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# ON 1-BRIDGE TORUS KNOTS

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ABSTRACT. A 1-bridge torus knot is a knot drawn on a standard torus in $S^3$ with 1-bridge. We introduce two types of normal forms to parametrize the family of 1-bridge torus knots that are similar to the Schubert's normal form and the Conway's normal form for 2-bridge knots. For a given Schubert's normal form we give a classificatoin of some sub-class of 1-bridge torus knots. We also give a description of the double brannced cover of $S^3$ branched along any 1-bridge torus knots by using the Conway's normal form and obtain an explicit formula for the first homology of the double cover.

## 1. INTRODUCTION

One of traditions in knot theory is to study a family of knots satisfying a certain condition. Examples of such families include the family of torus knots studied by Dehn and Schreier and the family of 2-bridge knots studies by Schubert, Montesinos and Conway. These classes can be referred as the classes of knots and links indexed by the pairs $(g, b)$ of non-negative integers as defined in [9]. A knot $K$ in a 3-manifold $M$ has a $(g, b)$-decomposition or is called a $(g, b)$-knot if for some heegaard splitting $M = U \cup V$ of genus $g$, each of $K \cap U$ and $K \cap V$ is consisted of trivial $b$ arcs. A collection of properly embedded arcs in a 3-manifold $W$ with boundary is *trivial* if arcs $\alpha$ in the collection together with arcs on $\partial W$ joining the two ends of the arcs bound mutually disjoint disks in $W$. A $(g, b)$-knot can be embedded in a heegaard surface of genus $g$ in $M$ except at $b$ over(or under)-bridges and vice versa. Torus knots are $(1, 0)$-knots and 2-bridge knots are $(0, 2)$-knots. Clearly the family of $(g, b)$-knots becomes strictly larger as $g$ or $b$ increase. Since an over-bridge can

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be removed by adding a handle and by embedding the over-bridge into the added handle, $(g,b)$-knots are contained in the family of $(g+1,b-1)$-knots.

In this article we study the family of 1-bridge torus knots, that is, $(1,1)$-knots in $S^3$. This family contains torus knots and 2-bridge knots and is contained in the family of double torus knots, that is, $(2,0)$-knots. Hill and Murasugi studied the family of double torus knots in [11, 12] and parametrized the family. Non-trivial knots with the trivial Alexander polynomial was found in the subfamily of double torus knots that separate the double torus. They also considered non-separating double torus knots and a subfamily of 1-bridge torus knots and found various double torus knots that are fibered.

The 1-bridge torus knot has the tunnel number one, but not all tunnel-number-one knots are 1-bridge torus knots. In [14], Morimoto, Sakuma and Yokota found tunnel-number-one knots that are not 1-bridge torus knots as confirmed by a condition on the Jones polynomial for a knot to admit a $(g,b)$-decomposition in [18]. In [15], they gave another criteria to determine whether a given knot has the tunnel number one and whether it is a 1-bridge torus knot.

Besides torus knots and 2-bridge knots, the family of 1-bridge torus knots includes Berge's double-primitive knots, 1-bridge braids that were classified by Gabai in [10] and satellite 1-bridge torus knots. Morimoto and Sakuma studied satellite 1-bridge torus knots and classified their unknotting tunnels in [13].

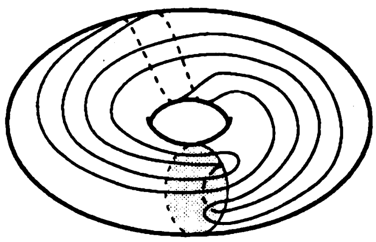

FIGURE 1. 1-bridge torus knot

In this article, we parameterize the family of 1-bridge torus knots using two kinds of normal forms as done for the family of 2-bridge knots. Schubert described a 2-bridge knots by a pair of integers of a certain condition from its top view. In the top view a 2-bridge knots is embedded in a plane except the two bridges. He in fact completely classified 2-bridge knots using this normal form [17]. Since a 1-bridge knot can be embedded in a standard torus except the bridge (See Figure 1), we will describe it by a 4-tuple of integers from this top view. We will call such a 4-tuple the *Schubert's normal form* of the 1-bridge torus knot determined by the 4-tuple. In Section 2, we introduce the Schubert's normal forms of 1-bridge torus knots and classify some subfamily of 1-bridge torus knots expressed the Schubert's normal forms.

On the other hand, a 2-bridge knot can also be viewed as a 4-plats as studied first in [2]. From this side view, it is easy to see that the composition of homeomorphisms of a four-punctured sphere that determines the 2-bridge knot. Using this description, Conway constructed a bijection between 2-bridge knots and lens spaces via double branched covers [8]. A similar description using the composition of homeomorphisms on a two-punctured torus is possible for 1-bridge torus knots and this will be called the *Conway's normal form.* In Section 3, we construct the double branched cover of $S^3$ branched along an 1-bridge torus knot given by the Conway's normal form and give a formula for the first homology of the branched double cover.

## 2. Schubert's normal forms

In this section, we introduce a notation describing a 1-bridge torus knot which is called Schubert's normal form and give a classification of subfamily of 1-bridge torus knots. The Schubert's normal form of a 1-bridge torus knot is an analogue of the Schubert's normal form of 2-bridge knot or link.

### 2.1. **Schubert's normal forms.**

**Theorem 2.1.** [6] *Any 1-bridge torus knots is represented by a 4-tuple* $(r, s, t, \rho)_\epsilon$*, where* $r, s, t$ *are non-negative integers,* $\rho$ *is an integer and* $\epsilon$ *is a sign* $\pm 1$.

In the Schubert's normal form of a 1-bridge torus knot, $r, s, t$ and $\epsilon$ determine the shape of the knot in the neighborhood of a meridian disk containing the bridge (See Figure 2), and $\rho$ means the rotation number (See Figure 3).

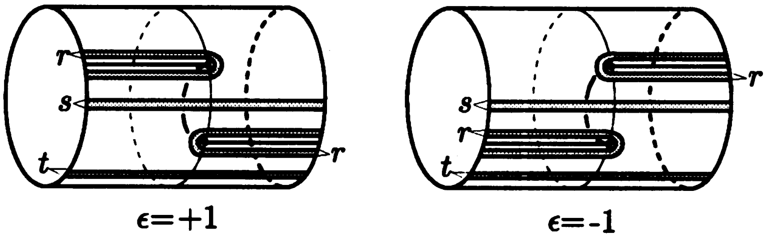


FIGURE 2

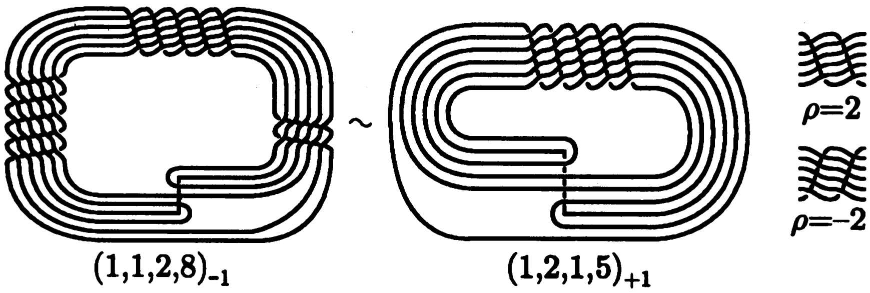


FIGURE 3. Schubert's normal forms of 1-bridge torus knots

**Remark 2.2.**

(1) $(r, s, t, \rho)_{+1} = (r, t, s, \rho + (2r + 1))_{-1}$ (See Figure 3).

(2) A 1-bridge torus knot with $(r, s, t, \rho)_{+1}$ is a mirror image of a 1-bridge torus knot with $(r, s, t, -\rho)_{-1}$.

(3) If $r = 0$ in the normal form, then it represents a 1-bridge braid(See [10]).

(4) A $(p, q)$-torus knot is a 1-bridge torus knot $(0, 0, p-1, -q)_{+1}$ or $(0, p-1, 0, -q+1)_{-1}$.

(5) Any 2-bridge knot in $S^3$ has a Schubert's normal form $B(\alpha, \epsilon\beta)$ (See Chapter 3 of [1]), where

$$\alpha > 0, \ 0 < \beta < \alpha, \ \epsilon = \pm 1, \ \gcd(\alpha, \beta) = 1, \text{ and } \alpha, \beta \text{ odd.}$$

A 2-bridge knot $B(\alpha,\epsilon\beta)$ is a 1-bridge torus knot $(\beta-1,\alpha-2\beta+1,0,\epsilon)_\epsilon$ (See Figure 4).

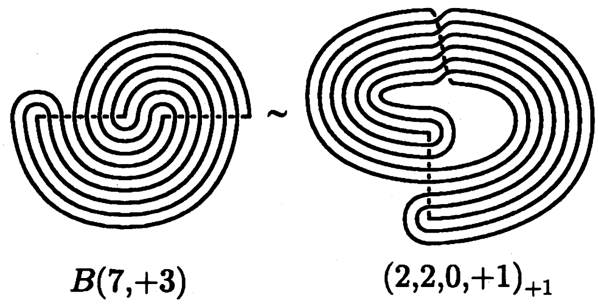


FIGURE 4

(6) K. Morimoto and M. Sakuma showed that any satellite knot which admits an unknotting tunnel is equivalent to a knot represented by $K(\alpha,\epsilon\beta;p,q)$ in [13], where

$$\alpha \text{ even integer}, p,q \text{ positive integers}, \ \epsilon=\pm 1 \text{ and } 0<\beta<\alpha/2.$$

The knot $K(\alpha,\epsilon\beta;p,q)$ is a 1-bridge torus knot $(\frac{\beta-1}{2},\frac{\alpha-2\beta}{2},\frac{\alpha}{2}p,\frac{\alpha}{2}q)_\epsilon$.

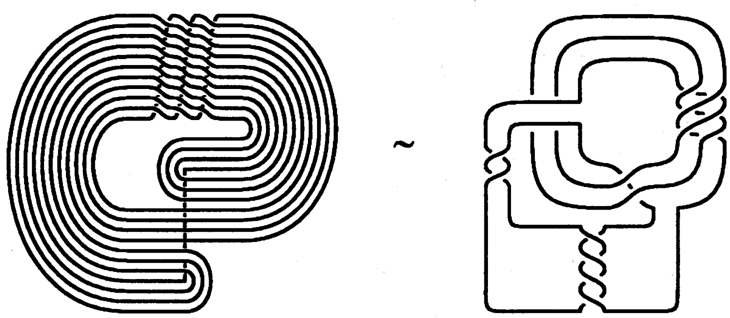

FIGURE 5. $K=(3,4,0,-3)_{-1}$

2.2. **sub-class $(r,s,0,\epsilon(s-1))_\epsilon$ of 1-bridge torus knots.** Consider $(r,s,0,\epsilon(s-1))_\epsilon$, where $r\geq 0$, $s>0$ are integers and $\epsilon=\pm 1$(See Figure 5).

**Lemma 2.3.** *$(r,s,0,\epsilon(s-1))_\epsilon$ is always the Schubert's normal form of 1-bridge torus knot. Furthermore, $(r,s,0,-(s-1))_{-1}$ is a mirror image of $(r,s,0,s-1)_{+1}$.*

*Proof.* For $(s-1, 0, 2(r+1), s)$, we get "1" from Compoment Counting Algorithm in [6], since $\gcd(1, 2r+s+1) = 1$. Therefore, $(r, s, 0, \epsilon(s-1))_\epsilon$ satisfies the conditions of Schubert's normal form. □

Since if $s = 1$ then $(r, 1, 0, 0)_\epsilon$ represents the unknot, we may assume that $s > 1$.

**Theorem 2.4.** [7] *Let $K_{r,s}$ be a 1-bridge torus knot $(r, s, 0, (s-1))_{+1}$. Then a genus of $K_{r,s}$ is*

$$\begin{cases} 2 + \dfrac{s(s-3)}{2} & \text{if } r \text{ is odd,} \\ \dfrac{s(s-1)}{2} & \text{if } r \text{ is even.} \end{cases}$$

*Furthermore, $K_{r,s}$ is fibred if and only if $r = 0$ or $1$.*

Using Theorem 2.4, we get the following corollary;

**Corollary 2.5.** *$K_{r,s}$ is not isotopic to $K_{\bar{r},\bar{s}}$ if $r \neq \bar{r}$ or $s \neq \bar{s}$.*

*Proof.* Suppose $K_{r,s}$ is isotopic to $K_{\bar{r},\bar{s}}$.

**Case 1)** $r = \bar{r}$

If $r$ and $\bar{r}$ are odd then by Theorem 2.4,

$$2 + \frac{s(s-3)}{2} = g(K_{r,s}) = g(K_{\bar{r},\bar{s}}) = 2 + \frac{\bar{s}(\bar{s}-3)}{2}$$

Therefore, $s = \bar{s}$ or $s + \bar{s} = 3$. Since $s \neq \bar{s}$, $s + \bar{s} = 3$ and so $s$ or $\bar{s}$ is 1 but this is impossible, since $s, \bar{s} > 1$.

If $r, \bar{r}$ are even then similarly, we meet a contradiction.

**Case 2)** $r \neq \bar{r}$

If $r$ and $\bar{r}$ are even(or odd), then by the method of Case 1, we meet a contradiction. So we may assume that $r$ is odd and $\bar{r}$ is even.

$$2 + \frac{s(s-3)}{2} = \frac{\bar{s}(\bar{s}-1)}{2}$$

Then integer solutions of the above equation are

$$\begin{cases} s = 1 \\ \bar{s} = 2 \end{cases}, \begin{cases} s = 1 \\ \bar{s} = 0 \end{cases}, \begin{cases} s = 1 \\ \bar{s} = -1 \end{cases} \text{ and } \begin{cases} s = 2 \\ \bar{s} = 2 \end{cases}$$

Therefore, the only possibility is the last solution. That is, $s = \bar{s} = 2$. Then $K_{r,s}$(or $K_{\bar{r},\bar{s}}$) is a 2-bridge knot $B(2r+s+1, r+1)$(resp. $B(2\bar{r}+\bar{s}+1, \bar{r}+1)$) (See 5. of Remark 2.2). Hence, $r = \bar{r}$. But this is impossible. □

**Theorem 2.6.** [7]

$$\Delta_{r,s}(t) \doteq \begin{cases} \dfrac{(t-1)(t^{s(s+1)}-1)}{(t^s-1)(t^{s+1}-1)} + \dfrac{r}{2}\dfrac{(t-1)^2(t^{s^2-1}-1)}{(t^{s+1}-1)} & \textit{if } r \textit{ is even,} \\ t^{(s^2-3s+4)/2} - \dfrac{r+1}{2}\dfrac{(t-1)^2(t^{(s-1)^2}-1)}{(t^{s-1}-1)} & \textit{if } r \textit{ is odd, } s=2 \textit{ or } 3, \\ \dfrac{t^{s-1}(t-1)(t^{(s-2)(s-1)}-1)}{(t^{s-2}-1)(t^{s-1}-1)} - \dfrac{r+1}{2}\dfrac{(t-1)^2(t^{(s-1)^2}-1)}{(t^{s-1}-1)} & \textit{if } r \textit{ is odd, } s \geq 4, \end{cases}$$

*where $\Delta_{r,s}(t)$ is the Alexander polynomial of $K_{r,s}$.*

**Corollary 2.7.** *$K_{r,s}$ is fibred if and only if its Alexander polynomial is monic.*

*Proof.* From Theorem 2.6,

$$\text{the leading coefficient of } \Delta_{r,s}(t) = \begin{cases} 1+\dfrac{r}{2} & \text{if } r \text{ is even,} \\ -\dfrac{r+1}{2} & \text{if } r \text{ is odd,} \end{cases}$$

and $K_{r,s}$ is fibred iff $r = 0$ or $1$ by Theorem 2.4. Hence, the proof is complete. □

Recently, K. Murasugi and M. Hirasawa conjectured the above statement for twisted torus knots. They proved that it is true for the type 1:1 non-separable double torus knots and M. Hirasawa showed that the statement is also true for the sub-class of twisted torus knots. Therefore, their conjecture is true for our class.

**Theorem 2.8.** [7] *For $r \geq 0$, $s \geq 2$,*

$$\text{(1)} \qquad V_{K_{r,s}}(t) = \frac{t^{3(s-1)\delta_r - s(s+1)/2}}{1-t^2}\Big(\sum_{i=0}^{r-1}(-t^{-1})^i A(t) + (-t^{-1})^r B(t)\Big) \quad \textit{if } r \geq 1,$$

$$\text{(2)} \qquad V_{K_{0,s}}(t) = \frac{t^{-s(s+1)/2}}{1-t^2}B(t),$$

*where $A(t) = 1 - t^{s+2} - t^{2(2-s)} + t^{2-s}$ and $B(t) = 1 - t^{1-s} - t^{s+2} + t$.*

**Corollary 2.9.** [7] *$K_{r,s}$ $r \geq 0$, $s \geq 2$ is non-amphicheiral except for $K_{1,2}$ which is a figure-eight knot.*

Corollary 2.5 and Corollary 2.9 give us the classification of the 1-bridge torus knots with normal forms $(r, s, 0, \epsilon(s-1))_\epsilon (r \geq 0,\ s \geq 2)$.

**Theorem 2.10.** *For any two 1-bridge torus knots $K, K'$ with normal forms $(r, s, 0, \epsilon(s-1))_\epsilon$ $(r', s', 0, \epsilon'(s'-1))_{\epsilon'}$, respectively,*

$$K \text{ is not isotopic to } K' \text{ if } \epsilon \neq \epsilon',\ r \neq r' \text{ or } s \neq s'$$

*except for $(1, 2, 0, 1)_{+1} = (1, 2, 0, -1)_{-1}$ which is a figure-eight knot.*

## 3. Conway's normal forms and Double branched covers

In this section, we concern about double branched covers of a 3-sphere branched along the 1-bridge torus knots. In order to this, we use an analogue of Conway's normal form of the 2-bridge knot (See Chapter 12. in [1] and Chapter 10. in [16]).

3.1. **Conway's normal forms of 1-bridge torus knots.** Let $K$ be a 1-bridge torus knot, $(V_1, t_1) \cup_h (V_2, t_2)$ a (1,1)-decomposition of $(S^3, K)$ and $\bar{t}_2$ be an arc on $\partial V_2$ such that $t_2 \cup \bar{t}_2$ bounds a disk in $V_2$. Then $h$ is a homeomorphism from $\partial V_2$ onto $\partial V_1$ which is isotopic to a homeomorphism $h_0$ sending a meridian(resp. longitude) in $\partial V_2$ to a longitude(resp. meridian) in $\partial V_1$, and $t_1 \cup h(\bar{t}_2)$ is isotopic to $K$. Then $\bar{h} = h h_0^{-1}$ is a homeomorphism on a torus isotopic to the identity.

The mapping class group $M(1,2)$ of a two-punctured torus is generated by $d_m$, $d_\ell$, $\tau_\ell$, $\tau_m$ and $\sigma$ (For exmaple, see Chapter 4. in [5]), where $d_m$(resp. $d_\ell$) is a Dehn-twist along the meridian(resp. longitude), $\sigma$ is a homeomorphism exchanging two punctures and $\tau_m$ (resp. $\tau_\ell$) is a homeomorphism sliding one of punctures along the meridian (resp. longitude) as illustrated in Figure 6. By forgetting the punctures, a homomorphimsm $j_*\colon M(1,2) \to M(1,0)$ into the mapping class group of a torus is induced. Then $\bar{h}$ is in $\ker j_*$. Therefore we can say that an element of $\ker j_*$ represents a (1,1)-decomposition of a 1-bridge torus knot.

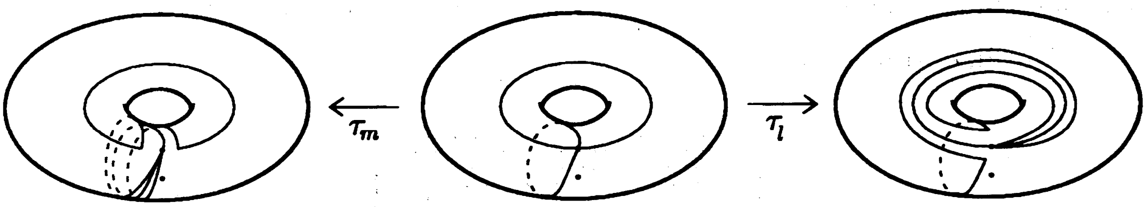


FIGURE 6

Consider the homoemorphisms $h_\ell = \tau_\ell \sigma \tau_\ell^{-1}$ and $h_m = \tau_m \sigma^{-1} \tau_m^{-1}$. Then

$$\tau_\ell^2 = h_\ell \sigma \qquad \text{and} \qquad \tau_m^2 = h_m^{-1} \sigma.$$

The homeomorphisms $h_\ell$ and $h_m$ have an effect on the arc $t_1$ in $V_1$ as illustrated in Figure 7.

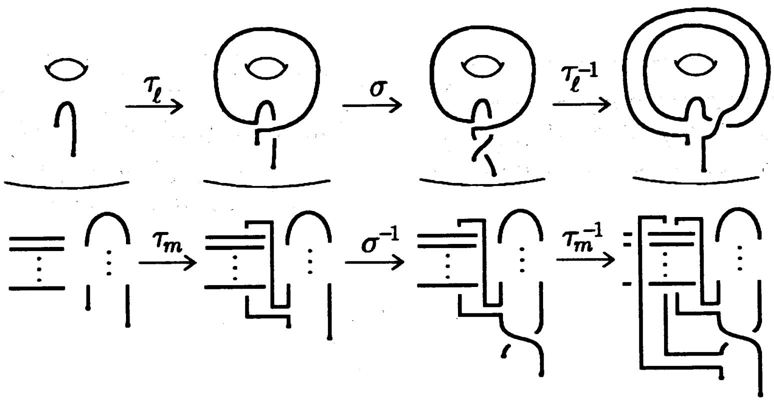


FIGURE 7

For integers $a_1, \ldots, a_m, b_1, \ldots, a_m$,

(3) $$[(a_1, b_1, a_2, b_2), (a_3, b_3, a_4, b_4), \ldots, (a_{m-1}, b_{m-1}, a_m, b_m)]$$

represents a 1-bridge torus knot in $S^3$ that has a (1,1)-decomposition $(V_1, t_1) \cup_h (V_2, t_2)$ such that $h = \bar{h} h_0$ and

$$\bar{h} = (h_\ell^{a_1} \sigma^{b_1} h_m^{a_2} \sigma^{b_2})(h_\ell^{a_3} \sigma^{b_3} h_m^{a_4} \sigma^{b_4}) \cdots (h_\ell^{a_{m-1}} \sigma^{b_{m-1}} h_m^{a_m} \sigma^{b_m}).$$

The above (1,1)-decomposition of a 1-bridge torus knot will be called a *Conway's normal form* of a 1-bridge torus knot.

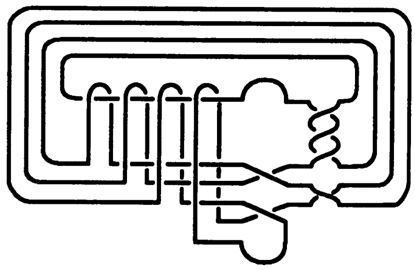

FIGURE 8. Conway's normal form $[(3,0,1,0),(-1,0,1,0)]$

**Theorem 3.1.** [6] *Every 1-bridge torus knot has a Conway's normal form.*

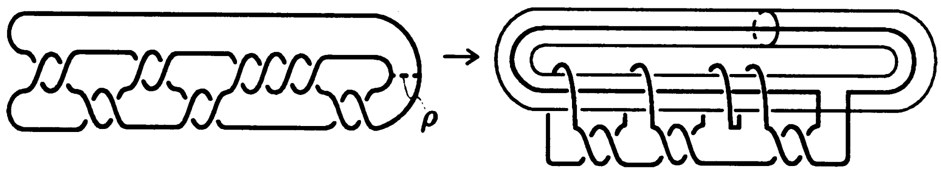


FIGURE 9

**Remark 3.2.** *A 2-bridge knot has the Conway's normal form* $[2a_1, 2a_2, \ldots, 2a_m]$ *as illustrated in Figure 9. We choose a (1,1)-tunnel* $\rho$ *as in Figure 9. Then we get a (1,1)-decomposition of it and the attaching homeomorphism of the (1,1)-decomposition is* $h_0$ $(\tau_\ell\sigma^{-1})(\sigma^{-2a_m}\tau_m^{a_{m-1}})\cdots(\sigma^{-2a_2}\tau_m^{a_1})$*(See Figure 9). By using the relations of* $\ker j_*$ *we can obtain a Conway's normal form of 1-bridge torus knot for the given 2-bridge knot.*

3.2. **Double branched covers along 1-bridge torus knots.** Consider a double branched cover $\Sigma$ of a solid torus $V$ branched along a trivial arc in $V$, which is a genus two handlebody (See Figure 10).

Then from Figure 10 and Figure 11, the following facts are evident;

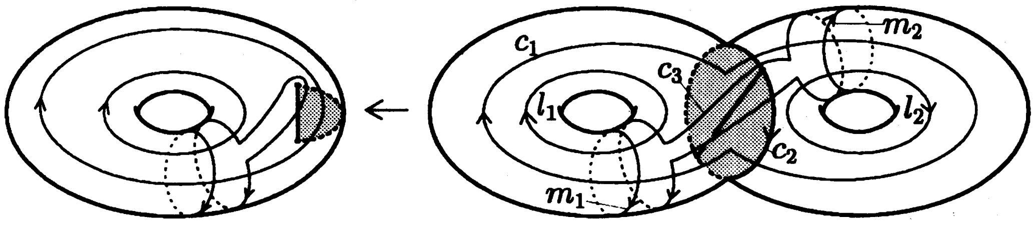


FIGURE 10. Double branched cover of a solid torus branched along an arc

(1) The lifting of $h_0$, $\tilde{h}_0$, is a homeomophism of $\partial\Sigma$ such that $\tilde{h}_0(m_1) = l_1$, $\tilde{h}_0(m_2) = l_2$, $\tilde{h}_0(l_1) = m_1$ and $\tilde{h}_0(l_2) = m_2$.

(2) The lifting of $\sigma$, $\tilde{\sigma}$, is $d_{c_2}$, where $c_2$ is a curve as shown in Figure 10.

(3) The lifting of $h_\ell$, $\tilde{h}_\ell$, is $d_{c_1}^{-1} d_{l_1}^2 d_{l_2}^2 d_{c_2}^{-1}$,

(4) The lifting of $h_m$, $\tilde{h}_m$, is $d_{c_3}^{-1} d_{m_1}^2 d_{m_1}^2 d_{c_2}$, where $c_i (i = 1, 2, 3)$ is a curve depicted at Figure 10.

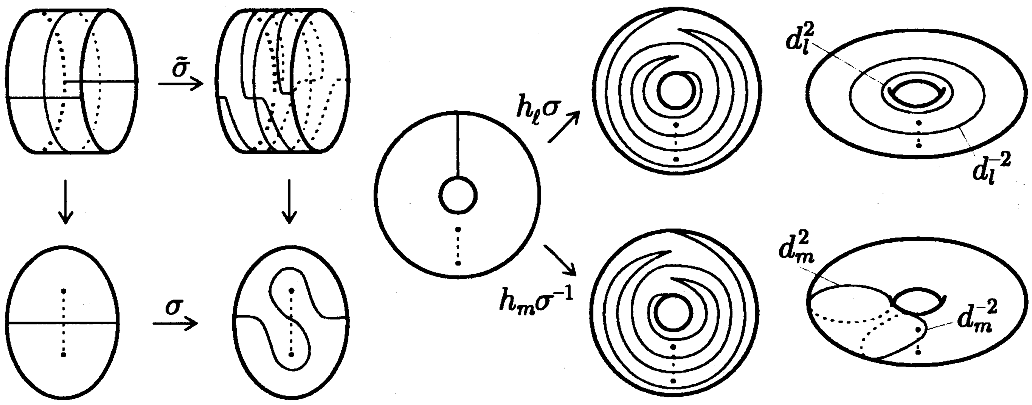


FIGURE 11. The lifting of $\sigma$ and the homoemorphisms $h_\ell$, $h_m$

Therefore, we can obtain the following theorem;

**Theorem 3.3.** *If a 1-bridge torus knot $K$ has a Conway's normal form*

$$[(a_1, b_1, a_2, b_2), (a_3, b_3, a_4, b_4), \ldots, (a_{m-1}, b_{m-1}, a_m, b_m)]$$

*then the double branched cover $X_2$ of $S^3$ branched along $K$ has a genus two $H$ splitting $\Sigma_1 \cup_{\tilde{h}} \Sigma_2$ such that $\Sigma_i (i = 1, 2)$ is a genus two handlebody and*

$$\tilde{h} = (\tilde{h}_\ell^{a_1}\tilde{\sigma}^{b_1}\tilde{h}_m^{a_2}\tilde{\sigma}^{b_2})(\tilde{h}_\ell^{a_3}\tilde{\sigma}^{b_3}\tilde{h}_m^{a_4}\tilde{\sigma}^{b_4})\cdots(\tilde{h}_\ell^{a_{m-1}}\tilde{\sigma}^{b_{m-1}}\tilde{h}_m^{a_m}\tilde{\sigma}^{b_m})\tilde{h}_0.$$

**Lemma 3.4.** [6]

$$\begin{aligned}\tilde{h}_*([m_1]) &= 1/2z_m[m_1] + (a_m z_m + 1/2(z_{m-1}+1))[l_1]\\ &\quad - 1/2z_m[m_2] - (a_m z_m + 1/2(z_{m-1}-1))[l_2],\end{aligned}$$

*where $z_m$ is a sequence such that $z_m = 2a_{m-1}z_{m-1} + z_{m-2}$, $z_0 = 0$ and $z_1 = 1$.*

**Proposition 3.5.** *Let $z_m$ be a sequence satisfying the following recursive formula;*

$$z_{m+1} = 2a_m z_m + z_{m-1}, z_0 = 0 \text{ and } z_1 = 1,$$

*where $a_m$ is a sequence. Then*

$$z_{m+1} = 2^m(a_1a_2\cdots a_m) + \sum_{t=1}^{[\frac{m}{2}]} 2^{m-2t} \sum_{\substack{(j_1,\ldots,j_t)\\ \in C_m^t}} A(j_1, j_2, \ldots, j_t), \tag{4}$$

*where*

$$C_m^t = \{(j_1, \ldots, j_t) \in \mathbf{N}^t \mid 1 \le j_1 < \cdots < j_t < m,\ j_k - j_{k-1} \ge 2,\ k = 1, \ldots, t\}$$

$$A(j_1, j_2, \ldots, j_t) = (a_1a_2\cdots a_{j_1-1})(a_{j_1+2}\cdots a_{j_2-1})\cdots(a_{j_t+2}\cdots a_m),$$

*and $A(1, 3, \cdots, m-1) = 1$ when $m$ is even.*

From Lemma 3.4 and Proposition 3.5, we caculate the first homology of $X_2$.

**Theorem 3.6.** *Let $K$ be a 1-bridge torus knot with the Conway's normal form*

$$[(a_1, b_1, a_2, b_2), (a_3, b_3, a_4, b_4), \ldots, (a_{m-1}, b_{m-1}, a_m, b_m)],$$

*and $X_2$ be a double branched cover of $S^3$ branched along $K$. Then $H_1(X_2) = \mathbf{Z}/|z,$ where $z_{m+1}$ is a sequence at the formula (4).*

*Proof.* By excision, $H_1(X_2) \cong H_1(\Sigma_1 \cup (B_1 \cup B_2))$, where $B_1, B_2$ are the tubular neighborhoods of meridian disks $D_1$, $D_2$ of $\Sigma_2$. And by Mayer-Vietoris sequence, $H_1(X_2) = \text{Coker}(f\colon H_1(A_1 \cup A_2) \to H_1(\Sigma_1))$, where $A_i = \partial D_i \times I (i = 1, 2)$. Since $[m_1]$ and $[m_2]$ generate $H_1(A_1 \cup A_2)$ and $H_1(\Sigma_1) = \langle l_i, m_i \mid m_i = 0, i = 1, 2 \rangle$, $f([m_i]) = \tilde{h}_*([m_i])$, $i = 1, 2$. From Lemma 3.4 and the periodic property of $X_2$,

$$f([m_1]) = (a_m z_m + 1/2(z_{m-1} + 1))[l_1] - (a_m z_m + 1/2(z_{m-1} - 1))[l_2],$$

$$f([m_2]) = -(a_m z_m + 1/2(z_{m-1} - 1))[l_2] + (a_m z_m + 1/2(z_{m-1} + 1))[l_1].$$

Therefore, $H_1(X_2) = \langle l_1, l_2 \,|\, R \rangle$, where

$$R = \begin{bmatrix} (a_m z_m + 1/2(z_{m-1} + 1)) & -(a_m z_m + 1/2(z_{m-1} - 1)) \\ -(a_m z_m + 1/2(z_{m-1} - 1)) & (a_m z_m + 1/2(z_{m-1} + 1)) \end{bmatrix}.$$

Hence, the proof is complete since

$$R \sim \begin{bmatrix} 1 & 0 \\ 0 & 2a_m z_m + z_{m-1} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & z_{m+1} \end{bmatrix}$$

□

**Corollary 3.7.** *$H_1(X_2)$ is a finite cyclic group and $|z_{m+1}| = |\Delta_K(-1)|$, where $\Delta_K(t)$ is the Alexander polynomial of $K$.*

**Corollary 3.8.** *Suppose $K$ is a 1-bridge torus knots with the Conway's normal form*

$$[(a_1, b_1, a_2, b_2), (a_3, b_3, a_4, b_4), \ldots, (a_{m-1}, b_{m-1}, a_m, b_m)].$$

(1) *If $a_{2i} = 0$ or $a_{2i-1} = 0$ for $i = 1, \ldots, m/2$ then $K$ is a trivial knot.*

(2) *If either $a_i > 0$ or $a_i < 0$ then $K$ is not a trivial knot.*

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