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On positivity and universality of templates induced from diffeomorphisms of the disk

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1. INTRODUCTION

In this note, we consider links induced from periodic orbits of orientation preserving automorphisms φ of D^2. We first present some basic terminologies. We denote the i-th iteration of φ by φ^i. We say that $x \in D^2$ is *a period* $k \in \mathbf{N}$ *periodic point* if $\varphi^k(x) = x$ and $\varphi^i(x) \neq x$ for $1 \leq i < k$. In particular, we say that x is *a fixed point* if x is a period 1 periodic point. For $x \in D^2$, $\{\varphi^i(x) \mid i \in \mathbf{N}\}$ is called *the orbit of* x and denoted by $O_\varphi(x)$. If x is a periodic point, then $O_\varphi(x)$ is called *the periodic orbit of* x.

Let $\Phi = \{\varphi_t\}_{0\leq t\leq 1}$ be an isotopy of D^2 such that $\varphi_0 = id_{D^2}$, $\varphi_1 = \varphi$. For a finite union of periodic orbits P of φ, we define a subset of $\widetilde{V} = D^2 \times S^1 (\cong D^2 \times I/(x,0) \sim (x,1))$, denoted by $\mathcal{S}_\Phi P$, as follows.

$$\mathcal{S}_\Phi P = \bigcup_{0\leq t\leq 1} (\varphi_t(P) \times \{t\})/(x,0) \sim (x,1).$$

$\mathcal{S}_\Phi P$ is called *a suspension of* P *by* Φ. Let V be a standardly embedded solid torus in the 3-sphere S^3. Then $h : \widetilde{V} \to V$ denotes a homeomorphism such that for a longitude $\tilde{\ell}$ on $\widetilde{V}$, $h(\tilde{\ell})$ is a knot with the linking number of $h(\tilde{\ell})$ and the core circle of V being 1 (see Figure 1). For each $i \in \mathbf{Z}$, $h^i(\mathcal{S}_\Phi P)$ is a link in S^3, where the orientation of $\mathcal{S}_\Phi P$ is induced from parametrization by t.

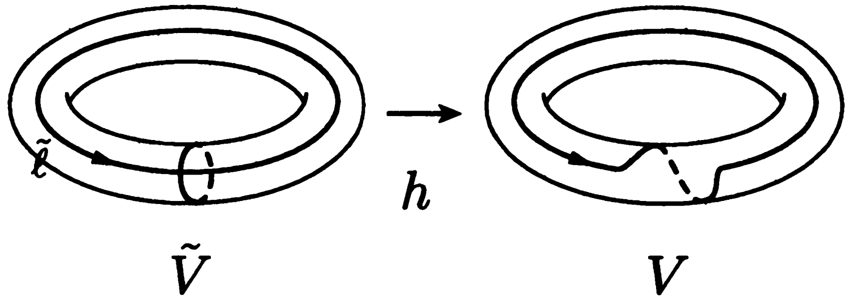

Figure 1

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Definition 1.1. Let $\varphi : D^2 \to D^2$ be an orientation preserving automorphism, and $\Phi = \{\varphi_t\}_{0 \le t \le 1}$ an isotopy of D^2 such that $\varphi_0 = id_{D^2}$, $\varphi_1 = \varphi$. We say that φ *induces all link types* if there exists an integer $i \in \mathbf{Z}$ satisfying the following conditions.

($*$) For each link L in S^3, there exists a finite union of periodic orbits P_L of φ such that $L = h^i(S_\Phi P_L)$.

We note that the definition does not depend on Φ. Moreover the number of integers i such that h^i satisfies ($*$) does not depend on Φ (see [8]). Hence we denote the number by $\overline{N}(\varphi)$, that is,

$$\overline{N}(\varphi) = \sharp\{i \in \mathbf{Z} \mid i \text{ satisfies } (*) \text{ for } \Phi\}.$$

The topological entropy $h_{top}(\varphi)$ for φ is a measure of its dynamical complexity (see [14] for a definition of the entropy). A result of Gambaudo-van Strien-Tresser ([3, Theorem A]) tells us that if $h_{top}(\varphi) = 0$, then φ does not induce all link types, i.e., $\overline{N}(\varphi) = 0$. It is natural to ask the following problem:

Problem 1.2. Which automorphism induces all link types ?

In [11], the second author researched the Smale horseshoe map [13] on Problem 1.2. The Smale horseshoe map is a fundamental example to study complicated dynamics since the invariant set is hyperbolic and is conjugate to the 2-shift, and such invariant sets are often observed in many dynamical systems [9] (see [12] for basic definitions of dynamical systems).

Theorem 1.3. [11] *Let H be the Smale horseshoe map. Then $\overline{N}(H) = \overline{N}(H^2) = 0$ and $\overline{N}(H^3) = 1$.*

Since $h_{top}(H)$ and $h_{top}(H^2)$ are positive, Theorem 1.3 shows the existence of diffeomorphisms not inducing all link types.

We will consider Problem 1.2 for generalized horseshoe maps G using *twist signature* $t(G)$ (see Definitions 2.1, 2.2). In Theorem 3.1, we completely determine the number $\overline{N}(G)$ by $t(G)$.

2. Generalized horseshoe map and twist signature

For definitions of generalized horseshoe map and twist signature, we first introduce some terminologies. Let $R = [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] \subset D^2$, and let S_0, S_1 be half disks as in Figure 2(a). For $c, c' \in [-\frac{1}{2}, \frac{1}{2}]$, we call $\ell_v = \{c\} \times [-\frac{1}{2}, \frac{1}{2}]$ (resp. $\ell_h = [-\frac{1}{2}, \frac{1}{2}] \times \{c'\}$) *a vertical* (resp. *a horizontal*) *line.* For $[c, d]$, $[c', d'] \subset [-\frac{1}{2}, \frac{1}{2}]$, we call $B = [c, d] \times [-\frac{1}{2}, \frac{1}{2}]$ (resp. $B' = [-\frac{1}{2}, \frac{1}{2}] \times [c', d']$) *a vertical* (resp. *a horizontal*) *rectangle.*

Let B_1, B_2 (resp. B'_1, B'_2) be disjoint vertical (resp. disjoint horizontal) rectangles. The notation $B_1 <_1 B_2$ (resp. $B'_1 <_2 B'_2$) means the first (resp. second) coordinate of a point in B_2 (resp. B'_2) is greater than that of B_1 (resp. B'_1). We denote the open rectangle which lies between B_1 and B_2 by (B_1, B_2).

Definition 2.1. Let $n \ge 2$ be an integer. *A generalized horseshoe map G of length n* is an orientation preserving diffeomorphism of D^2 satisfying the following: There exist vertical rectangles $B_1 <_1 B_2 <_1 \cdots <_1 B_n$ and horizontal rectangles $B'_1 <_2 B'_2 <_2 \cdots <_2 B'_n$ such that

(1) for each $1 \leq i \leq n$, $G(B_i) = B'_j$ for some $1 \leq j \leq n$,
(2) for each $1 \leq i \leq n-1$, $G((B_i, B_{i+1})) \subset S_k$ for some $k \in \{0,1\}$,
(3) G expands the part of horizontal lines which intersects each B_i uniformly, and contract the vertical lines in each B_i uniformly,
(4) $G|_{S_0} : S_0 \to S_0$ is contractive,
(5) if n is even (resp. odd), then $G(S_1) \subset Int\ S_0$ (resp. $G|_{S_1} : S_1 \to S_1$ is contractive) and
(6) G has no periodic points in $D^2 \setminus R$.

Definition 2.2. Let G be a generalized horseshoe map of length n. *Twist signature* $t(G)$ *of* G is the array of n integers $(a_1, \cdots, a_n)$ satisfying the following:
(1) $a_1 = 0$.
(2) For $2 \leq i \leq n$, $a_i = a_{i-1} + 1$ if $G(B_{i-1}) <_2 G(B_i)$ and $G((B_{i-1}, B_i)) \subset S_1$, or if $G(B_{i-1}) >_2 G(B_i)$ and $G((B_{i-1}, B_i)) \subset S_0$. Otherwise $a_i = a_{i-1} - 1$.

By the condition of generalized horseshoe maps G, $\Lambda = \bigcap_{m \in \mathbf{Z}} G^m(B_1 \cup \cdots \cup B_n)$ is hyperbolic which is conjugate to the n-shift.

Notice that the Smale horseshoe map is a generalized horseshoe map of length 2 with twist signature $(0,1)$.

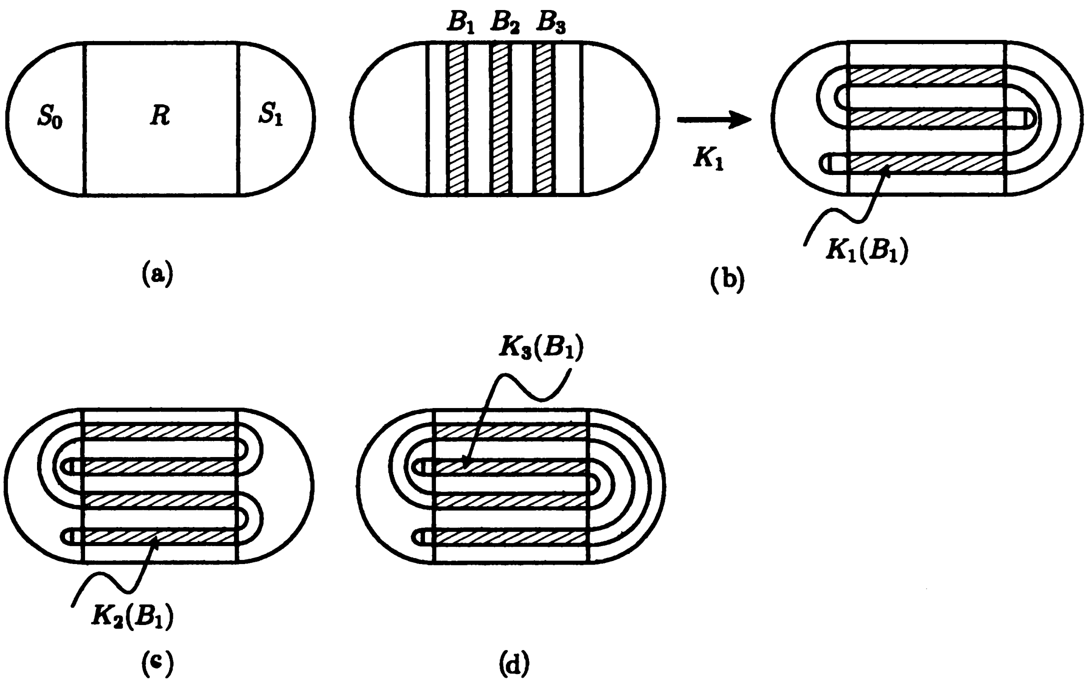

Figure 2

Example 2.3. (1) Let K_1 be a generalized horseshoe map of length 3 as in Figure 2(b). Then $t(K_1) = (0,1,2)$.
(2) Let K_2 be a generalized horseshoe map of length 4 as in Figure 2(c). Then $t(K_2) =$

$(0,1,0,-1)$.
(3) Let K_3 be a generalized horseshoe map of length 4 as in Figure 2(d). Then $t(K_3) = (0,-1,-2,-3)$.

3. Statement of results

Let G be a generalized horseshoe map with twist signature $(a_1, \cdots, a_n)$. We say that G is *positive* (resp. *negative*) if for any $i \in \{1, \cdots, n\}$, $a_i \geq 0$ (resp. $a_i \leq 0$). We say that G is *mixed* if G is neither positive nor negative. For example, K_1, K_2, K_3 in Example 2.3 are positive, mixed, negative respectively.

The following is Main theorem of this note:

Theorem 3.1. *For $x \in \mathbf{R}$, let $[x]$ be the greatest integer which does not exceed x. Let G be a generalized horseshoe map with twist signature $(a_1, \cdots, a_n)$. Let $M_+ = \max\{a_i | 1 \leq i \leq n\}$ and $M_- = \min\{a_i | 1 \leq i \leq n\}$. If G is positive, then $\overline{N}(G) = [\frac{M_+-1}{2}]$. If G is negative, then $\overline{N}(G) = [\frac{-M_--1}{2}]$. If G is mixed, then $\overline{N}(G) = [\frac{M_+-1}{2}] + [\frac{-M_--1}{2}] + 1$.*

The next corollary is a direct consequence of the above theorem:

Corollary 3.2. *Let G be a generalized horseshoe map, and M_+ and M_- be as in Theorem 3.1. Then G induces all link types, i.e., $\overline{N}(G) \geq 1$ if and only if G is one of the following types.*

- *G is positive and $M_+ \geq 3$.*
- *G is negative and $M_- \leq -3$.*
- *G is mixed.*

Recall that K_1, K_2, K_3 are generalized horseshoe maps in Example 2.3. By Theorem 3.1, $\overline{N}(K_1) = 0$, $\overline{N}(K_2) = 1$ and $\overline{N}(K_3) = 1$.

The proof of Theorem 3.1 is done by using the template theory ([2], [4], [5]).

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