数理解析研究所講究録 1230 巻 2001 年 33-37

INVOLUTIVE EQUIVALENCE BIMODULES AND INCLUSIONS OF C^*-ALGEBRAS WITH WATATANI INDEX2

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ABSTRACT. Let A be a unital simple C^*-algebra. We shall intorduce involutive A-A equivalence bimodules and prove that the all C^*-algebaras containing A with Watatani index 2 are constructed by an involutive A-A equivalence bimodule and A.

1. INTRODUCTION

V. Jones intorduced index theory for II_1 factors. As one of his motivations of his definition of index, there is Goldman's theorem, which says that if $[M : N] = 2$, there is a crossed product decomposition $M = \times_\alpha \mathbb{Z}/2\mathbb{Z}$.

Y. Watatani extended index theory to C^*-algebaras. He defined indices of conditional expectations in terms of quasi-basis, which is generalization of the Pimsner-Popa basis. There is an inclusion of unital simple C^*-algebaras with Watatani index 2, which is no written by the crossed product of a $\mathbb{Z}/2\mathbb{Z}$ action.

Equivalence bimodules for C^*-algebaras A and B are introduced by M. A. Riefell, wihich is a left Hilbert A-module as well as a right Hilbert B-module with full C^*-algebra valued inner products ${}_A\langle\ \rangle$ and $\langle\ \rangle_B$ such that $x_A\langle y, z\rangle = \langle x, y\rangle_B z$ holds.

Let A be a unital simple C^*-algebra. We shall intorduce involutive A-A equivalence bimodules and prove that the all C^*-algebaras containing A with Watatani index 2 are constructed by an involutive A-A equivalence bimodule and A.

2. PRELIMINARIES

2.1. **Some results for inclusions with index 2.** Let B be a unital C^*-algebra and A a C^*-subalgebra of B with a common unit. Let E be a conditional expectation of B onto A with $1 < \text{Index}E < \infty$. Then by Watatani [10] we have the C^*-basic construction $C^*\langle B, e_A\rangle$ where e_A is a projection induced by E. Let $\widetilde{E}$ be the dual conditional expectation of $C^*\langle B, e_A\rangle$ onto B defined by

$$\widetilde{E}(ae_Ab) = \frac{1}{t}ab \quad \text{for any} \quad a, b \in B,$$

where $t = \text{Index}E$. Let F be a linear map of $(1-e_A)C^*\langle B, e_A\rangle(1-e_A)$ to $A(1-e_A)$ defined by

$$F(a) = \frac{t}{t-1}(E \circ \widetilde{E})(a)(1-e_A)$$

for any $a \in (1-e_A)C^*\langle B, e_A\rangle(1-e_A)$. By a routine computation we can see that F is a conditional expectation of $(1-e_A)C^*\langle B, e_A\rangle(1-e_A)$ onto $A(1-e_A)$.

Lemma 2.1.1. *With the above notations, let* $\{(x_i, x_i^*)\}_{i=1}^n$ *be a quasi-basis for* E. *Then*

$$\{\sqrt{t-1}(1-e_A)x_je_Ax_i(1-e_A), \sqrt{t-1}(1-e_A)x_i^*e_Ax_j^*(1-e_A)\}_{i,j=1}^n$$

is a quasi-basis for F. *Furthermore* $\text{Index}F = (t-1)^2(1-e_A)$.

Proof. This is immediate by a direct computation. □

Date: May 31, 2001.

Corollary 2.1.1. *We suppose that* Index$E = 2$. *Then*

$$(1-e_A)C^*\langle B, e_A\rangle(1-e_A) = A(1-e_A) \cong A.$$

Proof. By Lemma2.1.1 there is a conditional expectation F of $(1-e_A)C^*\langle B, e_A\rangle(1-e_A)$ onto $A(1-e_A)$ and

$$\text{Index}F = (\text{Index}E - 1)^2(1-e_A).$$

Since Index$E = 2$, Index$F = 1 - e_A$. Hence by Watatani [10],

$$(1-e_A)C^*\langle B, e_A\rangle(1-e_A) = A(1-e_A).$$

If $a(1-e_A) = 0$, for $a \in A$, then $a = 2\widetilde{E}(a(1-e_A)) = 0$. Therefore the map $a \to a(1-e_A)$ is injective. And hence $A(1-e_A) \cong A$. Thus we obtain the conclusion. □

Lemma 2.1.2. *With the same asumptions as in Lemma2.1.1, we suppose that* Index$E =$ 2. *Then for any* $b \in B$,

$$(1-e_A)b(1-e_A) = E(b)(1-e_A).$$

Proof. By Corollary2.1.1 there exists $a \in A$ such that $(1-e_A)b(1-e_A) = a(1-e_A)$. Therefore

$$\begin{aligned} a &= 2\widetilde{E}(a(1-e_A)) \\ &= 2\widetilde{E}((1-e_A)b(1-e_A)) \\ &= 2\widetilde{E}(b - e_A b - b e_A + E(b)e_A) \\ &= 2(b - \frac{1}{2}b - \frac{1}{2}b + \frac{1}{2}E(b)) = E(b). \end{aligned}$$

Thus we obtain the conclusion. □

Proposition 2.1.1. *With the same asumptions as in Lemma2.1.1, we suppose that* Index$E$ 2. *Then there is a unitary element* $U \in C^*\langle B, e_A\rangle$ *satisfying the followings:*

(1) $U^2 = 1$,
(2) $UbU^* = 2E(b) - b$ *for* $b \in B$.

Hence if $\beta = Ad(U)|_B$, β *is an automorphism of* B *with* $\beta^2 = id$ *and* $B^\beta = A$.

Proof. By Lemma2.1.2, for any $b \in B$

$$\begin{aligned} (1-e_A)b(1-e_A) &= b - e_A b - b e_A + E(b)e_A \\ &= E(b)(1-e_A) = E(b) - E(b)e_A. \end{aligned}$$

Therefore

$$E(b) = b - e_A b - b e_A + 2E(b)e_A.$$

Let U be a unitary element defined by $U = 2e_A - 1$. Then by the above equation for any $b \in B$

$$\begin{aligned} UbU^* &= (2e_A - 1)b(2e_A - 1) \\ &= 4E(b)e_A - 2e_A b - b2e_A + b \\ &= 2(b - e_A b - b e_A + 2E(b)e_A) - b \\ &= 2E(b) - b. \end{aligned}$$

Thus we obtain the conclusion. □

Remark 2.1.1. By the above proposition, $E(b) = \frac{1}{2}(b + \beta(b))$.

Lemma 2.1.3. *Let* B *be a unital* C^**-algebra and* A *a* C^**-subalgebra of* B *with a common unit. Let* E *be a conditional expectation of* B *onto* A *with* Index$E = 2$. *Then we have*

$$C^*\langle B, e_A\rangle \cong B \times_\beta \mathbb{Z}_2.$$

Proof. We may assume that $B \times_\beta \mathbb{Z}_2$ acts on the Hilbert space $l^2(\mathbb{Z}_2, H)$ faithfully, where H is some Hilbert space on which B acts faithfully. Let W be a unitary element in $B \times_\beta \mathbb{Z}_2$ with $\beta = Ad(W)$, $W^2 = 1$. Let $e = \frac{1}{2}(W+1)$. Then e is a projection in $B \times_\beta \mathbb{Z}_2$ and $ebe = E(b)e$ for any $b \in B$. In fact,

$$\begin{aligned} ebe &= \frac{1}{4}(W+1)b(W+1) = \frac{1}{4}(Wb+b)(W+1) \\ &= (WbW + bW + Wb + b). \end{aligned}$$

On the other hand by Remark 2.1.1,

$$\begin{aligned} E(b)e &= \frac{1}{2}(b+\beta(b))\frac{1}{2}(W+1) = \frac{1}{4}(bW + b + \beta(b)W + \beta(b)) \\ &= \frac{1}{4}(WbW + bW + Wb + b). \end{aligned}$$

Hence $ebe = E(b)e$ for $b \in B$. Also $A \ni a \mapsto ae \in B \times_\beta \mathbb{Z}_2$ is injective. In fact, if $ae = 0$, $aW + a = 0$. Let $\widehat{\beta}$ be the dual action of β. Then $0 = \widehat{\beta}(aW + a) = -a + a$. Thus $2a = 0$, i.e., $a = 0$. Thus by Watatani[10, Proposition 2.2.11], $C^*\langle B, e_A\rangle \cong B \times_\beta \mathbb{Z}_2$. □

Remark 2.1.2. (1) By the proofs of Watatani[10, Propositions 2.2.7 and 2.2.11], we see that $\kappa(b) = b$ for any $b \in B$ where κ is the isomorphism of $C^*\langle B, e_A\rangle$ onto $B \times_\beta \mathbb{Z}_2$ in Lemma 2.1.3.
(2) The above lemma is obtained in Kajiwara and Watatani [5, Theorem 5.13]

By Lemma 2.1.3 and Remark 2.1.2, we regard $\widehat{\beta}$ as an automorphism of $C^*\langle B, e_A\rangle$ with $\widehat{\beta}(b) = b$ for any $b \in B, \widehat{\beta}^2 = id$ and $\widehat{\beta}(e_A) = 1 - e_A$.

Lemma 2.1.4. *With the same assumptions as in Lemma 2.1.3,*

$$C^*\langle B, e_A\rangle^{\widehat{\beta}} = B.$$

Proof. By Lemma 2.1.3 for any $x \in C^*\langle B, e_A\rangle$, we can write $x = b_1 + b_2 U$, where $b_1, b_2 \in B$, We suppose that $\widehat{\beta}(x) = x$. Then $b_1 - b_2 U = b_1 + b_2 U$. Thus $b_2 = 0$. Hence $x = b_1 \in B$. Since it is clear that $B \subset C^*\langle B, e_A\rangle^{\widehat{\beta}}$, we obtain the conclusion. □

2.2. **Involutive equivalence bimodules.** Let A be a unital C^*-algebra and $X (= {}_A X_A)$ a complete A-A equivalence bimodule. X is *involutive* if there exists a conjugate linear map $x \to x^\sharp$ on X, such that

(1) $(x^\sharp)^\sharp = x, \quad x \in X$,
(2) $(a \cdot x \cdot b)^\sharp = b^* x^\sharp a^*, \quad x \in X, a, b \in A$,
(3) ${}_A\langle x, y^\sharp\rangle = \langle x^\sharp, y\rangle_A, \quad x, y \in X$,

where ${}_A\langle , \rangle$ and $\langle , \rangle_A$ are the left and right A-valued inner products of X.

Lemma 2.2.1. *Let V be a map of X onto its dual bimodule $\widetilde{X}$ defined by $V(x) = \widetilde{x^\sharp}$. Then V is a bimodule isomorphism preserving the left and right A-valued inner products.*

Proof. By $a \cdot \widetilde{x} \cdot b = \widetilde{b^* \cdot x \cdot a^*}$, for $a, b \in A$ and $x \in X$,

$$\begin{aligned} V(a \cdot x \cdot b) &= \widetilde{(a \cdot x \cdot b)^\sharp} \\ &= \widetilde{b^* \cdot x^\sharp \cdot a^*} \\ &= a \cdot \widetilde{x^\sharp} \cdot b = a \cdot V(x) \cdot b. \end{aligned}$$

By ${}_A\langle x, y^\sharp\rangle = \langle x^\sharp, y\rangle_A$ and $(x^\sharp)^\sharp = x$, for $x, y \in X$,

$$\begin{aligned} {}_A\langle V(x), V(y)\rangle^{\sim} &= {}_A\langle \widetilde{x^\sharp}, \widetilde{y^\sharp}\rangle^{\sim} \\ &= \langle x^\sharp, y^\sharp\rangle_A \\ &= {}_A\langle x, (y^\sharp)^\sharp\rangle = {}_A\langle x, y\rangle. \end{aligned}$$

Similarly, $\langle V(x), V(y)\rangle_{\tilde{A}}^{\sim} = \langle x, y\rangle_A$. Thus we obtain the conclusion. □

3. Correspondence between involutive equivalence bimodules and inclusions of C^*-algebras with Watatani index2

Let A be a unital C^*-algebra and we denote by (B, E) a pair of a unital C^*-algebra B including A with a common unit and a conditional expectation E of B onto A with $\mathrm{Index}E = 2$. Let $\mathcal{L}$ be the set of all such pairs (B, E). We define an equivalence relation $\sim$ in $\mathcal{L}$ as follows: For $(B, E), (B_1, E_1) \in \mathcal{L}$, $(B, E) \sim (B_1, E_1)$ if and only if there is an isomorphism π of B onto B_1 such that $\pi(a) = a$ for any $a \in A$ and $E_1 \circ \pi = E$. We denote by $[B, E]$ the equivalence class of (B, E).

Let $\mathcal{M}$ be the set of all complete involutive A-A equivalence bimodules. We define an equivalence relation $\sim$ in $\mathcal{M}$ as follows: For $X, Y \in \mathcal{M}$, $X \sim Y$ if and only if there is a bimodule isomorphism ρ of X onto Y preserving the left and right A-valued inner products with $\rho(x^\sharp) = \rho(x)^\sharp$. We denote by $[X]$ the equivalence class of X. Then we have the next theorem.

Theorem 3.0.1. *There is a* 1-1 *correspondence between* $\mathcal{L}/\sim$ *and* $\mathcal{M}/\sim$.

4. Involutive equivalence bimodules for simple C^*-algebras

4.1. Construction of involutive equivalence bimodules by $2\mathbb{Z}$-inner C^*-dynamical systems. Let A be a simple unital C^*-algebra and α an automorphism of A and we suppose that $\alpha^2 = Ad(z)$ where z is a unitary element in A with $\alpha(z) = z$. Let X_α be the vector space A with the obvious left action of A on X_α and the obvious left A-valued inner product, but we define the right action of A on X_α by $x \cdot a = x\alpha^{-1}(a)$ for any $x \in X_\alpha$ and $a \in A$, and the right A-valued inner product by $\langle x, y\rangle_A = \alpha(x^*y)$ for any $x, y \in X_\alpha$.

Proposition 4.1.1. *With the above notations, Let B_{X_α} be a C^*-algebra defined by X_α and L the linking algebra for X_α as defined in Section 3. Then the following conditions are equivalent:*

(1) B_{X_α} *is simple,*
(2) $A' \cap B_{X_\alpha} = \mathbb{C} \cdot 1$,
(3) $B'_{X_\alpha} \cap L = \mathbb{C} \cdot 1$,
(4) α *is an outer automorphism of* A.

Let B be a unital C^*-algebra and A a C^*-subalgebra of B with a common unit. Let E be a conditional expectation of B onto A with $\mathrm{Index}E = 2$. For any $n \in \mathbb{N}$ let M_n be the $n \times n$-matrix algebra over $\mathbb{C}$ and $M_n(A)$ the $n \times n$-matrix algebra over A. Let $\{x_i, x_i^*)\}_{i=1}^n$ be a quasi-basis for E. We define $q = [q_{ij}] \in M_n(A)$ by $q_{ij} = E(x_i^* x_j)$. Then by Watatani [10], q is a projection and $C^*\langle B, e_A\rangle \simeq qM_n(A)q$. Let π be an isomorphism of $C^*\langle B, e_A\rangle$ onto $qM_n(A)q$ defined by

$$\pi(ae_A b) = [E(x_i^* a)E(bx_j)] \in M_n(A)$$

for any $a, b \in B$. Especially for any $b \in B$,

$$\pi(b) = [E(x_i^* b x_j)]$$

since $\sum_{i=1}^n x_i e_A x_i^* = 1$.

Proposition 4.1.2. *With the above notations, the following conditions are equivalent:*

(1) e_A *and* $1 - e_A$ *are equivalent in* $C^*\langle B, e_A\rangle$,

(2) *there exists a unitary element* $u \in B$ *such that* $\{(1,1),(u,u^*)\}$ *is a quasi basis for* E,

(3) *there exists a* $2\mathbb{Z}$*-inner* C^**-dynamical system* $(A, \mathbb{Z}, \alpha)$ *such that* $X_\alpha \sim X_B$.

Let θ be an irrational number in $(0,1)$ and A_θ the corresponding irrational rotation C^*-algebra. Let B be a unital C^*-algebra including A_θ as a C^*-subalgebra of B with a common unit. We suppose that there is a conditional expectation E of B onto A_θ with $\mathrm{Index}E = 2$ and that $A'_\theta \cap B = \mathbb{C} \cdot 1$

Proposition 4.1.3. *With the above notation there is a* $2\mathbb{Z}$*-inner* C^**-dynamical system* $(A_\theta, \mathbb{Z}, \alpha)$ *such that* $(B, E) \sim (A \times_{\alpha/2\mathbb{Z}} \mathbb{Z}, F)$, *where* F *is the canonical conditional expectation of* $A \times_{\alpha/2\mathbb{Z}} \mathbb{Z}$ *onto* A.

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