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Random Fields: Theory and Applications to Quantum Fields

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§1. Introduction

We shall discuss the analysis of random complex systems which have close connections with quantum dynamics. In particular, we analyse stochastic processes $X(t)$ and random fields $X(C)$, in a systematic manner. Actually, our aim is to study those systems by using white noise analysis.

The idea of the analysis is that we first provide a basic and standard system of random variables and to express the given system as a function of the system provided in adavance. Naturally follows the analysis of the function. The system of variables where we start involves *idealized elemental random variables* (abbr. i.e.r.v.). To take such a system is in line with the

Reductionism.

This thought seems to be similar to the atomism in physics. We may refer to the lecture given by P.W. Anderson at University of Tokyo in 1999. The title of his lecture included Emergence together with Reductionism.

The next step is to form a function of the elemental elements obtained by the reduction; namely

Synthesis.

The goal has to be the *analysis* of the function (may be called functional) to identify the random complex system in question.

The first step of taking suitable system of i.e.r.v.'s has been influenced by the way how to understand the notion of a stochastic process. We therefore have a quick review of the definition of a stochastic process starting from the idea of J. Bernoulli and Lévy on the definition of a stochastic process, where we are suggested to consider the *innovation* of a stochastic process. It is viewed as a system of i.e.r.v.'s, which will be specified to be a *white noise.*

The analysis of white noise functionals has many significant characteristics which are fitting for investigation of quantum mechnical phenomena. Thus, we shall be able to show examples to which white noise theory is efficintly applied.

§2. Review of defining a stochastic process and white noise analysis

There is a traditional, and in fact original way of defining a stochastic process. Let us refer to Lévy's definition of a stochastic process given in his book [3] Chapt. II. "une fonction aléatoire $X(t)$ du temps t dans lequel le *hasard* intervient à chaque instant". The hasard is expressed as an infinitesimal random variable $Y(t)$ which is independent of the observed values of $X(s), s \leq t$, in the past. The random variable $Y(t)$ is nothing but the innovation of the process $X(t)$.

Formally speaking the $Y(t)$, which is usually an infiniotesimal random variable, contains the information that was gained by the $X(t)$ during the time interval $[t, t + dt)$.

It would be fine if the given process is expressd as a functional of $Y(t)$ in the following manner:

$$X(t) = \Psi(Y(s), s \leq t, t),$$

where Ψ is a sure (non random) function. Such a trick may be called the **reductionism**. The expression is *causal* in the sense that the $X(t)$ is expressed as a function of $Y(s), s \leq t$, and never uses $Y(s)$ with $s > t$.

The collection $\{Y(s)\}$ is a system of i.e.r.v.'s so that the above expression is a realization of the synthesis. We are particularly interested in the case where the system of i.e.r.v.'s is taken to be a white noise. We are now ready to discuss white noise analysis.

First we note that the white noise analysis has many advantages.

1) It is an infinite dimensional analysis. Actually, our stochastic analysis can be systematically done by taking a white noise as a sytem of i.e.r.v.'s to express the given random complex systems. Indeed, the analysis is essentially infinite dimensional as will be seen in what follows.

2) Rotation group. The white noise measure supported by the space E^* of generalized functions on the parameter space R^d is invariant under the rotations of E^*. Hence a harmonic analysis arising from the group will naturally be discussed. The group contains significant subgroups which describes essentially infinite dimensional characters.

3) Random fields $X(C)$ parametrized by C is discussed in the similar manner to $X(t)$ so far as innovation is concerned. For concrete discussion, we assume that C is a closed smooth convex manifold like a contour or a surface. Needless to say, $X(C)$ enjoys more profound characteristic properties.

4) The so-called S-transform applied to white noise functionals provides a bridge connecting white noise functionals and classical functionals of ordinary functions. We can therefore appeal to the classical theory of functionals established in the first half of the twentieth century. Differential and integral calculus of white noise functionals, often generalized functionals, harmonic analysis including Fourie analysis, Laplacians, complexification and other theories are refered to the monograph [13] and others.

§3. Relations to Quantum Dynamics

We now explain briefly some topics in quantum dynamics to which white noise theory can be applied.

1) Representation of the canonical commutation relations for Boson field. This topic is well known. Let $\dot{B}(t)$ be a white noise and let ∂_t denote the $\dot{B}(t)$-derivative. Then it

is an annihilation operator and its dual operator ∂_t^* stands for the creation. They satisfy the commutation relations

$$[\partial_t, \partial_s] = [\partial_t^*, \partial_s^*] = 0,$$
$$[\partial_t, \partial_s^*] = \delta(t-s).$$

From these, a representation of the canonical commutation relations hold for Bosonic particle.

2) Reflection positivity (T-positivity). A stationary multiple Markov (say, N-ple Markov) Gaussian process has a spetral density function $f(\lambda)$ of particular type. Namely,

$$f(\lambda) = \sum_1^N \frac{c_k}{\lambda^2 + a_k^2}.$$

On the other hand, it is proved that

Proposition. The covariance function $\gamma(h)$ of a stationary T-positive Gaussian process is expressed in the form

$$\gamma(h) = \int_0^\infty \exp[-|h|x] dv(x),$$

where v is a positive finite measure.

By applying this assertion to the N-ple Markov Gaussian process we claim that T-positivity requires $c_k > 0$ for every k. Note that in the strictly N-ple Markov case this condition is not satisfied. It is our hope that this result would be generalized to the cases of general stochastic processes of multiple Markov properties.

3) A path integral formulation. One of the realization of Dirac-Feynman's idea of the path integral may be given by the following method using generalized white noise functionals. First we establish a class of possible trajectories when a Lagrangian $L(x, \dot{X})$ is given. Let x be the classical trajectory determined by the Lagrangian. As soon as we come to quantum dynamics we have to consider fluctuating paths y. We propose that they are given by

$$y(s) = x(s) + \sqrt{\frac{\hbar}{m}} B(s).$$

The average over the paths is replaced with the expectation with respect to the probability measure for which Brownian motion $B(t)$ is defined. Thus, the propagator $G(y_1, y_2, t)$ is given by

$$E\left\{N \exp\left[\frac{i}{\hbar}\int_0^t L(y, \dot{y}) ds + \frac{1}{2}\int_0^t \dot{B}(s)^2 ds\right] \delta(y(t) - y_2)\right\}.$$

With this setup, actual computations have been done to get exact formulae of the propagators. (L. Streit et al.)

4) Dirichlet forms in infinite dimensions. With the help of positive grneralized white noise functionals we prove criteria for closability of energy forms.

5) Random fields $X(C)$. We assume that $X(C)$ has a causal representation in terms of white noise.

5.1) Markov property and multiple Markov properties. We are suggested by Dirac's paper [1] to define Markov property. For Gaussian case we are given reasonable definition (see [14]) by using the canonical representation in terms of white noise. Some attempt-s have been made for some non Gaussian fields. It is an interesting question to fine conditions related to multiple Markov properties.

5.2) Stochastic variational equations of Langevin type. Let C runs through a class $\mathbf{C}$ of concentric circles. The equation is

$$\delta X(C) = -\lambda X(C) \int_C \delta n(s)ds + X_o \int_C v(s)\partial_s^* \delta n(s)ds.$$

The explicit solution is given by using the S-transform and the classical theory of functionals.

5.3) We have made an attempt to define a random field $X(C), C \in \mathbf{C}$ which satisfies conformal invariance. Reversibility can also be discussed.

§4. Concluding remarks

Some of future directions are proposed.

1. One is concerned with good applications of the Lévy Laplacian. Its significance is that it is an operator that is essentially infinite dimensional.

2. A two dimensional Brownian path is considered to have some optimality in occupying the territory. This property should reflect to the construction of a model of physical phenomena.

3. Systematic approach to invariance of random fields under transformation group will be discussed. The reversibility of a random field discussed in this line would suggest a generalization of the path integral method discussed in 3) of §3.

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