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The Symmetric Groups and Algebraic Central Limit Theorems

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Abstract

In this note, we review some of our results on central limit theorems in algebraic probability and report an attempt to develop their quantum aspects. We illustrate our approach with materials concerning the symmetric groups.

1 Introduction

Let G be a discrete group and S generate G with $S^{-1} = S$ (as a set) and $S \not\ni e$ (the identity element in G). (G, S) forms a Cayley graph $\mathcal{X}$, in which G is the vertex set and $x, y \in G$ are adjacent (denoted by $x \sim y$) if and only if $\exists s \in S$ such that $sx = y$. The adjacency operator A on $\mathcal{X}$ acts by definition on a suitable function space on G as

$$(Af)(x) = \sum_{y:y\sim x} f(y) \ , \qquad (f \in Fun(G)),$$

which is a formal expression when the degree $\kappa = |S| = \infty$. Let us take a normalised positive-definite function φ on G or let φ be a state on a suitable algebra $\mathcal{A}(G)$ generated by G. We are interested in asymptotic spectral structure of A on large $\mathcal{X}$ through some scaling limit. To be more precise, let $S^{(n)} \nearrow S$ with $|S^{(n)}| < \infty$ and $S^{(n)-1} = S^{(n)}$. The adjacency operator $A^{(n)}$ at a finite level is

$$(A^{(n)}f)(x) = \sum_{y:y\sim x, yx^{-1}\in S^{(n)}} f(y) \ , \qquad (f \in Fun(G)).$$

We can formulate our central limit theorem by considering convergence of moments or spectral distribution of

$$(A^{(n)} - \varphi(A^{(n)}))/\sqrt{\varphi((A^{(n)} - \varphi(A^{(n)}))^2)}$$

with respect to φ. The asymptotic is taken along the size n and possibly other addtional parameters contained in the state φ. (See later sections.) More generally, we can discuss several adjacency operators $A_{S_1}, A_{S_2}, A_{S_3}, \cdots$ associated with subsets $S_1, S_2, S_3, \cdots$ of S and their mixed moments or joint distribution (if A_i's are commuting) with respect to φ. It is straightforward to extend the consideration to other regular graphs than Cayley graphs.

In this note, we treat Cayley graphs of the symmetric groups $\mathcal{S}_n$ and distance-regular graphs appearing as homogeneous spaces of the symmetric groups. Spectral structure of these groups is at *finite* level studied well by using combinatorial and representation-theoretical technique. The algorithmic results, however, become very complicated as the size of the graph grows. In order to make the limiting procedure more transparent, we intend to apply *quantum decomposition* of an adjacency operator, which is a basic idea widely used in quantum probability.

2 Working on Johnson Graph

A Johnson graph is an important distance-regular graph as well-known as a Hamming graph. For $v, d \in N$, let $X = \{x \subset \{1, 2, \cdots, v\} \mid |x| = d\}$ be the d-subsets of a v-set. ($2d \leq v$ without loss of generality.) By definition two vertices $x, y \in X$ are adjacent if $|x \cap y| = d - 1$ in Johnson graph $J(v, d)$. It has diameter d and degree $\kappa = d(v-d)$. $J(v, d)$ is regarded as a homogeneous space $\mathcal{S}_d \times \mathcal{S}_{v-d} \backslash \mathcal{S}_v$. We fix a base point $x_0 \in X$. The vacuum state is defined as $\langle \Phi(0), \cdot\, \Phi(0) \rangle_{\ell^2(X)}$ where $\Phi(0) = \delta_{x_0}$.

In [7], we showed the following central limit theorem by using spectral data of Johnson graphs(e.g. seen in Bannai-Ito [3]).

Theorem 1 For a growing family of $J(v, d)$, the distribution of normalised adjacency operator $A/\sqrt{\kappa}$ with respect to the vacuum state converges weakly to:

$$\text{(i)} \quad e^{-(x+1)} I_{[-1,\infty)}(x) dx \qquad \text{as} \quad d \to \infty \text{ and } \frac{2d}{v} \to 1\, ,$$

$$\text{(ii)} \quad \sum_{l=0}^{\infty} \frac{2(1-p)}{2-p} \Big(\frac{p}{2-p} \Big)^l \delta_{-\frac{p}{\sqrt{p(2-p)}} + \frac{2(1-p)}{\sqrt{p(2-p)}} l} \quad \text{as} \quad d \to \infty \text{ and } \frac{2d}{v} \to p \in (0, 1].$$

(The original statement in [7] contained an extra condition in (ii), which proves to be inessential.)

We can extend Theorem 1 to a quantum central limit theorem by introducing quantum decomposition of the adjacency operator: $A = A^+ + A^-$. Let $\Gamma(\mathcal{X}) = \bigoplus_{n=0}^{d} \Phi(n)$ be the finite-dimensional Fock space associated with a Johnson graph $\mathcal{X}$, where $\Phi(n)$ is a normalised number vector. Let

$$\Gamma = \{(\xi_n) = \sum_{n=0}^{\infty} \xi_n e_n \in C^{\infty} \mid \sum_{n=0}^{\infty} (n!)^2 |\xi_n|^2 < \infty\}$$

be a 1-mode interacting Fock space. Let B^+, B^- and N denote the creator, the annihilator and the number operator on Γ. For an interacting Fock space and operators on it, we refer to Accardi-Bożejko [1] and Accardi-Obata [2].

Theorem 2 For a growing family of $J(v,d)$ such that $d \to \infty$ and $2d/v \to p \in (0,1]$, we have

$$\langle \Phi(l), \frac{A^{\epsilon_1}}{\sqrt{\kappa}}\frac{A^{\epsilon_2}}{\sqrt{\kappa}}\cdots\frac{A^{\epsilon_m}}{\sqrt{\kappa}}\Phi(j)\rangle_{\Gamma(\mathcal{X})} \longrightarrow \langle e_l, C^{\epsilon_1}C^{\epsilon_2}\cdots C^{\epsilon_m}e_j\rangle_\Gamma$$

for $\forall m \in \boldsymbol{N}$, $\forall\epsilon_1,\epsilon_2,\cdots,\epsilon_m \in \{+,-\}$, $\forall l,j \in \{0,1,2,\cdots\}$, where

$$C^{\pm} = C_p^{\pm} = B^{\pm} + \frac{1}{\sqrt{p(2-p)}}N\ .$$

Theorem 2 yields Theorem 1 as classical reduction with an interesting observation of relations to orthogonal polynomials. Including the definition of quantum decomposition $A = A^+ + A^-$, full details of Theorem 2 and more general version about distance-regular graphs will be included in [11] (partly announced in an IIAS workshop 20 – 22 / 2 / 2001). As for a quantum central limit theorem on Hamming graphs, we refer to Hashimoto-Obata-Tabei [6].

Motivated by Hashimoto [4], we introduced Gibbs state Φ_q on the adjacency algebra $\mathcal{A}(\mathcal{X})$ of a distance-regular graph $\mathcal{X}$ in [9]:

$$\Phi_q(Q) = \langle \Phi(0), \Big(\sum_{h=0}^{d} q^h A_h\Big)Q\Phi(0)\rangle \qquad (Q \in \mathcal{A}(\mathcal{X})).$$

Here A_i is the ith adjacency operator on $\mathcal{X}$. Φ_q becomes actually a state on $\mathcal{A}(\mathcal{X})$ for $0 \le q \le 1$ if the graph $\mathcal{X}$ is nice, e.g. if $\mathcal{X}$ is quadratically embedded into a Hilbert space. Then the temperature T of $\mathcal{X}$ is introduced as $T \propto -1/\log q$. In [9], we showed the following central limit theorem (low temperature limit).

Theorem 3 For a growing family of $J(2d,d)$, the distribution of

$$(A - \Phi_q(A))/\sqrt{\Phi_q((A-\Phi_q(A))^2)}$$

with respect to Φ_q converges weakly to:

(i) $e^{-(x+1)}I_{[-1,\infty)}(x)dx$ as $d \to \infty$ and $q = r/d^\alpha \to 0$ $(r \ge 0, \alpha > 1 :$ fixed)

(ii) $\sqrt{2r+1}e^{-(x\sqrt{2r+1}+2r+1)}J_0(i2\sqrt{r(x\sqrt{2r+1}+r+1)})I_{[-\frac{r+1}{\sqrt{2r+1}},\infty)}(x)dx$

as $d \to \infty$ and $q = r/d \to 0$ $(r \ge 0 :$ fixed), where

$$J_0(z) = \sum_{k=0}^{\infty}\frac{(-z^2/4)^k}{(k!)^2} \qquad (z \in \boldsymbol{C})$$

is the 0th Bessel function.

Seen from the viewpoint of Theorem 2, Theorem 3 can be interpreted as convergence of a superposition of matrix elements. Finding the limit distribution of (ii) is equivalent to computing the moments

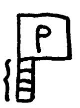
$$\sum_{n=0}^{\infty}\frac{r^n}{(n!)^2}\langle (B^+ + B^- + 2N)^p e_0, e_n\rangle_\Gamma \qquad (p \in \{0,1,2,\cdots\})$$

for operators B^+, B^- and N on the interacting Fock space Γ. It can be done through a combinatorial argument by using an appropriate Bratteli diagram. It turns out that the limit distribution is a translation of that of $X_0 + X_1 + \cdots + X_M$ where $X_0, X_1, X_2, \cdots$ are independent random variables obeying the exponential distribution $e^{-x}dx$ and M is also independent random variable of X_i's obeying Poisson distribution with parameter r. Details of these observations and computation of the moments will be contained in [10]. See Hashimoto [5] for the discussion of Haagerup states on the free group algebras.

3 Working on the Infinite Symmetric Group

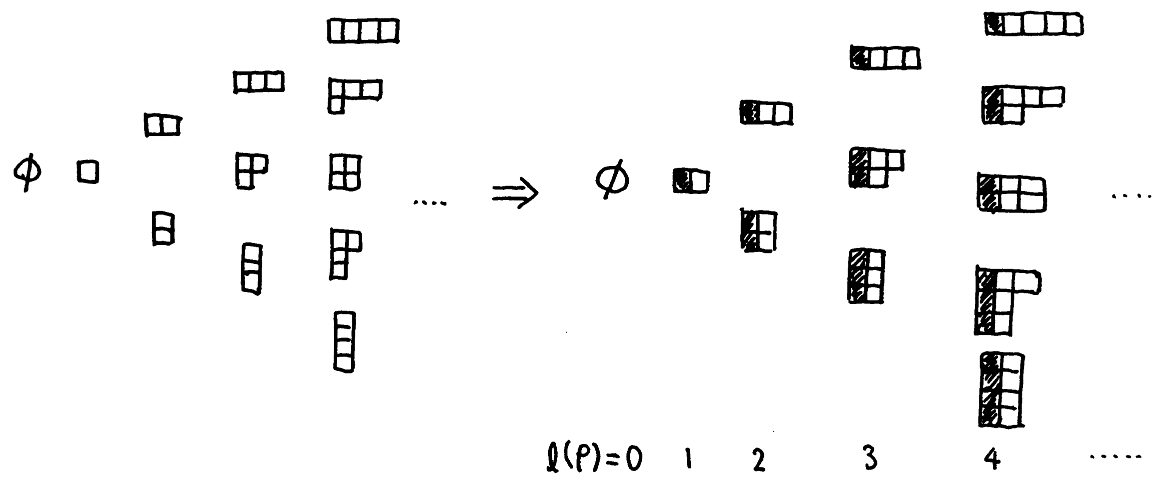
Let $\mathcal{S}_\infty = \bigcup_{n=1}^\infty \mathcal{S}_n$ be the infinite symmetric group with the identity element e. The nontrivial ($\neq \{e\}$) conjugacy classes of $\mathcal{S}_\infty$ are parametrised by $\mathcal{D}$, the set of the Young diagrams without a row consisting of only one box. Let C_ρ denote the conjugacy class corresponding to $\rho \in \mathcal{D}$. We use the cycle notation $\rho = (2^{k_2(\rho)}3^{k_3(\rho)} \ldots j^{k_j(\rho)} \ldots)$ which means that diagram $\rho \in \mathcal{D}$ contains $k_j(\rho)$ number of j-rows. Set $|\rho| = \sum_j jk_j(\rho)$, the number of boxes of ρ. Let $r(\rho)$ denote the number of rows of ρ and $l(\rho) = |\rho| - r(\rho)$ the "length function". In fact, for given $\rho \in \mathcal{D}$, taking sufficiently large n and $g \in C_\rho \cap \mathcal{S}_n$, and letting $[g]_n$ denote the number of cycles of $g \in \mathcal{S}_n$, we see the minimal number of the transpositions in $\mathcal{S}_n$ expressing g as their product is equal to

$n - [g]_n = n - \sharp$ of rows of



$$= (|\rho| + |\text{leg}|) - (r(\rho) + |\text{leg}|) = l(\rho) .$$

It is convenient to arrange the diagrams in $\mathcal{D}$ according to $l(\rho)$, which is induced by adding one column to the left side (as indicated below) of each diagram in the usual arrangement of the Young lattice.



The assignment of edges is in a different way from the Young lattice, as is mentioned

An adjacency operator is formally written as $A_\rho = \sum_{g\in C_\rho} g$ for $\rho \in \mathcal{D}$. Taking $n \geq |\rho|$ and setting $C_\rho^{(n)} = C_\rho \cap \mathcal{S}_n$, we get an adjacency operator at n-level, $A_\rho^{(n)} = \sum_{g\in C_\rho^{(n)}} g$. Let $\Phi = \langle \delta_e, \cdot\, \delta_e\rangle_{\ell^2(\mathcal{S}_\infty)}$ be the vacuum state. In [8], we showed the following central limit theorem. $H_k(x)$ denotes the Hermite polynomial of degree k obeying the recurrence formula

$$xH_k(x) = H_{k+1}(x) + kH_{k-1}(x) \ , \quad H_0(x) = 1 \ , \quad H_1(x) = x \ .$$

Theorem 4 For $\forall m \in \boldsymbol{N}$, $\forall \rho_1, \rho_2, \cdots, \rho_m \in \mathcal{D}$, $\forall r_1, r_2, \cdots, r_m \in \{0, 1, 2, \cdots\}$, we have

$$\lim_{n\to\infty} \Phi\Big(\Big(\frac{A_{\rho_1}^{(n)}}{\sqrt{|C_{\rho_1}^{(n)}|}}\Big)^{r_1}\Big(\frac{A_{\rho_2}^{(n)}}{\sqrt{|C_{\rho_2}^{(n)}|}}\Big)^{r_2}\cdots\Big(\frac{A_{\rho_m}^{(n)}}{\sqrt{|C_{\rho_m}^{(n)}|}}\Big)^{r_m}\Big)$$
$$= \prod_{j\geq 2}\int_{\boldsymbol{R}} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\Big(\frac{H_{k_j(\rho_1)}(x)}{\sqrt{k_j(\rho_1)!}}\Big)^{r_1}\Big(\frac{H_{k_j(\rho_2)}(x)}{\sqrt{k_j(\rho_2)!}}\Big)^{r_2}\cdots\Big(\frac{H_{k_j(\rho_m)}(x)}{\sqrt{k_j(\rho_m)!}}\Big)^{r_m} dx \ .$$

This result extended Kerov's theorem in [12]. Indeed, restricted to $\mathcal{S}_n$ such that $n \geq |\rho|$, the spectral decomposition of A_ρ acting on $\ell^2(\mathcal{S}_n)$ is given by

$$A_\rho = \sum_{\lambda\in\mathcal{Y}_n} \frac{|C_\rho^{(n)}|\chi_\rho^\lambda}{\dim\lambda} E_\lambda \ , \qquad \Phi(E_\lambda) = \frac{\dim^2\lambda}{n!} \quad \text{(Plancherel measure)}$$

where $\mathcal{Y}_n$ denotes the set of Young diagrams with n boxes and χ_ρ^λ is the value of the irreducible character corresponding to λ taken on C_ρ.

As an attempt to develop a quantum aspect of Theorem 4, let us discuss at first a decomposition of $A_{\square\!\square}$ for simplicity. For $e \neq g \in \mathcal{S}_n$, we define operators g^+ and g^- on $\ell^2(\mathcal{S}_n)$ as

$$g^+\delta_x = \begin{cases} \delta_{gx} & \text{if } [gx] < [x] \\ 0 & \text{otherwise,} \end{cases} \qquad g^-\delta_x = \begin{cases} \delta_{gx} & \text{if } [gx] > [x] \\ 0 & \text{otherwise.} \end{cases}$$

Set $A^\pm = \sum_{g\in C_{\square\!\square}\cap\mathcal{S}_n} g^\pm$. Clearly $A_{\square\!\square} = A^+ + A^-$. Let $p_{\tau\rho}^\sigma$ denote the intersection number of the group association scheme $\mathcal{X}(\mathcal{S}_n)$, namely, if $x, y \in \mathcal{S}_n$ and $x^{-1}y \in C_\sigma$,

$$p_{\tau\rho}^\sigma = |\{z \in \mathcal{S}_n | x^{-1}z \in C_\tau, z^{-1}y \in C_\rho\}| \ .$$

(This quantity does not depend on the choice of x, y whenever $x^{-1}y \in C_\sigma$.) The action of $A^\pm$ to number vectors is as follows.

Proposition 1 Set $v_\rho = \sum_{x\in C_\rho\cap\mathcal{S}_n} \delta_x$ for $\rho \in \mathcal{D}$. We have

$$A^\pm v_\rho = \sum_{\sigma : l(\sigma) = l(\rho)\pm 1} p_{\square\!\square\,\rho}^\sigma v_\sigma \ .$$

Proof Note that, if $g = (i\ j)$,

$$[gx] < [x] \iff [gx] = [x] - 1 \iff i \text{ and } j \text{ are contained in different cycles of } x,$$
$$[gx] > [x] \iff [gx] = [x] + 1 \iff i \text{ and } j \text{ are contained in the same cycle of } x.$$

$$
\begin{aligned}
A^{+}v_{\rho} &= \sum_{x\in C_{\rho}}\sum_{g\in C_{\square\square}} g^{+}\delta_{x} \\
&= \sum_{y\in\mathcal{S}_{n}} |\{(g,x)|x\in C_{\rho}, g\in C_{\square\square}, y=gx, [gx]=[x]-1\}|\,\delta_{y} \\
&= \sum_{\sigma}\sum_{y\in C_{\sigma}} |\{(g,x)|x\in C_{\rho}, g\in C_{\square\square}, y=gx, [gx]=[x]-1\}|\,\delta_{y} \\
&= \sum_{\sigma:l(\sigma)=l(\rho)+1}\ \sum_{y\in C_{\sigma}} |\{(g,x)|x\in C_{\rho}, g\in C_{\square\square}, y=gx\}|\,\delta_{y} \\
&= \sum_{\sigma:l(\sigma)=l(\rho)+1} p_{\square\square\,\rho}^{\ \sigma} v_{\sigma}\ .
\end{aligned}
$$

The action of A^{-} is similar. (Alternatively, use the adjoint relation.) QED

Let us assume that $\sigma,\rho\in\mathcal{D}$ satisfy $l(\sigma)=l(\rho)+1$ and consider $p_{\square\square\,\rho}^{\ \sigma}$ in sufficiently large $\mathcal{S}_{n}$. If $p_{\square\square\,\rho}^{\ \sigma}>0$, then we can specify the number j such that a j-row of σ splits into two rows of ρ (possibly 1-rows). According to the cases: (i) $|\sigma|=|\rho|+2$, (ii) $|\sigma|=|\rho|+1$, (iii) $|\sigma|=|\rho|$, we have

$$
p_{\square\square\,\rho}^{\ \sigma} = \begin{cases} \text{(i) } k_{2}(\sigma)=k_{2}(\rho)+1 \\ \text{(ii) } jk_{j}(\sigma)=j(k_{j}(\rho)+1) \\ \text{(iii) } C_{j}k_{j}(\sigma)=C_{j}(k_{j}(\rho)+1) \end{cases}
$$

where C_{j} is the number of cutting a j-polygon into two pieces so that each has at least two vertices. In particular, $p_{\square\square\,\rho}^{\ \sigma}$ does not depend on n if $l(\sigma)=l(\rho)+1$. This property enables us to assign multiplicity $p_{\square\square\,\rho}^{\ \sigma}$ for pair (ρ,σ) such that $l(\sigma)=l(\rho)+1$ in $\mathcal{D}$. (Especially, they are joined by definition if $p_{\square\square\,\rho}^{\ \sigma}>0$.) We define a normalised number vector $\Phi(\rho)=v_{\rho}/\sqrt{|C_{\rho}|}$ with respect to the usual inner product in $\ell^{2}(\mathcal{S}_{n})$ and a finite-dimensional Fock space $\Gamma(\mathcal{S}_{n})=\boldsymbol{C}\Phi(\emptyset)\oplus\bigoplus_{\rho\in\mathcal{D},|\rho|\le n}\boldsymbol{C}\Phi(\rho)$. Proposition 1 yields

$$
\frac{1}{\sqrt{|C_{\square\square}|}}A^{\pm}\Phi(\rho) = \sum_{\sigma:l(\sigma)=l(\rho)\pm 1} p_{\square\square\,\rho}^{\ \sigma}\sqrt{\frac{|C_{\sigma}|}{|C_{\square\square}|\,|C_{\rho}|}}\Phi(\sigma)\ .
$$

Clearly $|C_{\rho}|\asymp n^{|\rho|}$. If $l(\sigma)=l(\rho)+1$, only the term of $|\sigma|=|\rho|+2$ survives in the right hand side sum as $n\to\infty$, for which we have

$$
p_{\square\square\,\rho}^{\ \sigma}\sqrt{\frac{|C_{\sigma}|}{|C_{\square\square}|\,|C_{\rho}|}} \longrightarrow \sqrt{k_{2}(\rho)+1} \qquad (\sigma=\rho\sqcup\square\square).
$$

Using $|C_{\rho}|p_{\square\square\,\sigma}^{\ \rho}=|C_{\sigma}|p_{\square\square\,\rho}^{\ \sigma}$, we see, if $l(\sigma)=l(\rho)-1$, the only surviving term of $|\rho|=|\sigma|+2$ satisfies

$$
p_{\square\square\,\rho}^{\ \sigma}\sqrt{\frac{|C_{\sigma}|}{|C_{\square\square}|\,|C_{\rho}|}} \longrightarrow \sqrt{k_{2}(\rho)} \qquad (\sigma=\rho\setminus\square\square).
$$

Let us introduce a Fock space $\Gamma = \boldsymbol{C}\Psi(\emptyset) \oplus \bigoplus_{\rho\in\mathcal{D}} \boldsymbol{C}\Psi(\rho)$, where $\Psi(\emptyset)$ (the vacuum state) and $\Psi(\rho)$'s are normalised vectors. The above discussion leads to the following assertion.

Proposition 2 Associated with the decomposition $A_{\Box\!\Box} = A^+ + A^-$ on $\mathcal{S}_n$, the following quantum central limit theorem holds:

$$\langle \Phi(\sigma), \frac{A^{\epsilon_1}}{\sqrt{|C_{\Box\!\Box}|}}\frac{A^{\epsilon_2}}{\sqrt{|C_{\Box\!\Box}|}}\cdots\frac{A^{\epsilon_m}}{\sqrt{|C_{\Box\!\Box}|}}\Phi(\rho)\rangle_{\Gamma(\mathcal{S}_n)} \longrightarrow \langle \Psi(\sigma), a^{\epsilon_1}a^{\epsilon_2}\cdots a^{\epsilon_m}\Psi(\rho)\rangle_{\Gamma}$$

as $n \to \infty$ for $\forall m \in \boldsymbol{N}$, $\forall \epsilon_1, \epsilon_2, \cdots, \epsilon_m \in \{+,-\}$, $\forall \sigma, \rho \in \mathcal{D} \cup \{\emptyset\}$, where the limit operators $a^{\pm}$ are defined as

$$a^+\Psi(\rho) = \sqrt{k_2(\rho)+1}\,\Psi(\rho \sqcup \Box\!\Box) \ , \quad a^-\Psi(\rho) = \sqrt{k_2(\rho)}\Psi(\rho \backslash \Box\!\Box) \ .$$

Hence, if we start from the vacuum state $\Psi(\emptyset)$, $a^{\pm}$ do not take us out of $\boldsymbol{C}\Psi(\emptyset) \oplus \bigoplus_{k=1}^{\infty} \boldsymbol{C}\Psi((2^k))$ and act on it in the same way as the creator and the annihilator on a Boson Fock space. Obviously, classical reduction of Proposition 2 yields a well-known Gaussian limit (included in Kerov's theorem [12]).

Beyond the vacuum expectation, let us discuss a central limit theorem measured by the state associated with Kerov-Olshanski-Vershik's generalised regular representation of the infinite symmetric group $\mathcal{S}_\infty$. In [13], Kerov-Olshanski-Vershik introduced an interesting deformation of the regular representation of $\mathcal{S}_\infty$ by using a 1-cocycle containing a complex parameter z. The representation space is the L^2-space on a projective limit of probability spaces $(\mathcal{S}_n, \mu_t^n)$. Here $\mu_t^n(\{x\}) = t^{[x]}/(t)_n$ for $x \in \mathcal{S}_n$, $t = |z|^2$ and $(t)_n = t(t+1)\cdots(t+n-1)$. This representation gives a central positive-definite function ϕ_z on $\mathcal{S}_\infty$, which can be expressed on $\mathcal{S}_n$ as

$$\phi_z|_{\mathcal{S}_n}(x) = \sum_{u\in\mathcal{S}_n} z^{[x^{-1}u]-[u]}\frac{t^{[u]}}{(t)_n} = \sum_{\lambda\in\mathcal{Y}_n} M_z(\lambda)\frac{\chi_\rho^\lambda}{\dim\lambda} \quad (x \in C_\rho \cap \mathcal{S}_n),$$

$$M_z(\lambda) = \frac{1}{(t)_n}\prod_{(ij)\in\lambda} |z+j-i|^2 \frac{\dim^2\lambda}{n!} \qquad (z \in \boldsymbol{C}\setminus\{0\}).$$

We obtain a tracial state $\phi_z|_{\mathcal{S}_n}$ by $\boldsymbol{C}$-linear extension. The limiting case $z \to \infty$ corresponds to the regular representation, and then ϕ_∞ is interpreted as the vacuum vector state. Set $\phi_z(\rho) = \phi_z(w)$ where $w \in C_\rho$.

Let us work on $\mathcal{S}_n$. Our state ϕ_z has density matrix $\sum_{|\rho|\le n}\phi_z(\rho)A_\rho$. A simple compuation yields $\phi_z(\Box\!\Box) = (z+\overline{z})/(t+1)$. If $z+\overline{z}=0$, then $A_{\Box\!\Box}$ has mean 0 and variance

$$\phi_z(A_{\Box\!\Box}^2) = \frac{n(n-1)}{2}\Big\{1 + \frac{2(n-2)}{t+2} + \frac{(n-2)(n-3)}{(t+2)(t+3)}\Big\} \ .$$

This suggests a central limit theorem for the adjacency operator $A_{\Box\!\Box}$ with respect to ϕ_z as $n \to \infty$ and n/t converges to a nonzero value. (Compare it with Theorem 3.)

It is interesting to understand the limiting expectation with respect to ϕ_z as a superposition of matrix elements similarly to the discussion following Theorem 3. In that situation, $q^h\sqrt{\kappa_h}$ converged as $d \to \infty$ and $q = r/d \to 0$. However, we have now a difficulty that $\phi_z(\rho)\sqrt{|C_\rho|}$ *does not converge* in general as $n \to \infty$ with $t = |z|^2 \asymp n$. Assume that $z + \bar{z} = 0$. For special diagrams (2^l), we have

$$\phi_z((2^l)) = \begin{cases} \frac{1}{(t)_{2l}}\Big(\frac{l!}{(l/2)!}t^l + O(t^{l-1})\Big) & \text{if } l \text{ is even} \\ 0 & \text{if } l \text{ is odd.} \end{cases}$$

Hence $\phi_z((2^l))\sqrt{|C_{(2^l)}|}$ converges as $t \asymp n \to \infty$. On the other hand, for the cycles, we have

$$\phi_z((2k-1)) = \frac{\text{polynomial of degree } k \text{ in } t}{(t)_{2k-1}}, \quad \phi_z((2k)) = 0 .$$

Actually, we conjecture that, for $\rho = (2^{k_2}3^{k_3}4^{k_4}\cdots)$,

$$\begin{aligned} \phi_z(\rho) &= O(t^{k_2+2k_3+2k_4+3k_5+3k_6+4k_7+\cdots})/(t)_{|\rho|} \\ &= O(1/t^{k_2+k_3+2k_4+2k_5+3k_6+3k_7+4k_8+\cdots}) \end{aligned}$$

holds and hence $\phi_z(\rho)\sqrt{|C_\rho|}$ may possibly diverge by $n^{(k_3+k_5+k_7+\cdots)/2}$ as $t \asymp n \to \infty$. If we take another normalised number vector

$$\Phi'(\rho) = v_\rho / \sqrt{\frac{|C_\rho|}{n^{k_3(\rho)+k_5(\rho)+\cdots}}} ,$$

this problem is overcome and we can still control the convergence of the branching coefficients of the action of $A\pm$ under the appropriate normalisation. However, the limit operators $a^\pm$ in Proposition 2 have more complicated actions on the Fock space $C\Psi(\emptyset) \oplus \bigoplus_{\rho\in\mathcal{D}} C\Psi(\rho)$. Details will appear elsewhere later on.

References

[1] L.Accardi, M.Bożejko, Interacting Fock spaces and Gaussianization of probability measures, Infinite Dimen. Anal. Quantm Prob. **1** (1998), 663 – 670.

[2] L.Accardi, N.Obata, Introduction to algebraic probability (in Japanese), Nagoya Mathematical Lectures **2**, Nagoya Univ., 1999.

[3] E.Bannai, T.Ito, Algebraic combinatorics I, Menlo Park, California, Benjamin/Cummings, 1984.

[4] Y.Hashimoto, Deformations of the semicircle law derived from random walks on free groups, Prob. Math. Stat. **18** (1998), 399 – 410.

[5] Y.Hashimoto, Quantum decomposition in discrete groups and interacting Fock spaces, preprint, 2001.

[6] Y.Hashimoto, N.Obata, N.Tabei, A quantum aspect of asymptotic spectral analysis of large Hamming graphs, preprint, 2001.

[7] A.Hora, Central limit theorems and asymptotic spectral analysis on large graphs, Infinite Dimen. Anal. Quantum Prob. **1** (1998), 221 – 246.

[8] A.Hora, Central limit theorem for the adjacency operators on the infinite symmetric group, Commun. Math. Phys. **195** (1998), 405 –416.

[9] A.Hora, Gibbs state on a distance-regular graph and its application to a scaling limit of the spectral distributions of discrete Laplacians, Prob. Theory Relat. Fields **118** (2000), 115 – 130.

[10] A.Hora, in preparation.

[11] A.Hora, N.Obata, in preparation.

[12] S.Kerov, Gaussian limit for the Plancherel measure of the symmetric group, C. R. Acad. Sci. Paris **316**, Série I (1993), 303 – 308.

[13] S.Kerov, G.Olshanski, A.Vershik, Harmonic analysis on the infinite symmetric group. A deformation of the regular representation, C. R. Acad. Sci. Paris **316**, Série I (1993), 773 – 778.