数理解析研究所講究録 1229 巻 2001 年 10-18

A SURVEY OF KNOWN RESULTS ON 1-GENUS 1-BRIDGE KNOTS

日本女子大学 理学部 林忠一郎 CHUICHIRO HAYASHI
FACULTY OF SCIENCE
JAPAN WOMEN'S UNIVERSITY

We recall the definition of 1-genus 1-bridge knots. A properly imbedded arc t in a solid torus V is called *trivial* if it is boundary parallel, that is, there is a disc C imbedded in V such that $t \subset \partial C$ and $C \cap \partial V = \mathrm{cl}\,(\partial C - t)$. We call such a disc a *cancelling disc* of the trivial arc t. Let M be a closed connected orientable 3-manifold, and K a knot in M. The knot K is called a 1-*genus* 1-*bridge knot* in M if M is a union of two solid tori V_1 and V_2 glued along their boundary tori ∂V_1 and ∂V_2 and if K intersects each solid torus V_i in a trivial arc t_i for $i = 1$ and 2. The splitting $(M, K) = (V_1, t_1) \cup_H (V_2, t_2)$ is called a 1-*genus* 1-*bridge splitting* of (M, K), where $H = V_1 \cap V_2 = \partial V_1 = \partial V_2$, the torus. We call also the splitting torus H a 1-*genus* 1-*bridge splitting*. We say $(1,1)$-knots and $(1,1)$-splitting for short.

1-genus 1-bridge knots are very important in light of Heegaard splittings and Dehn surgeries as shown in the theorems below.

Theorem 0.1. (T. Kobayashi [15]) *Let M be a closed orientable connected 3-manifold of genus 2. Suppose that M admits a non-trivial torus decomposition. Then either (i) M is a union os an exterior of a $(1,1)$-knot and a Seifert fibered manifold over a disc with 2-exceptional fibers, or (ii)–(v), which we omit here.*

Let $(M,K) = (V_1,t_1) \cup_H (V_2,t_2)$ be a $(1,1)$-splitting. If there are an essential simple closed curve ℓ in the torus H and cancelling discs C_i of t_i in V_i for $i=1$ and 2 such that $C_i \cap \ell = \emptyset$, then we say that the knot (M,K) has a *satellite diagram* on the $(1,1)$-splitting torus H. At this time, the knot K has a 1-bridge diagram on an annulus in H. We say that the satellite diagram is of meridional (resp. longitudinal) slope if ℓ is of meridional (resp.longitudinal) slope of V_1 or V_2.

Theorem 0.2. (K. Morimoto and M. Sakuma [19]) *Let K be a satellite knot in the 3-sphere S^3 of tunnel number one. Then K is a satellite $(1,1)$-knot such that K has a satellite diagram of non-meridional and non-longitudinal slope on the $(1,1)$-splitting torus.*

It is well-known that all the $(1,1)$-knots are of tunnel number one.

Theorem 0.3. (D. Gabai [4]) *Let V be a solid torus, and K a knot in the interior of V. Suppose that a Dehn surgery on K yields a solid torus. Then K is a 1-bridge braid, that is, isotopic to a union of an arc α on ∂V and a trivial arc in a meridian disc D of V such that all the intersection points of α and ∂D are of the same sign.*

Note that K forms a $(1,1)$-knot when we imbed the 1-bridge braid (V,K) in a standard manner in a 3-manifold of genus 1.

Theorem 0.4. (A. Thompson [27]) *Let M be a closed connected orientable 3-manifold, and $M = W_1 \cup_H W_2$ a Heegaard splitting of genus 2. Suppose that this splitting has the disjoint curve property, that is, there are an essential simple closed curve ℓ in H and essential discs D_i of the handlebody W_i such that $\ell \cap (D_1 \cap D_2) = \emptyset$ Then M is non-hyperbolic or a result of a Dehn surgery on a $(1,1)$-knot.*

These theorems show that $(1,1)$-knots are important. There are many researches on $(1,1)$-knots as below. In the following, we assume that M is not homeomorphic to $S^2 \times S^1$ for simplicity.

Let V be a solid torus, and t a trivial arc in V. We call a disc D properly imbedded in V a *t-compressing disc* if D is disjoint from t and ∂D is essential in $\partial V - \partial t$.

Let $(M,K) = (V_1,t_1) \cup_H (V_2,t_2)$ be a $(1,1)$-splitting. The splitting is called *K-reducible* if there are t_i-compressing D_i in (V_i,t_i) for $i=1$ and 2 such that $\partial D_1 = \partial D_2$ in H.

Theorem 0.5. (H. Doll [3]) *Let M be a closed connected orientable 3-manifold of genus 1, and (M,K) a $(1,1)$-knot. Then the next three conditions are equivalent.*
(1) The knot K is split, that is, the exterior of K contains an essential 2-sphere.
(2) The $(1,1)$-splitting is K-reducible.
(3) K is the trivial knot, that is, it bounds an imbedded disc in M.

He has studied more general case of g-genus n-bridge knots.

Theorem 0.6. ([9]) *Let (S^3,K) be a $(1,1)$-knot. Then K is a trivial knot if and only if the $(1,1)$-splitting is K-reducible.*

Theorem 0.7. ([9], [13], [11]) *Let (M,K) be a $(1,1)$-knot. Then K is a core knot, that is, the exterior is a solid torus if and only if for $(i,j)=(1,2)$ or $(2,1)$ there are a meridian disc D of V_i such that $D \cap t_i = \emptyset$ and a cancelling disc C of t_j in V_j such that ∂C intersects ∂D transversely in a single point.*

Let V be a solid torus, and t a trivial arc in V. We call a meridian disc D of V a *meridionally compressing disc* if D intersects t transversely in a single point.

Let $(M, K) = (V_1, t_1) \cup_H (V_2, t_2)$ be a $(1,1)$-splitting. The splitting is called *weakly K-reducible* if there are properly imbedded discs D_i in V_i for $i = 1$ and 2 such that $\partial D_1 \cap \partial D_2 = \emptyset$ in H.

Lemma 0.8. ([10]) *Let (M, K) be a $(1,1)$-knot. Suppose that the $(1,1)$-splitting is weakly K-reducible. Then either (1) K is a core knot in a lens space, (2) K is a (maybe trivial) 2-bridge knot in S^3 or (3) K is a composite knot of a core knot and a 2-bridge knot.*

Theorem 0.9. (H. Doll [3]) *Let K be a $(1,1)$-knot. If K is a composite knot, then the $(1,1)$-splitting is weakly K-reducible.*

Theorem 0.10. (T. Kobayashi and O. Saeki [16]) *Let K be a 2-bridge knot in the 3-sphere S^3. Then any $(1,1)$-splitting of K is weakly K-reducible.*

Theorem 0.11. (K. Morimoto [18]) *Let K be a non-trivial non-core torus knot, where "torus" knot means that K can be isotoped into a Heegaard splitting torus. Then any $(1,1)$-splitting of K is cancellable, that is, there are cancelling discs C_i of t_i in V_i for $i = 1$ and 2 such that $\partial C_1 \cap \partial C_2 = \partial t_1 = \partial t_2$.*

We can push K along the discs C_1 and C_2 into the splitting torus.

Theorem 0.12. ([9]) *Let (M, K) be a $(1,1)$-knot. Suppose that K is a cabled knot, that is, there is a solid torus V in M such that $K \subset \partial V$ and that any meridian disc of V intersects K in two or more points. Then either (1) the $(1,1)$-splitting is K-reducible or weakly K-reducible, (2) K is a torus knot, or (3) K has a 1-bridge diagram on an annulus A in the splitting torus H such that each bridge is an essential arc in A.*

Theorem 0.13. ([10]) *Let (M, K) be a $(1,1)$-knot. Note that M may be a lens space. If K is a satellite knot, then the $(1,1)$-split admits a satellite diagram of a non-meridional non-longitudinal slope.*

Theorem 0.14. (H. Matsuda [17]) *Let* (S^3, K) *be a non-trivial* $(1,1)$*-knot. Suppose that* K *bounds a Seifert surface* F *of genus* 1. *Then either (1)* K *is a* 2*-bridge knot and* F *is a plumbing sum of two twisted unknotted annulus or (2)* F *is obtained from an essential annulus* A *in the* $(1,1)$*-splitting torus* H *by adding a twisted band along an essential arc in* $H - \partial A$.

Theorem 0.15. (M. Hirasawa and C. Hayashi [12]) *Let* $(M, K) = (V_1, t_1) \cup_H (V_2, t_2)$ *be a* $(1,1)$*-splitting. Let* F' *be a closed connected orientable surface of genus* 2 *imbedded in* M *such that* K *is contained in* F' *and that* F *intersects the knot exterior in an incompressible and boundary incompressible surface. Then* F' *can be isotoped to intersect each solid torus* V_i *in zero or some number of* ∂*-parallel annuli disjoint from* K *and one of the surfaces of four types (a)–(b) as below :*
(a) ∂*-parallel once punctured torus which contains the arc* t_i*,*
(b) an annulus A *which is parallel to an annulus* A' *in* ∂V*, contains the arc* t_i*, and added a non-twisted band* B *along an essential arc in* A'*, so that* $A \cup B$ *forms a once punctured torus,*
(c) a pair of pants P *such that* P *is* ∂*-parallel in* ∂V_i*, that* P *contains the arc* t_i*, that precisely two components of* ∂P *is essential in* ∂V*, and that* ∂t_i *is contained in the other component of* ∂P*,*
(d) an annulus Z *which is parallel to an annulus* Z' *in* ∂V*, contains the arc* t_i*, and added a non-twisted band* C *along an inessential arc in* A'*, so that* $Q = Z \cup C$ *forms a pair of pants and that the inessential component of* ∂Q *contains* ∂t_i*.*

These theorems are on $(1,1)$-splittings of special $(1,1)$-knots. How about $(1,1)$-splittings of general $(1,1)$-knots?

Following theorem helps study of $(1,1)$-splittings. This is a generalization of a result by H. Rubinstein and M.Scharlemann [22].

Theorem 0.16. (T. Kobayashi and O. Saeki [16]) *Let M be a closed connected orientable 3-manifold. Let L be a link in M. Suppose that M has a 2-fold branched covering with the branched set L. Let H_i be a (g_i, n_i)-splitting of (M, L) for $i = 1$ and 2. Suppose that the splittings are not weakly L-reducible. Then after an adequate isotopy H_1 and H_2 intersect each other transversely in a non-empty collection of L-essential loops, that is, none of the loops $H_1 \cap H_2$ bounds a disc D in H_1 or H_2 such that D is disjoint from L or intersects L in a single point.*

There are some notes on the above theorem.
(1) A $(1,1)$-splitting is a special case of a (g,n)-splitting.
(2) The condition "non-empty" is very important because we can isotope H_1 and H_2 to be disjoint from each other.
(3) The projective space $\mathbb{R}P^3$ does not have a branched covering with the branched set a core knot, for example.
(4) The author expect that the above theorem holds when there is not such a branched covering.

Theorem 0.17. ([11]) *Let M be the 3-sphere S^3 or a lens space. Let K be a knot in M. Let H_1 and H_2 be $(1,1)$-splitting tori of (M, K). Suppose that H_1 and H_2 intersect each other transversely in a non-empty collection of K-essential loops. Then after an adequate isotopy either*
(1) H_1 and H_2 are isotopic to each other in (M, K),
(2) one of the splittings H_1 and H_2 is weakly K-reducible,
(3) K is a satellite knot, or
(4) H_1 and H_2 intersect each other transversely in 1 or 2 K-essential loops.

Theorem 0.18. ([11]) *In case (4) in the previous theorem, after an adequte isotopy at least one of the next four conditions (a)–(d) holds.*
(a) One of (1)–(3) in the conclusion of the previous theorem holds.
(b) (M, K) is a sum of two tangles (B, T) and (X, S) as below. (B, T)

is a trivial 2-string tangle. X is a once punctured lens space and S is a disjoint union of two arcs s_1 and s_2 properly imbedded in X such that $E_i = cl(X - N(s_i))$ is a solid torus and that s_j is parallel to the boundary ∂E_i for $(i,j) = (1,2)$ or $(2,1)$. The $(1,1)$-splitting torus H_i is obtained from ∂X by applying a tubing operation along the arc s_i for $i = 1$ and 2.

(c) One of the splittings H_1 and H_2 admits a satellite diagram of a longitudinal slope.

(d) There is a solid torus V in M as below. The exterior of the solid torus is also a solid torus. The knot K intersects V in two arcs. There are disjoint union of two discs D_1 and D_2 in ∂V as below. There are disjoint union of two balls B_1 and B_2 such that $B_i \cap V = D_i$, that $K \cap B_i$ is an arc, that K intersects the solid torus $V \cup B_i$ in a trivial arc, and that H_i is isotopic to $\partial V \cap B_i$ for $i = 1$ and 2.

In case (c), the knot K is obtained from a component L_1 of a 2-bridge link $L_1 \cup L_2$ by a Dehn surgery on the other component L_2.

The author is not satisfied with the conclusion (d).

References

1. J. Berge *The knots in $D^2 \times S^1$ with non-trivial Dehn surgery yielding $D^2 \times S^1$*, Topology Appl. 38 (1991) 1–19
2. D. H. Choi and K. H. Ko *On 1-bridge torus knots*, preprint
3. H. Doll, *A generalized bridge number for links in 3-manifolds*, Math. Ann. 294 (1992) 701–717.
4. D. Gabai *Surgery on knots in solid tori*, Topology 28 (1989) 1–6
5. D. Gabai *1-bridge braids in solid tori*, Topology Appl. 37 (1990) 221–235
6. H. Goda and C. Hayashi *Genus two Heegaard splittings of exteriors of 1-genus 1-bridge knots*, preprint
7. H. Goda, C. Hayashi and N. Yoshida *Genus two Heegaard splittings of exteriors of knot and the disjoint curve property*, to appear in Kobe J. Math.

8. L. Grasselli and M. Mulazzani *Genus one* 1-*bridge knots and Dunwoody mainfolds*, to appear in Forum Math.
9. C. Hayashi, *Genus one* 1-*bridge positions for the trivial knot and cabled knots*, Math. Proc. Camb. Phil. Soc. 125 (1999), 53–65.
10. C. Hayashi *Genus one* 1-*bridge positions for the trivial knot and cabled knots*, Math. Proc. Camb. Phil. Soc. 125 (1999) 53–65
11. C. Hayashi *Satellite knots in* 1-*genus* 1-*bridge positions*, Osaka J. Math. 36 (1999) 203–221
12. C. Hayashi and M. Hirasawa *Essential positions on Heegaard splitting genus two surface for* 1-*genus* 1-*bridge knots*, in preparation
13. J. Hempel *3-manifolds as viewed from the curve complex*, Topology 40 (2001), 631–657
14. P. Hoidn *On* 1-*bridge genus of small knots*, preprint
15. T. Kobayashi *Structures of the Haken manifolds with Heegaard splitting of genus two*, Osaka J. Math. 21 (1984) 437–455
16. T. Kobayashi and O. Saeki *Rubinstein-Scharlemann graphic of 3-manifold as the discriminant set of a stable map*, Pacific J. Math. 195 (2000) 101–156
17. H. Matsuda *Genus one knots which admit (1,1)-decompositions*, to appear in Proc. Am. Math. Soc.
18. K. Morimoto *On minimum genus Heegaard splittings of some orientable closed* 3-*manifolds*, 321–355 (1989) 12 No.2 Tokyo J. Math.
19. K. Morimoto and M. Sakuma *On unknotting tunnels for knots*, Math. Ann. 289 (1991) 143–167
20. K. Morimoto, M. Sakuma and Y.Yokota *Examples of tunnel number one knots which have the property '1+1=3'*, Math. Proc. Camb. Phil. Soc. 119 (1996) 113–118
21. M. Mulazzani *Cyclic presentations of groups and branched cyclic coverings of* $(1,1)$-*knots*, preprint
22. H. Rubinstein and M. Scharlemann, *Comparing Heegaard splittings of non-Haken 3-manifolds*, Topology 35 (1996), 1005-1026.
23. T. Saito *Genus one 1-bridge knot as viewed from the curve complex*, in preparation
24. H. J. Song *Dunwoody* $(1,1)$-*decomposable knots*, preprint
25. H. J. Song and K. H. Ko *Spatial* θ-*curve associated with Dunwoody* $(1,1)$-*decomposable knots*, preprint

26. H. J. Song and S. H. Kim *Dunwoody 3-manifolds and* $(1,1)$*-decomposible knots*, preprint
27. A. Thompson *The disjoint curve property and genus 2 manifolds*, Topology Appl. 97 (1999) 237–279
28. Y-Q. Wu *∂-reducing Dehn surgeries and 1-bridge knots* Math. Ann. 295 (1992) 319–331
29. Y-Q. Wu *Incompressible surfaces and Dehn surgery on 1-bridge knots in handlebodies*, Math. Proc. Cambridge Phil. Soc. 120 (1996) 687–696

〒112－8681
東京都文京区目白台2－8－1
日本女子大学理学部数物科学科
林忠一郎

Chuichiro Hayashi:
Department of Mathematical and Phisical Sciences, Faculty of Science,
Japan Women's University,
Mejiro, Toshima-ku, Tokyo, 171-8588, Japan.

e-mail : hayashic@fc.jwu.ac.jp