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Topology of Lagrangian Submanifolds

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Y. Eliashberg gave a talk on topology of Lagrangian submanifolds at a conference held at RIMS from 9 to 12 May 2000. Here we note only a part of his talk.

The content of Sections 1 and 2, except Theorem 1.4 can be found in [1]. Theorem 1.4 is joint with L. Polterovich and is contained in [2]. Results stated in Section 3 are extracted from a joint with M. Gromov paper [3].

1 Unknotting of Lagrangian surfaces in symplectic 4-manifold

Let (M^{2n},ω) be a symplectic manifold. An n-dimensional submanifold L is called a *Lagrangian submanifold* if $\omega|_L = 0$.

Example $M = \mathbb{R}^{2n} = \mathbb{C}^n, \omega_0 = \sum_{i=1}^{n} dx^i \wedge dy^i$, where $(z_1,\cdots,z_n) = (x_1 + iy_1,\cdots,x_n+iy_n)$ is the standard coordinate of $\mathbb{C}^n$, is a symplectic manifold. In this case, a linear n-dimensional plane L is Lagrangian if and only if $iL \perp L$. If instead we have $iL \pitchfork L$, then L is called *totally real*. General totally real submanifolds are defined in an obvious manner.

We will treat $n=2$ case of the above example. The first result we will mention is the following *unknottedness theorem*.

Theorem 1.1. *Let* $\mathbb{R}^4_+ = \{y_2 \geq 0\}$ *and assume that a 2-disk* Δ *is embedded in* $\mathbb{R}^4_+$ *as* $(\Delta,\partial\Delta) \subset (\mathbb{R}^4_+,\partial\mathbb{R}^4_+)$ *and* $\partial\Delta = \{|z_1| = 1, z_2 = 0\}$. *Then, if we have* $\omega|_\Delta \geq 0$, *then* Δ *is unknotted, i.e. we can isotope* Δ *relative to* $\partial\Delta$ *to a disk in* $\partial\mathbb{R}^4_+$.

The proof of this theorem relies on the method of *filling with holomorphic discs* and we quot the necessary result here. We first define the pseudoconvexity of an oriented hypersurface Σ of general symplectic manifold $(M^{2n},\omega$

Let J be an almost complex structure on M tamed by ω. Then, for every point x on Σ, the tangent space T_xM has a J-invariant (2n - 2) dimensional subspace $T_x^J\Sigma$. $\bigcup_{x\in M} T_x^J\Sigma$ is a (2n - 2) dimensional subbundle T^JM of TM. Since Σ is oriented and each $T_x^J\Sigma$ has a natural orientation as a complex vector space, the quotient 1-dimensional bundle $T\Sigma/T^J\Sigma$ is also orientable, i.e. trivial. In particular, there is a trivial sub-line bundle $\underline{\mathbb{R}}$ of $T\Sigma$ such that $T\Sigma = \underline{\mathbb{R}} \oplus T^J\Sigma$. Choosing a non-vanishing section η of $\underline{\mathbb{R}}$ fixes a 1-form α on Σ satisfying $\alpha|_{T^J\Sigma} = 0$ and $\alpha(\eta) > 0$.

Definition 1.1. Σ is called *J – convex*, or *pseudoconvex* if the quadratic form $t \mapsto d\alpha(t, Jt)$ on $T^J\Sigma$ is positive definite.

With this preparation, we can state the following result.

Theorem 1.2. *Let Ω be a domain in $\mathbb{R}^4$ such that $\partial\Omega$ is pseudo convex w.r.t. some almost complex structure J tamed by ω_0. Let F be a surface with boundary embedded in $\partial\Omega$ such that F has a unique complex point which is elliptic, and J is integrable near that point. Moreover, assume that there is a J-holomorphic disc Δ with $\partial F = \partial\Delta$ and which is transversal to $\partial\Omega$ along $\partial\Delta$. Then $F \cup \Delta$ can be filled with a family of embedded, disjoint J-holomorphic discs $\{D_t\}$.*

Now we explain the outline of the proof of the unknottedness theorem. First, we take a large sphere S in $\mathbb{R}^4$ with the center on the y_2-axis which intersects with the z_1-plane along $\partial\Delta$, and let B be the interior domain of S. We can take a disk F in S whose boundary coincides with $\partial\Delta$ and has a unique complex point which is elliptic, and moreover it is isotopic to a disk on $\partial\mathbb{R}^4_+$ relative to the boundary. On the otherhand, the disk Δ can be slightly deformed by a boundary fixing isotopy so that $\omega|_\Delta > 0$. Taking B large enough, we can suppose that Δ is contained in B. Then, there is an almost complex structure J tamed by ω_0 for which Δ is J-holomorphic. Moreover J can be chosen integrable near the elliptic point of F. This will allow us to apply the filling with holomorphic disc technique to the triple $(\Omega = B, F, \Delta)$, and thus will supply us with the isotopy mentioned in the theorem.

Using the same technique, we can prove the next theorem.

Theorem 1.3. *Let Π_0 and Π_1 denote the hyperplanes $\{y_2 = 0\}$ and $\{y_2 = 1\}$, and let L_0 be the Lagrangian cylinder $\{|z_1| = 1, x_2 = 0, 0 \le y_2 \le 1\}$. Suppose L is another Lagrangian cylinder between Π_0 and Π_1 having the same boundary as L_0. Then, L is Lagrangian isotopic to L_0 relative to the boundary in $\mathbb{R}^4 \setminus (D_+ \cup D_- \cup R_+)$, where $D_+ = \{|z_1| \le 1, z_2 = 0\}$, $D_- = \{|z_1| \le 1, z_2 = 1\}$, and $R_+ = \{y_2 \ge 1, x_2 = z_1 = 0\}$.*

(Outline of the proof) We again replace the plane Π_0 by a boundary $\partial\Omega$ of a large convex domain Ω such that $\partial\Omega$ intersects with the z_1-plane along the unit circle C. As before, we can take a disk F whose boundary coincides with C and which has a unique complex point which is elliptic. On the otherhand, we can modify the cylinder Δ by a boundary fixing isotopy, as well as gluing a disk on the top of it, so that the resulting disk Δ will have the boundary C, on which the symplectic form is positive. Then, as before, we can choose an almost complex structure J integrable near the elliptic point of F, tamed by ω_0, with respect to which Δ is holomorphic, and then apply the filling with holomorphic disks technique to (F, Δ). This will supply the isotopy we want.

The next is the unknottedness result for Lagrangian knots in $\mathbb{R}^4$.

Theorem 1.4. *There is no knotted Lagrangian plane in* $\mathbb{R}^4$. *That is, if* $\phi : \mathbb{R}^2 \longrightarrow (\mathbb{R}^4, \omega_0)$ *is a Lagrangian embedding which coincides with the inclusion* $i : \mathbb{R}^2 \longrightarrow \mathbb{C}^2$ *defined by* $(x, y) \mapsto (x, 0, 0, y)$ *outside of a compact set, then there is a compact supported Lagrangian isotopy between* ϕ *and* i.

(outline of the proof) This theorem is a consequence of the following two results.

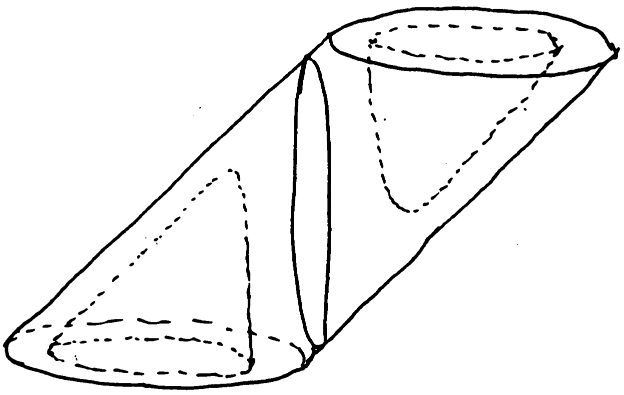
Proposition 1. *If a Lagrangian knot* L *in* $\mathbb{R}^4$ *is contained in some simple hypersurface* Q, *then* L *is Lagrangian isotopic to the flat plane.*

Proposition 2. *For every Lagrangian knot* L *in* $\mathbb{R}^4$, *there is a simple hypersurface* Q *containing it.*

We first explain the word *simple hypersurface*. Let R be a oriented hypersurface in $(\mathbb{R}^4, \omega_0)$. Then, the symplectic form ω_0 restricted to R defines an oriented 1-dimensional distribution on R by $Ker\omega_0$. R integrates into a 1-dimensional foliation. We call this foliation *characteristic.*

Definition 1.2. A hypersurface Q in $\mathbb{R}^4$ is called *simple* if each leaf of its characteristic foliation is diffeomorphic to $\mathbb{R}$ and outside a compact set of Q, each leaf coincide with a part of one of parallel straight lines of a given direction.

The proof of proposition 1 is carried out by constructing a 2-dimensional foliation $\{M_t\}_{t\in\mathbb{R}}$ on Q such that each leaf is a Lagrangian diffeomorphic to $\mathbb{R}^2$, $M_0 = L$ and M_t are embedded standard $\mathbb{R}^2$s for $t < -1, t > 0$. It can be done using the characteristic foliation. As for the proof of proposition 2, we need the filling with holomorphic disks technique. Namely, one first takes a 2-dimensional foliation whose leaves consist of trajectories of the characteristics foliation which intersect at $-\infty$ a line, parallel to a given direction. The constructed foliation is not flat at $+\infty$, but can be flatten via an appropriate Hamiltonian isotopy. We first fix some notations. Let (u, v, x, y) be the coordinate for $\mathbb{R}^4$, Q_0 be the hyperplane $\{v = 0\}$, L_0 be the standard Lagrangian plane $\{(u, 0, 0, y)\}$ and $\Sigma_0 = L_0 \cap C$. Let $C = \{(x-u)^2 + y^2 \leq 1\}$ and $K = \{(x-u)^2 + y^2 \leq 1/2\}$ be two cylinders contained in $\mathbb{R}^3 = \{(u, x, y)\}$. There is a convex domain V_δ defined by $V_\delta = \{-\delta\phi(u, x, y) < v < \delta\phi(u, x, y)\}$ where $\delta > 0$ and $\phi(u, x, y) = 1-(x-y)^2-y^2$. It satisfies $\partial V_\delta \supset \partial C$. Then, by a suitable dilatation, we can suppose that our Lagrangian knot L coincides with L_0 outside of K and is contained in V_δ. We now isotope $C \cap \{-1 \leq u \leq 1\}$ to a set like the figure below.

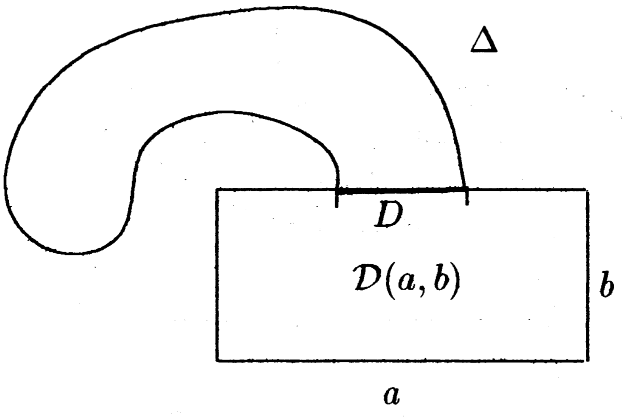


We denote this map by Φ. This can be done so that the images of the disks $\{t\} \times D^2$ are symplectic. We call the image of the discs by N. Then, there is a symplectic embedding χ from a neighbourhood of N to V such that $\chi(\Sigma_0) = V \cap L$ and χ is the identity outside K. We can define an almost complex structure J on $\mathbb{R}^4$ tamed by ω_0 such that the image of the disks $\{t\} \times D^2$ by the map $\chi \circ \Phi$ are J-holomorphic and flat near ∂V and outside of a compact set in $\mathbb{R}^4$. Then, since ∂C is contained in a pseudo convex boundary, examining the Maslov class of the generator of the first homology group of ∂C, we see that we can extend $\chi \circ \Phi$ to the whole cylinder C in a way that images of the discs $\{t\} \times D^2$, $t \in \mathbb{R}$ are J-holomorphic and for $|t|$ larger than 1, the map on $\{t\} \times D^2$ is the identity. If we call this map F, then $Q = (Q_0 - C \cap \{-1 \leq u \leq 1\}) \cup F(\{-1 \leq u \leq 1\})$ is the required simple hypersurface.

2 Invariants of S^2-knots in $\mathbb{R}^4$ via symplectic geometry

Let $f : S^2 \hookrightarrow \mathbb{R}^4$ be an embedding, and $\alpha := [f]$ the isotopy class of f. Let us denote by $\mathcal{D}(a,b)$ the polydisc $\{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| \leq a,\ |z_2| \leq b\}$.

We say that the class α admits a *(a,b)-realization for $a > 1$, $b > 0$* if α can be represented by an embedded sphere $S = \Delta \cup D \subset \mathbb{R}^4$ where $D = \{|z_1| \leq 1,\ z_2 = b\}$ and Δ is a 2-disk satisfying the following properties: $(\Delta, \partial\Delta) \subset (\mathbb{C}^2 \setminus \mathrm{Int}\mathcal{D}(a,b), \partial\mathcal{D}(a,b))$ intersects $\partial\mathcal{D}(a,b)$ transversely along the circle $\partial\Delta = \{|z_1| = 1,\ z_2 = b\}$, and $\omega|_\Delta > 0$.



Lemma 2.1. *For any isotopy class α of embeddings $S^2 \hookrightarrow \mathbb{R}^4$, there exist $a > 1$, $b > 0$ such that α admits a (a,b)- realization.*

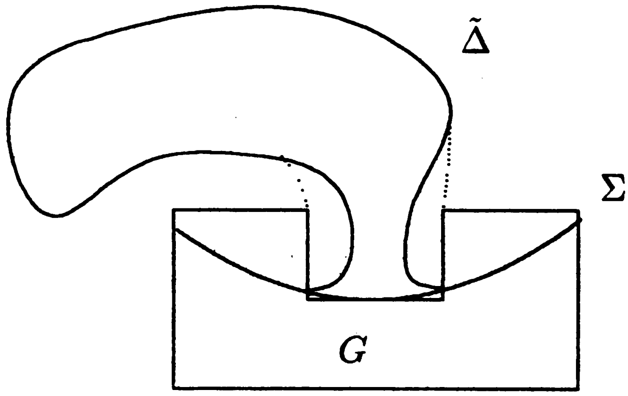
The following theorem asserts that a symplectic 2-disc cannot be knotted not only in the half-space but even in the complement of a sufficiently large polydisc.

Theorem 2.2. *If $[f]$ admits a $(3,2)$-realization, then it is trivial.*

We sketch the proof of this theorem. Set the following notations:

$$\begin{aligned}
\Omega &= \{x_2 \le \varepsilon|z_1|^2/(1-\varepsilon)^2\} \text{ where } z_2 = x_2 + iy_2 \\
\Sigma &= \partial\Omega \cap \mathcal{D}(a,b) \\
A_{c,d} &= \{|z_1| \le c,\ |y_2| \le d\} \\
\Sigma_{c,d} &= A_{c,d} \cap \Sigma \\
G &= \mathcal{D}(a,b) \setminus (A_{1,\varepsilon} \cap \Omega) \\
S &= \{y_2 = 0,\ |z_1| \le 1-\varepsilon\} \cap \Sigma.
\end{aligned}$$

Deform Δ into the following form, and denote the resulting disc by $\tilde{\Delta}$.



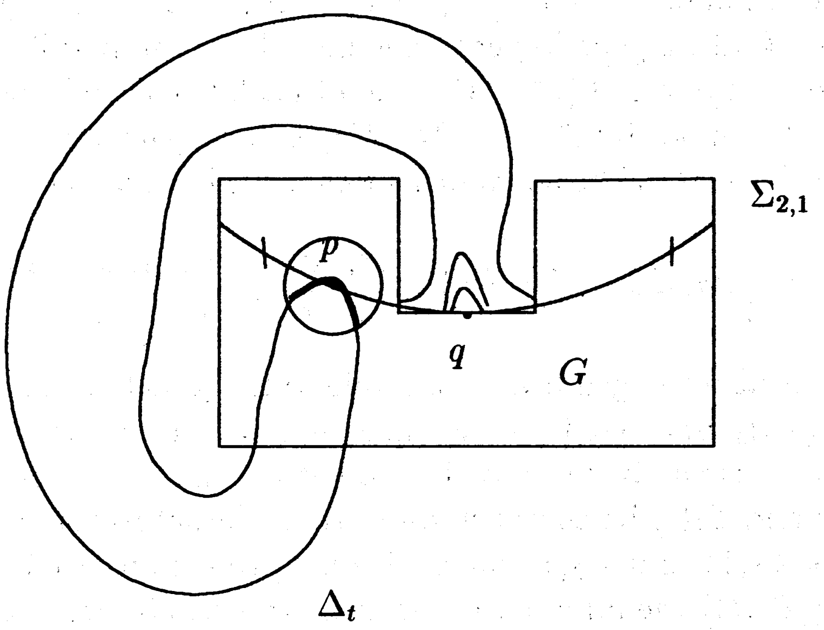
The disc $\tilde{\Delta}$ intersects Σ transversely along $\partial\tilde{\Delta} = \{|z_1| = 1-\varepsilon,\ z_2 = \varepsilon\}$. We can assume that $\omega|_{\tilde{\Delta}} > 0$ and $\tilde{\Delta}$ is holomorphic near $\partial\tilde{\Delta}$ (with respect to the standard complex structure on $\mathbb{C}^2$). Let us choose an almost complex structure J on $\mathbb{R}^4$ such that:

- J is tamed by ω.
- J is standard on G, near Σ and at infinity.
- $\tilde{\Delta}$ is J-holomorphic.

Then, the theorem can be deduced from the following:

Lemma 2.3. *The pair $(S, \tilde{\Delta})$ can be filled with J-holomorphic discs.*

Let $q \in S$ be the elliptic point of S, and $\{\Delta_t\}_t$ be a Bishop family of J-holomorphic disks developing from q. To show the lemma, it is sufficient to prove that $\mathrm{Int}\Delta_t \cap \Sigma_{1,\varepsilon} = \emptyset$. We want to eliminate the following case.



Notice that no disk can be tangent to a strictly pseudoconvex hypersurface from a convex side.

Suppose that some disc Δ_t is tangent to $\Sigma_{2,1}$ at a point p from the concave side. Observe that for any t we have

$$\int_{\Delta_t} \omega < \int_S \omega = \pi(1-\varepsilon)^2 \qquad \text{by Stokes' theorem.}$$

On the other hand, holomorphic curves have the following monotonicity property:

Lemma 2.4. *Let C be a properly embedded holomorphic curve in the open ball B of radius r in $\mathbb{C}^n$. Suppose that C contains the center of B. Then* Area $C \geq \pi r^2$.

We apply this lemma to $C = \Delta_t$, $B = B_{1-\varepsilon}(p)$. By assumption, $B \cap \Delta_t$ is contained in G, and J is standard on G. Therefore

$$\pi(1-\varepsilon)^2 \leq \mathrm{Area}(\Delta_t \cap B) \leq \int_{\Delta_t} \omega.$$

This contradicts the inequality $\int_{\Delta_t} \omega < \pi(1-\varepsilon)^2$.

3 Legendrian linking problem

Let V be a manifold and $PT^*(V)$ the projetivized cotangent bundle, i.e., the space of all tangent hyperplanes in $T(V)$. The manifold $PT^*(V)$ has a contact structure $\eta \subset T(PT^*(V))$ such that lift of each hypersurface $W \subset V$ to $PT^*(V)$, denote by $\mathcal{L}_W \subset PT^*(V)$, is a Legendrian submanifold for η. Moreover, let $W \subset V$ be a smooth submanifold of positive codimension. Put

$$\mathcal{L}_W := \left\{(w, H_w) \in PT^*(V) \;\middle|\; \begin{array}{l} H_w \text{ is a hypersurface such that} \\ T_w(W) \subset H_w \subset T_w(V) \end{array} \right\}.$$

Then $\mathcal{L}_W$ is also a Legendrian submanifold for η. Let W_1 and W_2 be submanifolds properly immersed into V such that they intersect transversely. Here "properly" means "being closed as a subset in V". Then $\mathcal{L}_{W_1} \cap \mathcal{L}_{W_2} = \emptyset$. Let $\mathcal{L}_1(t)$ and $\mathcal{L}_2(t)$ be compact supported contact isotopies of $\mathcal{L}_{W_1}$ and $\mathcal{L}_{W_2}$ such that $\mathcal{L}_1(1)$ and $\mathcal{L}_2(1)$ have disjoint projections to V. We denote by $\sharp(\mathcal{L}_1(t) \underset{reg}{\times} \mathcal{L}_2(t))$ the minimal number of crossings between all (compact supported) contact isotopies $\mathcal{L}_1(t)$ and $\mathcal{L}_2(t)$ which intersect transeversely and move $\mathcal{L}_1(0)$ and $\mathcal{L}_2(0)$ to $\mathcal{L}_1(1)$ and $\mathcal{L}_2(1)$.

Theorem 3.1. *Suppose $W_1 \cap W_2$ is compact, then we have*

$$\sharp(\mathcal{L}_1(t) \underset{reg}{\times} \mathcal{L}_2(t)) \geq \frac{1}{2} \operatorname{rank} H_*(W_1 \boxtimes W_2),$$

where $W_1 \boxtimes W_2$ denote the set $\{(w_1, w_2) \in W_1 \times W_2 \mid w_1 = w_2\}$.

Let $V = W \times \mathbb{R}$, $W_1 \subset W \times \mathbb{R}$, and the projection $W_1 \to W$ has non-zero degree. Here we assume W and W_1 connected orientable manifolds of the same dimension. One can drop the orientability condition if works with coefficient $\mathbb{Z}_2$. Moreover let $W_2 \subset W$ be a compact submanifold which lies on the left of W_1, i.e., $W_1 \cap \{(w_2, t_2 + t) \in W \times \mathbb{R} | (w_2, t_2) \in W_2, t \leq 0\} = \emptyset$.

Theorem 3.2. *If the projection of $\mathcal{L}_2(1)$ to V lies on the right of the projection $\mathcal{L}_1(1)$, then we have*

$$\sharp(\mathcal{L}_1(t) \underset{reg}{\times} \mathcal{L}_2(t)) \geq \operatorname{rank} H^*(W_2).$$

The proofs of these theorems rely on the generating functions and the stable Morse theory.

Postscript. In this lecture note we could note only a part of Eliashberg's talk. He mentioned many other topics on symplectic field theory (SFT), symplectic cobordisms, compactness properties, generalized Viterbo's theorem, Lagrangian skeletons, Lagrangian tori in $\mathbb{R}^4$ and so on.

References

[1] Y. Eliashberg, Topology of 2-knots in $\mathbb{R}^4$ and symplectic geometry, The Floer memorial volume, 335-353, 1995.

[2] Y. Eliashberg and L. Polterovich, Local Lagrangian 2-knots are trivial, Ann. of math., **144** (1996), 61-76.

[3] Y. Eliashberg and M. Gromov, Lagrangian intersection theory: Finite-dimensional approach, Amer. Math. Soc. Transl. (2) **186** (1998), 27-116.