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REDUCING DEHN FILLINGS AND x-FACES

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ABSTRACT. In this paper we investigate the distances between Dehn fillings on a hyperbolic 3-manifold that yield 3-manifolds containing essential small surfaces. We study the situations where one filling creates an essential sphere, and the other creates an essential sphere, annulus or torus.

1. INTRODUCTION

Let M be a compact, connected, orientable 3-manifold with a torus boundary component $\partial_0 M$. Let γ be a *slope* on $\partial_0 M$, that is, the isotopy class of an essential simple closed curve on $\partial_0 M$. The 3-manifold obtained from M by γ-*Dehn filling* is defined to be $M(\gamma) = M \cup V_\gamma$, where V_γ is a solid torus glued to M along $\partial_0 M$ in such a way that γ bounds a meridian disk in V_γ.

By a *small* surface we mean one with non-negative Euler characteristic including non-orientable surfaces. Such surfaces play a special role in the theory of 3-dimensional manifolds. We say that a 3-manifold M is *hyperbolic* if M with its boundary tori removed admits a complete hyperbolic structure of finite volume with totally geodesic boundary. Thurston's geometrization theorem for Haken manifolds [Th] asserts that a hyperbolic 3-manifold M with non-empty boundary contains no essential small surfaces.

If M is hyperbolic, then the Dehn filling $M(\gamma)$ is also hyperbolic for all but finitely many slopes [Th], and a good deal of attention has been directed towards obtaining a more precise quantification of this statement. Following [Go2], we say that a 3-manifold is of

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type S, D, A or T, if it contains an essential sphere, disk, annulus or torus, respectively. We denote by $\Delta(\gamma_1, \gamma_2)$ the distance, or minimal geometric intersection number, between two slopes γ_1, γ_2 on $\partial_0 M$.

Suppose that $M(\gamma_i)$ for $i = 1, 2$ contains an essential small surface $\widehat{F}_i$. Then we may assume that $\widehat{F}_i$ meets the attached solid torus V_{γ_i} in a finite collection of meridian disks, and is chosen so that the number of disks n_i is minimal among all such surfaces in $M(\gamma_i)$. Since M is hyperbolic, n_i is positive.

In this paper we investigate the distances between two Dehn fillings where one filling creates an essential sphere, and the other creates an essential sphere, annulus or torus. For the first case which is called the reducibility theorem, we give the same result but a simpler proof, and for the other two cases we announce stronger results. The main point is that the generic parts of the proofs of all cases use the same argument.

Theorem 1.1 (Gordon-Luecke). *Suppose that M is hyperbolic. If $M(\gamma_1)$ and $M(\gamma_2)$ are of type S then $\Delta(\gamma_1, \gamma_2) \leq 1$.*

Theorem 1.2. *Suppose that M is hyperbolic. If $M(\gamma_1)$ is of type S and $M(\gamma_2)$ is of type A then $\Delta(\gamma_1, \gamma_2) \leq 1$, or $\Delta(\gamma_1, \gamma_2) = 2$ with $n_2 = 2$.*

Theorem 1.3. *Suppose that M is hyperbolic. If $M(\gamma_1)$ is of type S and $M(\gamma_2)$ is of type T then $\Delta(\gamma_1, \gamma_2) \leq 2$, or $\Delta(\gamma_1, \gamma_2) = 3$ with $n_2 = 2$.*

2. Graphs of surface intersections

Hereafter M is a hyperbolic 3-manifold with a torus boundary component $\partial_0 M$. In this section we describe how essential small surfaces $\widehat{F}_1$ and $\widehat{F}_2$, in $M(\gamma_1)$ and $M(\gamma_2)$ respectively, give rise to labelled intersection graphs $G_i \subset \widehat{F}_i$ for $i = 1, 2$ in general context.

As in Section 1, let $\widehat{F}_i$ be an essential small surface in $M(\gamma_i)$ with $n_i = |\widehat{F}_i \cap V_{\gamma_i}|$ minimal. Then $F_i = \widehat{F}_i \cap M$ is a punctured surface properly embedded in M, each of whose n_i boundary components has slope γ_i. Recall that n_i is positive. Note that the minimality of n_i guarantees that F_i is incompressible and ∂-incompressible in M.

We use i and j to denote 1 or 2, with the convention that, when both appear, $\{i, j\} = \{1, 2\}$. By an isotopy of F_1, we may assume that F_1 intersects F_2 transversely. Let G_i be the graph in $\widehat{F}_i$ obtained by taking as the (fat) vertices the disks $\widehat{F}_i - \mathrm{Int}F_i$ and as edges the arc components of $F_i \cap F_j$ in $\widehat{F}_i$. We number the components of ∂F_i as $1, 2, \cdots, n_i$ in the order in which they appear on $\partial_0 M$. On occasion we will use 0 instead of n_i in short. This gives a numbering of the vertices of G_i. Furthermore it induces a labelling of the end points of edges in G_j in the usual way (see [CGLS]). We can assume that G_1 and G_2 have neither trivial loops nor circle components bounding disks in their punctured surfaces, since all surfaces are incompressible and boundary-incompressible. For the sake of simplicity we will say that F_i and G_i are of type X_i if $M(\gamma_i)$ is of type X_i.

The rest of this section will be devoted to several definitions and well known lemmas. Let x be a label of G_i. An *x-edge* in G_i is an edge with label x at one endpoint, and an xy-edge is an edge with label x and y at both endpoints.

An *x-cycle* is a cycle of positive x-edges of G_i which can be oriented so that the tail of each edge has label x. A *Scharlemann cycle* is an x-cycle that bounds a disk face of G_i, only when $n_j \geq 2$. Each edge of a Scharlemann cycle has the same label pair $\{x, x+1\}$, so we refer as $\{x, x+1\}$-Scharlemann cycle. The number of edges in a Scharlemann cycle, σ, is called the *length* of σ. In particular, a Scharlemann cycle of length two is called an *S-cycle* in short.

An *extended S-cycle* is the quadruple $\{e_1, e_2, e_3, e_4\}$ of mutually parallel positive edges in succession and $\{e_2, e_3\}$ form an S-cycle, only when $n_j \geq 4$.

Lemma 2.1. *If G_j is of type S, A or T, and G_i contains a Scharlemann cycle, then $\widehat{F}_j$ must be separating, and so n_j is even. Furthermore, when G_j is of type A or T (but $M(\gamma_j)$ is not of type S), the edges of the Scharlemann cycle cannot lie in a disk in $\widehat{F}_j$.*

Proof. Let E be a disk face bounded by a Scharlemann cycle with labels, say, $\{1, 2\}$ in G_i. Let V_{12} be the 1-handle cut from V_{γ_j} by the vertices 1 and 2 of G_j. Then tubing $\widehat{F}_j$ along ∂V_{12} and compressing along E gives such a new $\widehat{F}_j$ in $M(\gamma_j)$ that intersects V_{γ_j} fewer times than the old one.

Note that if $\widehat{F}_j$ is non-separating, then the new one is also non-separating, and so essential. Furthermore, if the edges of the Scharlemann cycle lie in a disk D in an annulus or torus $\widehat{F}_j$, then $\mathrm{nhd}(D \cup V_{12} \cup E)$ is a once punctured lens space. The incompressibility of $\widehat{F}_j$ means that $M(\gamma_j)$ must be reducible. □

Lemma 2.2. *If G_j is of type S or A, then G_i cannot contain Scharlemann cycles on distinct label pairs.*

Proof. Without loss of generality assume that σ_1 is a Scharlemann cycle with label pair $\{1,2\}$ bounding a face E in G_i.

First, assume that G_j is of type S. Consider a punctured lens space $L = \mathrm{nhd}((\widehat{F}_j - \mathrm{Int}D) \cup V_{12} \cup E)$ where D is a disk in $\widehat{F}_j$ separated by the edges of σ_1. Since ∂L is a reducing sphere in $M(\gamma_j)$, $2|(\widehat{F}_j - \mathrm{Int}D) \cap V_{\gamma_j}| - 2 = |\partial L \cap V_{\gamma_j}| \geq n_j$ by the minimality of n_j. Thus $|\mathrm{Int}D \cap V_{\gamma_j}| \leq \frac{n_j}{2} - 1$, i.e. the interior of D contains at most $\frac{n_j}{2} - 1$ vertices of G_j.

Suppose there is another Scharlemann cycle σ_2 with distinct label pair. Since the edges of σ_2 lie in the same disk component of $\widehat{F}_j$ separated by the edges of σ_1, the union of the edges of σ_2 and the corresponding two vertices should be contained in a disk containing at most $\frac{n_j}{2}$ vertices (note that σ_1 and σ_2 can share one label). This implies that the other $\frac{n_j}{2}$ vertices of G_j lie in the same disk component of $\widehat{F}_j$ separated by the edges of σ_2, contradicting the same argument of the preceding paragraph.

Next, assume that G_j is of type A (but $M(\gamma_j)$ is not of type S). We give a brief of the proof of [Wu3, Lemma 5.4(2)]. By Lemma 2.1 the union of the edges of σ_1 and the corresponding two vertices cannot lie in a disk, so cuts $\widehat{F}_j$ into two annuli A_1, A_2 and some disk components. Consider the manifold $Y = \mathrm{nhd}((\widehat{F}_j - \mathrm{Int}A_1) \cup V_{12} \cup E)$. Claim in Wu's proof guarantees that the frontier Q of Y is an essential annulus in $M(\gamma_j)$. Then $2|(\widehat{F}_j - \mathrm{Int}A_1) \cap V_{\gamma_j}| - 2 = |Q \cap V_{\gamma_j}| \geq n_j$ by the minimality of n_j. Thus $|\mathrm{Int}A_1 \cap V_{\gamma_j}| \leq \frac{n_j}{2} - 1$.

Suppose there is another Scharlemann cycle σ_2 with distinct label pair. Since the edges of σ_2 cannot lie in a disk, we may assume that these edges are contained in the annulus A_1. Like before, this implies that one (not contained in A_1) of two annuli in $\widehat{F}_j$ separated by the edges of σ_2 contains at least $\frac{n_j}{2}$ vertices of G_j in its interior, contradicting the previous argument. □

Lemma 2.3. *If G_j is of type S, A or T, then G_i cannot contain an extended S-cycle.*

Proof. All three cases follow from [Wu1, Lemma 2.3], [Wu3, Lemma 5.4(3)] and [BZ, Lemma 2.10], respectively.

In fact one could avoid these reference to [Wu1, Wu3] for the cases of type S and A. Note that an extended S-cycle is the simplest one in x-faces (defined in Section 3) where x is not a label of a Scharlemann cycle. Then these two cases follow immediately from Theorems 3.4 and 3.5. □

The *reduced graph* $\overline{G}_i$ of G_i is defined to be the graph obtained from G_i by amalgamating each family of parallel edges into a single edge.

3. x-FACE

In this section we assume that G_j is of type S or A. By Lemma 2.2 we may say that G_i contains only 12-Scharlemann cycles.

A disk face of the subgraph of G_i consisting of all the vertices and positive x-edges of G_i is called an *x-face*. Remark that the boundary of an x-face D may be not a circle, that is, ∂D may contain a double edge, and more than two edges of ∂D may be incident to a vertex on ∂D (see Figure 2.1 in [HM]). A cycle in G_i is a *two-cornered cycle* if it is the boundary of a face containing only 01-corners, 23-corners and positive edges, and additionally it contains at least one edge of a 12-Scharlemann cycle. Recall that 0 denotes n_j. A two-cornered cycle must contain both corners and a 03-edge because G_i has only 12-Scharlemann cycles. A *cluster* C is a connected subgraph of G_i satisfying that

(i) C consists of 12-Scharlemann cycles and two-cornered cycles,

(ii) every 12-edge of C belongs to both a Scharlemann cycle and a two-cornered cycle, and

(iii) C contains no cut vertex.

The notions of a two-cornered cycle and a cluster were used firstly in [Ho]. In the paper Hoffman showed that the disk bounded by a great x-cycle contains a pair of specific two-cornered cycles, called a seemly pair, and it can be used to find a new essential sphere meeting the attached solid torus in fewer times, leading to a contradiction.

Lemma 3.1. *Suppose that $n_j \geq 3$. An x-face, $x \neq 1, 2$, in G_i contains a cluster C.*

Proof. Let Γ_D be the subgraph of G_i in an x-face D. There is a possibility that ∂D is not a circle as mentioned before. Since we will find a pair of two-cornered cycles within D, we can cut formally the graph $G_i \cap D$ along double edges of ∂D and at vertices to which more than two edges of ∂D are incident to so that ∂D is deformed into a circle. (See also Figure 5.1 in [HM].) Thus we may assume that ∂D is a circle and Γ_D has no vertex in the interior of D. We may assume that the labels appear in anticlockwise order around the boundary of each vertex.

Suppose that D has a diagonal edge d with a label pair $\{a, b\}$, which are not of 12-Scharlemann cycles, as in Figure 1(a). Note that these labels must differ from x. Assume without loss of generality that $b < a < x$, i.e. three labels b, a, x appear in anticlockwise order in usual sence. Formally construct a new x-face D' as follows. Keep all corners and edges of Γ_D to the right of d (when d is directed from a to b), discard all corners and edges to the left of d, and then insert additional edges to the left of d, and parallel to d, until you first reach label x at one end of this parallel family of edges, as in Figure 1(b). In particular, these additional edges contain no edges of two-cornered cycles or Scharlemann cycles of the graph on the new x-face D'.

Repeat the above process for every diagonal edges which are not of 12-Scharlemann cycles, then we get a new x-face E and a graph Γ_E in E so that all diagonal edges are of 12-Scharlemann cycles and all (and only) boundary edges are x-edges. Furthermore the additional edges contain no edges of Scharlemann cycles or two-cornered cycles of Γ_E. Remark that an xx-edge can appear on the boundary of the graph because we cannot guarantee that $\widehat{F}_j$ is separating.

Claim 3.2. *Γ_E contains a 12-Scharlemann cycle, so does Γ_D.*

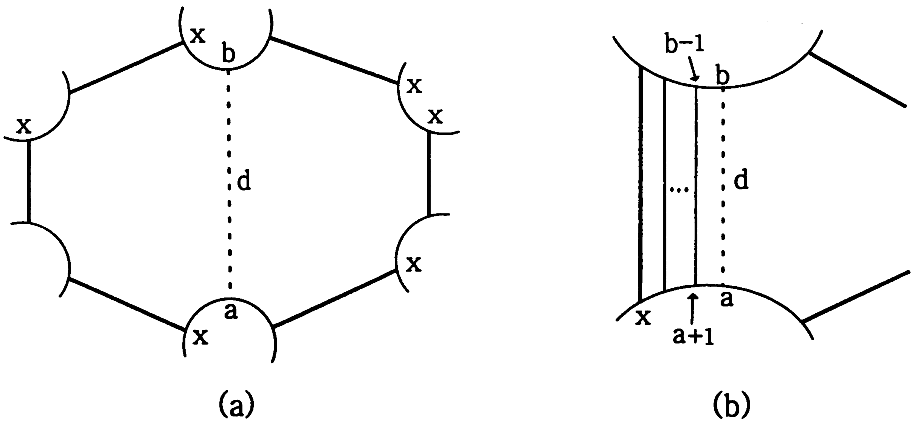

FIGURE 1. Split along a diagonal edge

Proof. Assume that Γ_E contains no 12-Scharlemann cycles, and so no diagonal edges. We show first that if for some vertex v of Γ_E two boundary edges are incident to v with label x, then these should be xx-edges. For, in Γ_E at least $n_j + 1$ edges are incident to v. If n_j is even, more than $\frac{n_j}{2}$ mutually parallel edges are incident to v, and so one of these edges should be a yy-edge, $y \neq x$, (recall that Γ_E cannot contain a Scharlemann cycle), a contradiction. If n_j is odd, two families of $\frac{n_j+1}{2}$ mutually parallel edges are incident to v, and the boundary edge of each family should be an xx-edge by the same reason above.

Consider the cycle σ consisting of boundary x-edges of Γ_E. Assume that σ has an x-edge which is not an xx-edge. So only one end has label x at v_1, say. By the fact we just proved, another x-edge incident to v_1 does not have label x at v_1. Thus this edge has label x at the other end v_2, say. After repeating this process, we are led to show that σ is a great x-cycle in the terminology of [CGLS]. (If all boundary edges are xx-edges, still we have a great x-cycle.) By the same argument in the proof of Lemma 2.6.2 of [CGLS], Γ_E contains a Scharlemann cycle, a contradicton. $\square$

This means that $\widehat{F}_j$ must be separating and n_j is even by Lemma 2.1. The parity rule guarantees that each edge of Γ_E connects vertices with one label even and the other label odd, and so there is no xx-edges.

Any 12-edge of a Scharlemann cycle does not belong to $\partial\Gamma_E$. Consider the face E_1 of Γ_E adjacent to the 12-edge which does not bound the Scharlemann cycle. It is possible that E_1 contains more than one 12-edges of Scharlemann cycles. Let $\{a_k, a_k+1\}$, $k = 1, \cdots, n$, be the consecutive label pairs of the corners between two consecutive 12-edges of Scharlemann cycles when runs clockwise around ∂E_1. Note that $a_1 = 2$ and $a_n = 0$.

Assume for contradiction that ∂E_1 is not a two-cornered cycle. Since some a_k then is neither 0 nor 2, there are indices l and m so that $a_k = 0$ or 2 when $1 \leq k < l$ or $k = m$, and $a_k \neq 0, 2$ when $l \leq k < m$.

Consider the edges of the parallelism class containing each $\{a_{k-1}+1, a_k\}$-edge for $l \leq k \leq m$. Since there is no Scharlemann cycles among these edges, one finds that $x \leq a_k < a_{k-1} + 1 \leq x$, or $x \leq a_k \leq a_{k-1} < x$. And so $x \leq a_m \leq a_{m-1} \leq \cdots \leq a_l \leq a_{l-1} < x$. This is impossible because $a_{l-1}, a_m = 0$ or 2 and all a_k's are even by the parity rule. Thus ∂E_1 is a two-cornered cycle. Let C be the union of all the Scharlemann cycles and all the two-cornered cycles adjacent to each 12-edges of the Scharlemann cycles. After cutting along cut vertices, a connected component of C is then a desired cluster in Γ_E and so in Γ_D. □

Let $\widetilde{F}_j$ be the twice-punctured sphere obtained from $\widehat{F}_j$ by deleting two fat vertices 1 and 2 (if G_j is of type A, then use $\widehat{F}_j$ after capping off two boundary circles by disks). The family of all 12-edges of a Scharlemann cycle in the cluster C separates $\widetilde{F}_j$ into disks, and one of those disks contains both vertices 0 and 3 of G_j because of the existence of 03-edges in C. The two 12-edges bounding such a disk is called *good edges* of C. Thus each Scharlemann cycle in C has exactly two good edges.

Let Λ be the maximal dual graph of C whose vertices are dual to Scharlemann cycles and two-cornered cycles containing good edges, and edges are dual to good edges of C as depicted in Figure 2. Thus in Λ, a vertex dual to a Scharlemann cycle has valency 2, and a vertex dual to a two-cornered cycle has valency the number of good edges of the two-cornered cycle. Furthermore Λ is a tree according to the construction of C. This implies that each 12-edge, which is not a good one, of two-cornered cycles related to vertices of

Λ contributes to the number of components of Λ by adding 1. Consequently there is a component Λ_g of Λ so that all 12-edges of its dual two-cornered cycles are good.

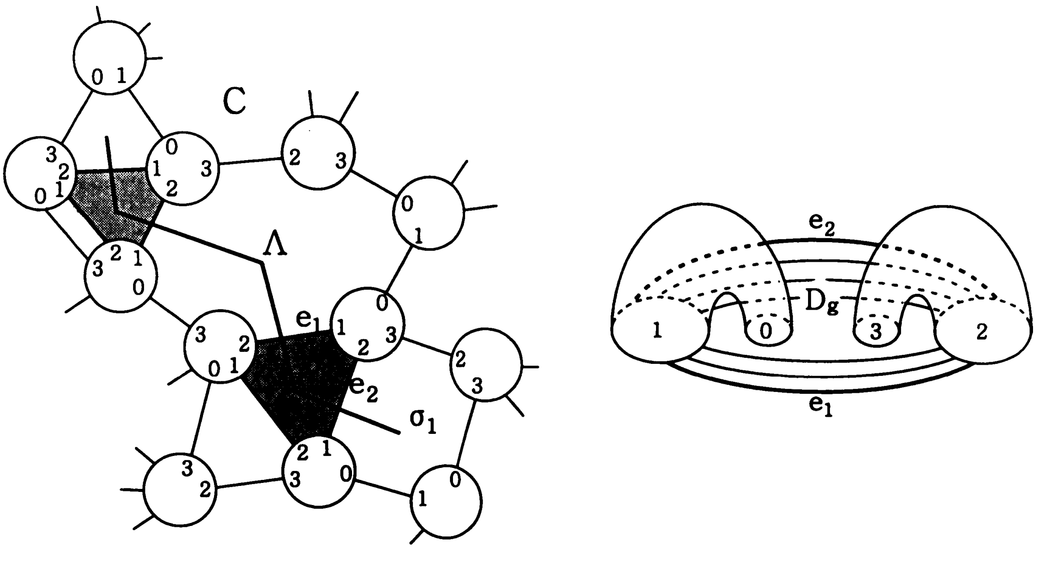

FIGURE 2. A cluster and a seemly pair

Hereafter, we consider the subgraph C_g of C, dual to Λ_g. Say, C_g contains n Scharlemann cycles, and so $2n$ good edges and $n+1$ two-cornered cycles. A two-cornered cycle dual to an end vertex of the tree Λ_g has only one good edge. Choose one e_1 of the nearest edges to vertex 0 (or 3) among them, i.e. there is no such good edges between e_1 and vertex 0 in $\widetilde{F}_j$. Let σ_g and σ_1 be the Scharlemann cycle and two-cornered cycle adjacent to e_1 respectively. Then σ_g has another good edge e_2, and e_1 and e_2 bound a disk D_g containing 0 and 3 in $\widetilde{F}_j$. Note that all Scharlemann cycles are parallel on a torus obtained from $\widetilde{F}_j$ by attaching an annulus ∂V_{12}. Thus exactly $n-1$ out of $2n$ good edges are not contained in D_g. Therefore we have another two-cornered cycle σ_2 all of whose 12-edges lie in D_g. Consequently all edges (consisting of 01-edges, 12-edges, 23-edges and 03-edges) of σ_1 and σ_2 lie in D_g. Furthermore if σ_2 has only one good 12-edge, then the two good edges of σ_1 and σ_2 lie on different sides of the vertices 0 and 3 in D_g. Such σ_1, σ_2 are called a *seemly pair*. Then we can say;

Lemma 3.3. *There is a seemly pair of two-cornered cycles in C.*

From now we apply the argument in [Ho, Section 6] to get the following two theorems.

Theorem 3.4. *If G_j is of type S with $n_j \geq 3$, then G_i cannot contain an x-face for $x \neq 1$ or 2.*

Proof. Suppose that G_i contains such an x-face. We continue the preceding argument. Recall that $|\mathrm{Int}D_g \cap V_{\gamma_j}| \leq \frac{n_j}{2} - 1$ as in the proof of Lemma 2.2. Let $X'_D = \mathrm{nhd}(D_g \cup V_{01} \cup V_{23})$ and $X'_F = \mathrm{nhd}(\widehat{F}_j \cup V_{01} \cup V_{23})$. Then X'_D is a genus two handlebody and X'_F is a once-punctured genus two handlebody. The genus two torus component of $\partial X'_F$ is referred to as the outer boundary of X'_F. Let E_i be the face bounded by the two-cornered cycle σ_i for $i = 1, 2$. The point is that all edges of σ_i are contained in D_g. Let $X_D = X'_D \cup \mathrm{nhd}(E_1) \cup \mathrm{nhd}(E_2)$ and $X_F = X'_F \cup \mathrm{nhd}(E_1) \cup \mathrm{nhd}(E_2)$ as in Figure 3. Since σ_1 is non-separating on $\partial X'_D$ and $\partial X'_F$, both ∂X_D and the outer components of ∂X_F are either a 2-sphere or the disjoint union of a 2-sphere and a torus, simultaneously. Note that the latter case occurs only when σ_1 and σ_2 are parallel.

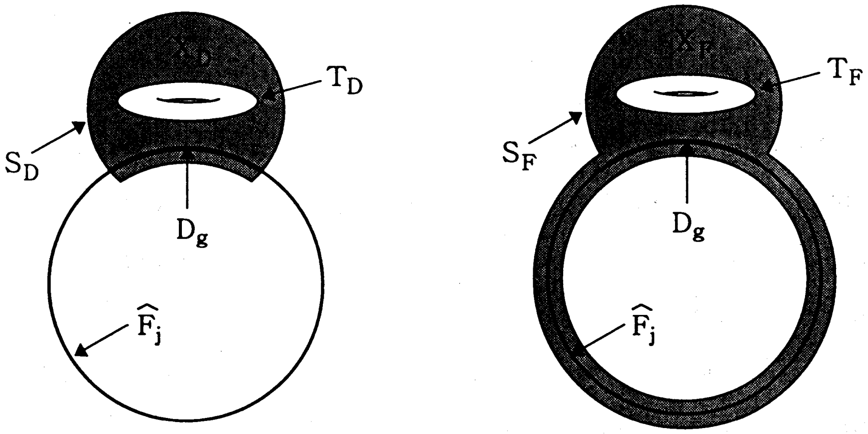

FIGURE 3. X_D and X_F

First, assume that ∂X_D is a 2-sphere S_D, and the outer component of ∂X_F is a 2-sphere S_F. If X_D is not a 3-ball, then S_D is a reducing sphere. By the previous remark $|S_D \cap V_{\gamma_j}| = 2|D_1 \cap V_{\gamma_j}| \leq n_j - 2$, contradicting the minimality of n_j. Thus it should be a 3-ball, and so X_F is homeomorphic to $S^2 \times I$. Thus S_F is isotopic to $\widehat{F}_j$, contradicting the minimality of n_j again.

Next, assume that ∂X_D is the disjoint union of a 2-sphere and a torus, $S_D \cup T_D$ and the outer components of ∂X_F are also the disjoint union of a 2-sphere and a torus, $S_F \cup T_F$. Recall that this case occurs whenever σ_1 and σ_2 cobound an annulus in $\partial X'_D$ and $\partial X'_F$. This is possible only when σ_2 corresponds to an end vertex of Λ_g with the same number of 01-corners and 23-corners as that of σ_1. Let D' be the intersection of ∂X_D and the inner sphere component of ∂X_F. By the choice of the seemly pair, this annulus in $\partial X'_D$ contains D', so does S_D. Similarly S_F contains a pushoff of $\widetilde{F}_j - D_g$.

If S_D is non-separating in $M(\gamma_j)$, then it is a reducing sphere with $|S_D \cap V_{\gamma_j}| \leq n_j - 2$, contradicting. Thus S_D, and hence T_D are separating in $M(\gamma_j)$. Let X''_D be the manifold bounded by S_D containing T_D in $M(\gamma_j)$. If X''_D is not a 3-ball, then S_D is a reducing sphere with less intersection with V_{γ_j}. Thus it should be a 3-ball, and so S_F is isotopic to $\widehat{F}_j$, contradicting again. This completes the proof. □

Theorem 3.5. *If G_j is of type A with $n_j \geq 3$, then G_i cannot contain an x-face for $x \neq 1$ or 2.*

Proof. Suppose that G_i contains such an x-face. By Theorem 3.4 $M(\gamma_j)$ is irreducible. Again we continue the argument stated before Lemma 3.3. We distinguish three cases;

First, suppose D_g contains $\partial\widehat{F}_j$. Then the edges of the Scharlemann cycle σ_g lie in a disk in $\widehat{F}_j$, contradicting Lemma 2.1.

Second, suppose D_g does not contain any component of $\partial\widehat{F}_j$. Then D_g is contained in $\widehat{F}_j$. The proof is similar to that of the preceding theorem. Let $X_D = \text{nhd}(D_g \cup V_{01} \cup V_{23} \cup E_1 \cup E_2)$ and $X_F = \text{nhd}(\widehat{F}_j \cup V_{01} \cup V_{23} \cup E_1 \cup E_2)$. Then ∂X_D is either a 2-sphere or the disjoint union of a 2-sphere and a torus, and the (similarly defined) outer components of the frontier of X_S are either an annulus or (by the choice of the seemly pair) the disjoint union of an annulus and a torus. The latter two cases occur only when σ_1 and σ_2 are parallel as previous. Since $M(\gamma_j)$ is irreducible, the 2-sphere component of ∂X_D should bound a 3-ball. Thus the annular component of the outer components of the frontier of X_F is isotopic to $\widehat{F}_j$, contradicting the minimality of n_j.

Finally suppose D_g contains exactly one component of $\partial\widehat{F}_j$. Let A_1 be the annalus $D_g \cap F_j$. Note that $|\mathrm{Int}A_1 \cap V_{\gamma_j}| \leq \frac{n_j}{2} - 1$ as in the proof of Lemma 2.2. Let $X_A = \mathrm{nhd}(A_1 \cup V_{01} \cup V_{23} \cup E_1 \cup E_2)$ and $X_F = \mathrm{nhd}(\widehat{F}_j \cup V_{01} \cup V_{23} \cup E_1 \cup E_2)$ again. Then ∂X_A, and also the outer components (which is not a pushoff of $\widehat{F}_j$ on the other side of attached handles V_{01} or V_{23}) of the frontier of X_F, is either an annulus or the disjoint union of an annulus and a torus. Let A_A and A_F be the corresponding annular components of the boundaries of X_A and X_F respectively. If A_A is essential in $M(\gamma_j)$, then $|A_A \cap V_{\gamma_j}| = 2|\mathrm{Int}A_1 \cap V_{\gamma_j}| \leq n_j - 2$, contradicting the minimality of n_j. Remark that A_A is incompressible because the central curve of A_A is isotopic to the central curve of $\widehat{F}_j$ and $\widehat{F}_j$ is incompressible. Thus A_A should be ∂-compressible, i.e. one of the manifolds X_A and $M(\gamma_j) - \mathrm{Int}X_A$ contains a ∂-compressing disk. Since $M(\gamma_j)$ is irreducible and ∂-irreducible, the manifold with the ∂-compressing disk is a solid torus with A_A as a longitudinal annulus. Through this manifold one can isotope $\widehat{F}_j$ to A_F which intersects V_{γ_j} fewer times than $\widehat{F}_j$. □

Remark that an extended S-cycle is the boundary cycle of the simplest x-face. Thus two theorems above guarantee that if G_j is of type S or A, then G_i cannot contain an extended S-cycle.

4. Proofs

In this section we prove the main theorems. Assume that $M(\gamma_1)$ is of type S, $M(\gamma_2)$ is one of three types, and $\Delta \geq 2$. Note that $n_1 \geq 3$ because M does not contain essential small surfaces.

Proposition 4.1. *G_1 contains a connected subgraph Λ so that it has a disk support D in $\widehat{F_1}$ such that $D \cap G_1^+ = \Lambda$, and each boundary vertex, except at most one vertex y_0, has degree at least $(\Delta - 1)n_2 + \chi(\widehat{F}_2)$ in Λ.*

Proof. If a vertex x of G_1 has more than $n_2 - \chi(\widehat{F}_2)$ negative edges, then G_2 contains more than $n_2 - \chi(\widehat{F}_2)$ positive x-edges by the parity rule. Thus the subgraph Γ_x of G_2 consisting of all vertices and positive x-edges of G_2 has n_2 vertices and more than $n_2 - \chi(\widehat{F}_2)$ edges. Then an Euler characteristic calculation shows that Γ_x contains a disk face, that is, an

x-face. Thus by Theorem 3.4 each vertex $x \neq 1, 2$, the labels of Scharlemann cycles of G_2 if it exist, has at least $(\Delta - 1)n_2 + \chi(\widehat{F}_2)$ positive edges.

Let G_1^+ denote the subgraph of G_1 consisting of all vertices and positive edges of G_1. Let Λ' be an extremal component of G_1^+. That is, Λ' is a component of G_1^+ having a disk support D such that $D \cap G_1^+ = \Lambda'$. In Λ', a block Λ with at most one cut vertex is called an *extremal block*. If Λ' has no cut vertex, then Λ' itself is an extremal block, and if Λ' has a cut vertex, then it has at least two extremal blocks.

Furthermore, if G_2 contains a 12-Scharlemann cycle, then $\widehat{F}_1$ is separating, and so G_1^+ is disconnected. Then there is an extremal component of G_1^+ which contains at most one of vertices 1 and 2. Now choose an extremal block so that it contains at most one such vertex or a cut vertex which is called a *ghost vertex* y_0. Then Λ is the desired subgraph. □

Proof of Theorem 1.1. $M(\gamma_2)$ is of type S. By Proposition 4.1, G_1 contains an extremal block Λ, each of whose boundary vertex except y_0 has at least $n_2 + 2$ consecutive edge endpoints, and so has all different n_2 labels.

If there is no ghost y_0, choose any label x' but 1 and 2, the labels of Scharlemann cycles of G_1. Or if such y_0 exists and it has more than two edges incident there, then choose x' among the labels of y_0 except 1 and 2. Let $\Gamma_{x'}$ be the subgraph of Λ consisting of all vertices and x'-edges. Then in $\Gamma_{x'}$ the number of edges cannot be less than the number of vertices. Again an Euler characteristic calculation of $\Gamma_{x'}$ on the disk guarantees the existence of x'-face, $x' \neq 1, 2$, contradicting Theorem 3.4. For the remaining case, if y_0 has only two edges incident there, then delete y_0 and the two edges from Λ. Since all labels still appear on each vertex of this new Λ, we can proceed the same argument above to get a contradiction. □

Proof of Theorem 1.2. $M(\gamma_2)$ is of type A. For the case $n_2 = 1$, G_2 can have only positive edges. Thus it contains an x-face, $x \neq 1, 2$, contradicting Theorem 3.4.

We assume therefore that $n_2 \geq 3$. Again by Proposition 4.1, G_1 contains an extremal block Λ, each of whose boundary vertex except y_0 has at least n_2 consecutive edge endpoints, i.e. all different n_2 labels. Then it contains a 12-Scharlemann cycle as in Claim

3.2. By Lemma 2.1 n_2 is even and this Scharlemann cycle divides the annulus $\widehat{F}_2$ into two disjoint annuli A_1 and A_2. Note that there must be y_0 with only two edges incident to at labels 1 and 2 in Λ, and there is no interior vertex of Λ. Otherwise, an Euler characteristic calculation shows that Λ contains an x-face for $x \neq 1, 2$, so does G_1. But this contradicts Theorem 3.5. Furthermore $n_2 = 4$ and the two edges incident to y_0 are indeed 14, 23-edges, otherwise Λ after deleting y_0 and these two edges also contains an x-face for $x \neq 1, 2$, contradicting as previous. By the same reason, each boundary vertices of Λ except y_0 has exactly four edges incident to. From the facts above one can conclude that Λ has a 34-edge. That means vertices 3 and 4 lie on the same annulus A_1 or A_2.

But, one vertex, say x', of Λ has four consecutive negative edges attached in G_1. By the parity rule these four edges produce two disjoint x'-cycles in G_2, each of which can not lie in a disk in $\widehat{F}_2$. Furthermore one connects vertices 1 and 3, and the other connects vertices 2 and 4. Thus vertices 3 and 4 can not lie on the same annulus A_1 or A_2, contradicting.

For the case $n_2 = 1$, done by [Wu3, Theorem 5.1]. □

Proof of Theorem 1.3. $M(\gamma_2)$ is of type T. Assume that $\Delta \geq 3$. For the case $n_2 = 1$, G_2 can have only positive edges. Thus it contains an x-face, $x \neq 1, 2$, contradicting Theorem 3.4.

We now assume that $n_2 \geq 3$. By Proposition 4.1, G_1 contains an extremal block Λ, each of whose boundary vertex except y_0 has at least $2n_2$ consecutive edge endpoints.

There is a label x of G_1 which is not of S-cycles of G_1. For, if not, all labels of G_1 are of S-cycles. Then by Lemma 2.1, $n_2 \geq 4$, and so there are two S-cycles σ_1 and σ_2 with disjoint label pairs. Let $\{\alpha_k, \alpha_k + 1\}$ be the labels of σ_k, and let E_k be the face of G_1 bounded by σ_k, $k = 1, 2$. Then shrinking V_{α_k, α_k+1} to its core in $V_{\alpha_k, \alpha_k+1} \cup E_k$ gives a Möbius band B_k such that ∂B_k is the loop on $\widehat{F}_2$ formed by the edges of σ_k. By Lemma 2.1, ∂B_1 is isotopic to ∂B_2. Then taking the union of B_1 and B_2 would give a Klein bottle in $M(\gamma_2)$. This is impossible because of [LOT].

Consider the subgraph Λ^x consisting of all vertices and x-edges of Λ. Since every boundary vertex of Λ, except y_0, has degree at least $2n_j$, it has at least two edges attached with label x. Note that Λ^x may be disconnected. Choose an extremal block Λ' of Λ^x (in a disk support D). Let v, e and f be the numbers of vertices, edges, and disk faces of Λ', respectively. Also let v_i, v_∂ and v_g be the numbers of interior vertices, boundary vertices and ghost vertices. Hence $v = v_i + v_\partial$ and $v_g = 0$ or 1.

Because of Lemma 2.3, each face of Λ' is a disk with at least 3 sides. Thus we have $3f + v_\partial \leq 2e$. Since it has only disk faces, combined with $v - e + f = \chi(D) = 1$, we get $e \leq 3v_i + 2v_\partial - 3$. On the other hand we have $2(v_\partial - v_g) + \Delta v_i \leq e$ because each boundary vertex of Λ', except y_0, has at least two edges attached with label x. These two inequalities give us that $3 \leq 2v_g$, a contradiction.

For the case $n_2 = 2$, done by [Oh, Theorem 1.1] and [Wu2, Theorem 1]. □

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