ and $f(t) = \sum_m f_m t^m$ satisfy

$$\begin{aligned} f(t) - 1 &= t^2 (f(t) + \frac{F(t)}{\sqrt{2}}) f(t), \\ F(t) - 1 &= t^2 F(t)^2, \end{aligned}$$

hence

$$t^2 F(t)^3 + t^2 F(t)^2 - 2F(t) + 2 = 0.$$

The Cauchy transform $G(t)$ of the distribution associated with the combinatorial moments (5.16) is given as a solution of

$$tG(t)^3 + G(t)^2 - 2tG(t) + 2 = 0.$$

(2) The limit distribution associated with (5.16) coincides with the one in *Examples 1.5* (1.16) and (1.17) of [15], up to the variance, where the anti-commutation $ab + ba$ of semi-circular elements a, b which are free from each other is observed. Indeed we see

that

$$\frac{1}{\sqrt{v}} \sum_{w_{ij}\in\Sigma_{2,\gamma}^{(N)}} \pi(w_{ij}) \approx \left(\frac{\sigma_1+\cdots+\sigma_{\gamma N}}{\sqrt{\gamma N}}\right)\left(\frac{\sigma_{\gamma N+1}+\cdots+\sigma_N}{\sqrt{(1-\gamma)N}}\right) + \left(\frac{\sigma_{\gamma N+1}+\cdots+\sigma_N}{\sqrt{(1-\gamma)N}}\right)\left(\frac{\sigma_1+\cdots+\sigma_{\gamma N}}{\sqrt{\gamma N}}\right).$$

which is nothing but the anti-commutation of semi-circular elements in the limit $N \to \infty$.

6. Commutative association scheme and its quantum decomposition

It is quite natural to apply our approach of the quantum decomposition to isotropic random walks on homogeneous graphs. In fact, it will be seen that any adjacency operator of a certain graph is decomposed into a sum of a 'creation', an 'annihilation' and a function of 'number' operators. Such a decomposition is motivated by the standard theory of quantum stochastic evolutions [14]. This idea is useful for quantum limit theorems associated with random walks on graphs, particularly, on distance regular graphs. Indeed, in [21][9] we carried out our idea on the Hamming scheme as follows. Let F be a set of $n+1$ points. A *Hamming graph* $H(d,n+1) = F^d$ is a distance regular graph with a distance $\partial(x,y)$ defined by

$$\partial(x,y) := \#\{i \mid x_i \neq y_i\}$$

for $x = (x_1,\ldots,x_d)$, $y = (y_1,\ldots,y_d) \in H(d,n+1)$. Let $A_{(d,n+1)}$ be the adjacency matrix of the Hamming graph $H(d,n+1)$. The adjacency matrix acts on the finite dimensional complex Hilbert space $\mathcal{H}$ spanned by $\{|x\rangle \mid x \in H(d,n+1)\}$ with the inner product uniquely determined by

$$\langle x|y\rangle = \delta_{xy}, \quad x,y \in H(d.n+1).$$

The action is given by

$$A|x\rangle = \sum_{x\sim y} |y\rangle,$$

where $x \sim y$ means that x is adjacent to y. We fix an arbitrary point $o \in H(d,n+1)$ as an origin. Then we have a partition

$$H(d,n+1) = \bigcup_{k=0}^{d} H_k$$

where $H_k := \{x \in H(d,n+1) \mid \partial(o,x) = k\}$. We define number vectors

$$|\Phi(k)\rangle := \frac{1}{\sqrt{\#H_k}} \sum_{x\in H_k} |x\rangle$$

for $k = 1,\ldots,d$. The orthonormal system $\{|\Phi(k)\rangle\}$ in $\mathcal{H}$ gives a one-mode Fock space

$$\Gamma_{(d,n+1)} := \bigoplus_{k=1}^{d} |\Phi(k)\rangle.$$

By the definition of the adjacency matrix, we have

$$A_{(d,n+1)}|x\rangle = \sum_{a\in H_+^{(d,n+1)}(x)} |a\rangle + \sum_{b\in H_-^{(d,n+1)}(x)} |b\rangle + \sum_{c\in H_o^{(d,n+1)}(x)} |c\rangle \tag{6.1}$$

Here we put for $\varepsilon = +, -$ and $\circ$,

$$H_\varepsilon^{(d,n+1)}(x) := \{y \sim x \mid \partial(y,o) = \partial(x,o) + \eta\}$$

with $\eta = +1, -1$ and 0 respectively. The equation (6.1) gives raise to a natural decomposition of the adjacency matrix $A_{(d,n+1)}$:

$$A_{(d,n+1)} = A^+_{(d,n+1)} + A^-_{(d,n+1)} + A^\circ_{(d,n+1)},$$

where each factor is determined by

$$A^\varepsilon_{(d,n+1)}|x\rangle = \sum_{y \in H_\varepsilon^{(d,n+1)}(x)} |y\rangle.$$

It follows from the distance regularity of $H(d, n+1)$ that as $n, d \to \infty$ with $n/d \to \tau \geq 0$ we have

$$\begin{aligned}
\frac{A^+_{(d,n+1)}}{\sqrt{nd}}|\Phi(k)\rangle &\approx \sqrt{(k+1)}|\Phi(k+1)\rangle \\
\frac{A^-_{(d,n+1)}}{\sqrt{nd}}|\Phi(k)\rangle &\approx \sqrt{k}|\Phi(k-1)\rangle \\
\frac{A^\circ_{(d,n+1)}}{\sqrt{nd}}|\Phi(k)\rangle &\approx k\sqrt{\tau}|\Phi(k)\rangle.
\end{aligned}$$

As a consequence, the asymptotic behavior of $A_{(d,n+1)}$ is described in terms of a one-mode Boson Fock space $\Gamma_b = \oplus \mathbf{C}|\Psi(k)\rangle$ and the canonical creation and annihilation operators $B^\pm$ defined by

$$B^+|\Psi(k)\rangle = \sqrt{k+1}|\Psi(k+1)\rangle$$

and

$$B^-|\Psi(k)\rangle = \sqrt{k}|\Psi(k-1)\rangle, \quad k \geq 1, \qquad B^-|\Psi(0)\rangle = 0.$$

Theorem 6.1. *Let $\tau \geq 0, m \geq 1$ and $\varepsilon_1, \ldots, \varepsilon_m \in \{+, -, \circ\}$ be given. Then for any $k, l = 0, 1, \ldots,$ we have*

$$\lim_{d,n\to\infty, n/d\to\tau} \left\langle \Phi(k) \left| \frac{A^{\varepsilon_1}_{(d,n+1)}}{\sqrt{nd}} \cdots \frac{A^{\varepsilon_m}_{(d,n+1)}}{\sqrt{nd}} \right| \Phi(l) \right\rangle_{\Gamma_{(d,n+1)}} = \langle \Psi(k) | B^{\varepsilon_1} \cdots B^{\varepsilon_m} | \Psi(l) \rangle_{\Gamma_b}.$$

In particular, we have a classical central limit theorem

$$\lim_{d,n\to\infty, n/d\to\tau} \left\langle \Phi(k) \left| \left(\frac{A_{(d,n+1)}}{\sqrt{nd}} \right)^m \right| \Phi(l) \right\rangle_{\Gamma_{(d,n+1)}} = \langle \Psi(k) | \left(\sqrt{\tau} B^+ B^- + B^+ + B^- \right)^m | \Psi(l) \rangle_{\Gamma_b}.$$

As a result, we reproduce one of Hora's results [13] on the asymptotic distribution of eigen values of the adjacency matrix associated with a Hamming graph.

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