数理解析研究所講究録 1233 巻 2001 年 18-50

Towards the Classification of Atoms of Degenerations, I

Splitting Criteria via Configurations of Singular Fibers

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VERSION: July 31, 2001

Abstract

Motivated by the classification problem of atomic degenerations, in our series of papers, we make a systematic study for splitting deformations of degenerations of complex curves. We provide various new methods to construct splitting deformations, and deduce many splitting criteria of degenerations, which will be applied to the classification of atomic degenerations. Roughly, our criteria are separated into two types; in the first type the criteria are expressed in terms of the configuration of a singular fiber, and in the second type, in terms of sub-divisors of a singular fiber. In both types, our constructions are 'visible', in that we can view how the singular fiber is deformed. In the present paper, we demonstrate splitting criteria of the first type.

thematical Subject Classification: Primary 14D05, 14J15; Secondary 14H15, 32S30
words: Degeneration of complex curves, Complex surface, Singular fiber, Riemann surf
ormation of complex structures, Splittings of singular fibers, Atomic degeneration, Monodr

Introduction

This paper constitutes one part of our series of papers on degenerations. By a degeneration, we mean a proper surjective map $\pi : M \to \Delta$ from a smooth complex surface M to the unit disk Δ such that the fiber over the origin is singular and any other fiber is a smooth curve of genus g ($g \geq 1$). A deformation of a degeneration is called a *splitting deformation*, provided that it induces a splitting of its singular fiber. We notice that it may occur that a degeneration admits no splitting deformation at all, in which case the degeneration is called *atomic*. Our main problem is to classify atomic degenerations of arbitrary genera (see [Re]). The classification has been known only for the very low genus cases; for the genus 1 case, by Moishezon [Mo], and for the genus 2 case, by Horikawa [Ho] (see also §6.3), where they used the double covering method for constructing splitting deformations.

Recent progress for the genus 3 case was made by Ashikaga and Arakawa [AA], who obtained results on the classification of atomic degenerations of hyperelliptic curves of genus 3. Their method is also based on the double covering method. Unfortunately, this method fails to work for degenerations of non-hyperelliptic curves. Some new idea is needed for constructing splitting deformations of degenerations of non-hyperelliptic curves even for the genus 3 case (note that for the genus 1 and 2 cases, all curves are hyperelliptic, but this is not the case for genus ≥ 3). In our series of papers we develop completely different methods for constructing splitting deformations, and apply them to the classification of atomic degenerations for the genus 3,4 and 5 cases [Ta,III, Ta]. The aim of this paper is to study the relation between the configurations of singular fibers and the existence of splitting deformations. We first show that two types of degenerations are atomic.

Theorem 2.0.2 *Let $\pi : M \to \Delta$ be a degeneration of curves such that the singular fiber X is either* (I) *a reduced curve with one node, or* (II) *a multiple of a smooth curve of multiplicity at least* 2. *Then $\pi : M \to \Delta$ is atomic.*

We remark that the proof of Theorem 2.0.2 carrries over to arbitrary dimensions to show that a degeneration of type (II) is atomic, i.e. letting $\pi : M \to \Delta$ be a degeneration of compact complex manifolds of arbitrary dimension, if the singular fiber X is a multiple of a smooth complex manifold, then $\pi : M \to \Delta$ is atomic.

Next, we shall state results on existence of splitting deformations. We demonstrate several splitting criteria via the configuration of the singular fiber. Roughly, these criteria are classified into two types; the first one is in terms of some singularities on the singular fiber and the second one is in terms of the existence of irreducible components of multiplicity 1 satisfying certain properties (see the list of splitting criteria in the bottom of this introduction). Most of our criteria also give the explicit description of splittings of singular fibers. We note that the commutativity of some topological monodromies follows from one of these criteria (see Proposition 6.1.2). From our criteria, we will see that many degenerations with non-star-shaped[1] singular fibers always admit splitting deformations. Together with Theorem 2.0.2 it is

[1]See §4.

interesting to know whether the following is true or not.

Conjecture 6.3.1 *A degeneration is atomic if and only if its singular fiber is either a reduced curve with one node, or a multiple of a smooth curve.*

See [Ta,III], [Ta] for results on this conjecture. (Actually, this conjecture seems too optimistic for higher genus cases. A more reasonable conjecture is given by replacing 'atomic' by 'absolutely atomic', where a degeneration $\pi : M \to \Delta$ is *absolutely atomic* provided that all degenerations with the same topological type as $\pi : M \to \Delta$ are atomic.) In order to classify atomic degenerations, the results of this paper enable us to use the induction with respect to genus g (see §6.3 for details); let Λ_g be a set of degenerations $\pi : M \to \Delta$ of curves of genus g such that

(1) the singular fiber X has a multiple node[2] (here we exclude the case where X is a reduced curve with only one node), or

(2) X contains an irreducible component Θ_0 of multiplicity 1 satisfying the following condition[3]: if $X \setminus \Theta_0$ is connected, then either genus$(\Theta_0) \geq 1$, or Θ_0 is a projective line intersecting other irreducible components at at least two points.

As a consequence of our splitting criteria, we obtain the following.

Theorem 6.3.2 *Suppose*[4] *that Conjecture* 6.3.1 *is valid for genus* $\leq g-1$. *If* $\pi : M \to \Delta$ *is a degeneration in* Λ_g*, then* π *is not atomic.*

Hence, if the assumption of this theorem is fulfilled, to determine atomic degenerations of curves of genus g, it suffices to check the splittability of degenerations $\pi : M \to \Delta$ such that

(A) $X = \pi^{-1}(0)$ is star-shape, or

(B) X is not star-shaped and (B.1) X has no multiple node and (B.2) if X has an irreducible component Θ_0 of multiplicity 1, then Θ_0 is a projective line, and intersects other irreducible components of X only at one point.

In [Ta,III], we develop another method for constructing splitting deformations, which uses 'barkable' sub-divisors in singular fibers. This method is quite powerful and works for degenerations satisfying (A) or (B).

List of splitting criteria via configurations of singular fibers

(In most cases, we assume that a degeneration is normally minimal (see §1). This assumption is not restrictive at all. See §1. We notice that in some cases, two different criteria are applicable to one degeneration.)

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[2]A multiple node is either an intersection point of two irreducible components of the same multiplicity, or a self-intersection point of an irreducible component.

[3]If $X \setminus \Theta_0$ is not connected, we pose no condition.

[4]This assumption is valid for $g = 2$ and 3.

Criterion 5.1.2 *Let $\pi : M \to \Delta$ be normally minimal such that the singular fiber X has a multiple node of multiplicity at least 2. Then there exists a splitting deformation of $\pi : M \to \Delta$, which splits X into X_1 and X_2, where X_1 is a reduced curve with one node and X_2 is obtained from X by replacing the multiple node by a multiple annulus.*

Criterion 5.1.3 *Let $\pi : M \to \Delta$ is normally minimal such that the singular fiber X contains a multiple node (of multiplicity ≥ 1). Then $\pi : M \to \Delta$ is atomic if and only if X is a reduced curve with one node.*

Criterion 5.2.2 *Let $\pi : M \to \Delta$ be relatively minimal. Suppose that the singular fiber X has a point p such that a germ of p in X is either*

(1) *a multiple of a plane curve singularity*[5] *of multiplicity at least 2, or*

(2) *a plane curve singularity such that if it is a node, then $X \setminus p$ is not smooth.*

Then $\pi : M \to \Delta$ admits a splitting deformation.

Criterion 6.1.1 *Let $\pi : M \to \Delta$ be normally minimal. Suppose that the singular fiber X contains an irreducible component Θ_0 of multiplicity 1 such that $X \setminus \Theta_0$ is (topologically) disconnected. Denote by $Y_1, Y_2, \ldots, Y_l$ $(l \geq 2)$ all connected components of $X \setminus \Theta_0$. Then $\pi : M \to \Delta$ admits a splitting deformation which splits X into $X_1, X_2, \ldots, X_l$, where X_i $(i = 1, 2, \ldots, l)$ is obtained from X by 'smoothing' $Y_1, Y_2, \ldots, \check{Y}_i, \ldots, Y_l$. Here $\check{Y}_k$ is the omission of Y_i.*

Criterion 6.2.1 *Let $\pi : M \to \Delta$ be normally minimal such that the singular fiber X contains an irreducible component Θ_0 of multiplicity 1. Let $\pi_1 : M_1 \to \Delta$ be the restriction of π to a tubular neighborhood M_1 of $X \setminus \Theta_0$ in M. Suppose that $\pi_1 : M_1 \to \Delta$ admits a splitting deformation Ψ_1 which splits $Y^+ := M_1 \cap X$ into $Y_1^+, Y_2^+, \ldots, Y_l^+$. Then $\pi : M \to \Delta$ admits a splitting deformation Ψ which splits X into $X_1, X_2, \ldots, X_l$, where X_i is obtained from Y_i^+ by gluing Θ_0^- along the boundary.*

Acknowledgment. I would like to express my deep gratitude to Professor Tadashi Ashikaga for valuable discussions and warm encouragement. It is also my great pleasure to thank Professor Fumio Sakai for valuable advice and suggestions after he read the early draft of this paper. I also would like to thank Professors Toru Gocho and Mizuho Ishizaka for fruitful discussions. I also would like to thank the Max-Planck-Institut für Mathematik at Bonn, and the Research Institute for Mathematical Sciences at Kyoto University for their hospitality and financial support.

1 Preparation

In this paper, $\Delta := \{s \in \mathbb{C} : |s| < 1\}$ stands for the unit disk. Let $\pi : M \to \Delta$ be a proper surjective holomorphic map from a smooth complex surface M to Δ,

[5] In this paper a plane curve singularity always means a reduced one.

such that $\pi^{-1}(0)$ is singular, and $\pi^{-1}(s)$, $(s \neq 0)$ is a smooth complex curve of genus g $(g \geq 1)$. We say that $\pi : M \to \Delta$ is a *degeneration* of complex curves of genus g with the *singular fiber* $X := \pi^{-1}(0)$. Two degenerations $\pi_1 : M_1 \to \Delta$ and $\pi_2 : M_2 \to \Delta$ are called *topologically equivalent* if there are orientation preserving homeomorphisms $H : M_1 \to M_2$ and $h : \Delta \to \Delta$, which make the following diagram commutative:

$$\begin{tikzcd} M_1 \arrow[r, "H"] \arrow[d, "\pi_1"'] & M_2 \arrow[d, "\pi_2"] \\ \Delta \arrow[r, "h"] & \Delta. \end{tikzcd}$$

Next, we introduce basic terminology concerned with deformations of degenerations. We set $\Delta^\dagger := \{t \in \mathbb{C} : |t| < \delta\}$, where δ is sufficiently small. Suppose that $\mathcal{M}$ is a smooth complex 3-manifold, and $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a proper surjective holomorphic map. We set $M_t := \Psi^{-1}(\Delta \times \{t\})$ and $\pi_t := \Psi|_{M_t} : M_t \to \Delta \times \{t\}$. Since M is smooth and $\dim\Delta^\dagger = 1$, the composite map $\mathrm{pr}_2 \circ \Psi : \mathcal{M} \to \Delta^\dagger$ is a submersion, and so M_t is smooth. We say that $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a *deformation* of $\pi : M \to \Delta$ if $\pi_0 : M_0 \to \Delta \times \{0\}$ coincides with $\pi : M \to \Delta$. For consistency, we mainly use the notation Δ_t instead of $\Delta \times \{t\}$.

We introduce a special class of deformations of a degeneration. Suppose that $\pi : M \to \Delta$ is *relatively minimal*, i.e. its singular fiber contains no (-1)-curve (exceptional curve of the first kind). A deformation $\Psi : M \to \Delta \times \Delta^\dagger$ is said to be a *splitting deformation* of $\pi : M \to \Delta$, provided that for $t \neq 0$, $\pi_t : M_t \to \Delta_t$ has at least two singular fibers. In this case, if $X_1, X_2, \ldots, X_l$ $(l \geq 2)$ are singular fibers of $\pi_t : M_t \to \Delta_t$, then we say that X *splits into* $X_1, X_2, \ldots, X_l$. We note that a splitting of the singular fiber induces a factorization of the topological monodromy γ of $\pi : M \to \Delta$. Letting γ_i be the topological monodromy around X_i in $\pi_t : M_t \to \Delta_t$, we have $\gamma = \gamma_1\gamma_2\cdots\gamma_l$.

Next, we define the notion of splitting deformations for a degeneration $\pi : M \to \Delta$ which is *not* relatively minimal. We first introduce some notation. Let us take a sequence of blow down maps

$$M \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \xrightarrow{f_3} \cdots \xrightarrow{f_r} M_r,$$

and degenerations $\pi_i : M_i \to \Delta$ $(i = 1, 2, \ldots, r)$ where

(1) $f_i : M_{i-1} \to M_i$ is a blow down of a (-1)-curve in M_{i-1} and the map $\pi_i : M_i \to \Delta$ is naturally induced from $\pi_{i-1} : M_{i-1} \to \Delta$, and

(2) $\pi_r : M_r \to \Delta$ is a relatively minimal.

Given a deformation $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ of $\pi : M \to \Delta$, we shall construct a deformation $\Psi_r : \mathcal{M}_r \to \Delta \times \Delta^\dagger$ of the relatively minimal degeneration $\pi_r : M_r \to \Delta$. First, recall that by Kodaira's stability theorem [Ko2], any (-1)-curve in a complex surface is preserved under an arbitrary deformation of the surface. Thus, there exists a family of (-1)-curves in $\mathcal{M}$. We blow down them simultaneously to obtain a deformation $\Psi_1 : \mathcal{M}_1 \to \Delta$ of $\pi_1 : M_1 \to \Delta$. Again, by Kodaira's stability, there exists a family of (-1)-curves in M_2, which we blow down simultaneously to obtain a deformation $\Psi_2 : \mathcal{M}_2 \to \Delta$ of $\pi_2 : M_2 \to \Delta$. We repeat this process and finally obtain a deformation $\Psi_r : \mathcal{M}_r \to \Delta$ of $\pi_r : M_r \to \Delta$. Namely, given a deformation $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ of $\pi : M \to \Delta$, we obtain a deformation $\Psi_r : \mathcal{M}_r \to \Delta \times \Delta^\dagger$ of $\pi_r : M_r \to \Delta$. We say that $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a *splitting deformation* of $\pi : M \to \Delta$, provided that $\Psi_r : \mathcal{M}_r \to \Delta \times \Delta^\dagger$ is a splitting deformation of the relatively minimal degeneration $\pi_r : M_r \to \Delta$. We say that a degeneration is *atomic* if it admits no splitting deformation at all.

In this paper, instead of relatively minimal degenerations, we mainly use normally minimal degenerations, because they reflect the topological type (or topological monodromies) of degenerations. See §4. Recall that $\pi : M \to \Delta$ is *normally minimal* if X satisfies the following conditions:

(1) the reduced part $X_{\text{red}} := \sum_i \Theta_i$ is normal crossing, and

(2) if Θ_i is a (-1)-curve, then Θ_i intersects other irreducible components at at least three points.

In this case, we also say that the singular fiber X is normally minimal. The following lemma is useful.

Lemma 1.0.1 *Let $\pi : M \to \Delta$ be a normally minimal degeneration of complex curves of genus g. Suppose that $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a deformation of $\pi : M \to \Delta$ such that $\pi_t : M_t \to \Delta$ $(t \neq 0)$ has at least two normally minimal singular fibers. Then $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a splitting deformation of $\pi : M \to \Delta$.*

Proof. We first show the statement for the case $g \geq 2$. Let $\pi_r : M_r \to \Delta$ be the relatively minimal model of $\pi : M \to \Delta$, and let $\Psi_r : \mathcal{M}_r \to \Delta \times \Delta^\dagger$ be the deformation of π_r, which is determined from Ψ. Suppose that Y_1 and Y_2 are normally minimal singular fibers of $\pi_{r,t} : M_{r,t} \to \Delta_t$. Then the image of Y_i $(i = 1, 2)$ in $M_{r,t}$ is also singular, because the topological monodromy of π_r around Y_i is nontrivial (see [MM2], and also [ES, Im, ST]). If $g = 1$, this argument is valid except that none of Y_1 and Y_2 is a multiple of a smooth elliptic curve, in which case, the topological monodromy is trivial. However, a multiple of a smooth elliptic curve is clearly relatively minimal (it contains no projective line at all), so we completes the proof. □

2 Atomic degenerations

In this section, we exhibit two types of atomic degenerations.

Theorem 2.0.2 *Let $\pi : M \to \Delta$ be a degeneration of curves such that the singular fiber X is either* (I) *a reduced curve with one node, or* (II) *a multiple of a smooth curve of multiplicity at least* 2. *Then $\pi : M \to \Delta$ is atomic.*

We notice that in the type (I), X has one or two irreducible components, in the later case, two smooth irreducible components intersecting at one point transversally. The type (II) means that X is of the form $m\Theta$, where $m \geq 2$, and Θ is a smooth curve.

Remark 2.0.3 We remark that the proof of Theorem 2.0.2 carrries over to arbitrary dimensions to show that a degeneration of type (II) is atomic, i.e. letting $\pi : M \to \Delta$ be a degeneration of compact complex manifolds of arbitrary dimension, if the singular fiber X is a multiple of a smooth complex manifold, then $\pi : M \to \Delta$ is atomic.

We first demonstrate that if X is a reduced curve with one node, then $\pi : M \to \Delta$ is atomic. We prove this by contradiction. Assume that $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a splitting deformation of π which splits X into $X_1, X_2, \ldots, X_l$ ($l \geq 2$). We notice that a deformation of a node is either equisingular, or smoothing. Hence X_i is an equisingular deformation of X, and so it is also a reduced curve with one node. Since M is diffeomorphic to M_t, we have $\chi(M) = \chi(M_t)$, where $\chi(M)$ stands for the topological Euler characteristic of M. From this equation, we deduce the following relation of Euler characteristics (see [BPV] p97):

$$\chi(X) - (2 - 2g) = \sum_{i=1}^{l} [\chi(X_i) - (2 - 2g)]. \tag{2.0.1}$$

Since X and $X_1, X_2, \ldots, X_l$ are reduced curves with one node, we have

$$\chi(X) = \chi(X_1) = \cdots = \chi(X_l) = 2 - 2g + 1.$$

Then (2.0.1) implies that $1 = l$, which gives the contradiction.

Note: We can also show the above statement purely analytically by the computation of Ext^1 (cf. [Pal]). In fact, if X splits into $X_1, X_2, \ldots, X_l$ ($l \geq 2$), then the node (A_1-singularity) of X splits into l nodes. However, an A_1-singularity does not admit any splitting. This gives a contradiction.

3 The proof of Theorem 2.0.2 for the type (II)

Next, we shall demonstrate that if X is a multiple of a smooth curve, then $\pi : M \to \Delta$ is atomic. The proof is quite intricate and long, so we separate the statement into several claims to clarify the main step of the proof; for a deformation $\pi_t : M_t \to \Delta_t$ of $\pi : M \to \Delta$, we first construct an unramified covering $p_t : \widetilde{M}_t \to M_t$, and then show that the Stein factorization of $\pi_t \circ p_t$ factors through a smooth family over a disk.

3.1 Preparation

First, we construct an unramified cyclic m-covering of M. For this purpose, we consider a line bundle $L = \mathcal{O}_M(\Theta)$ on M. Notice that $L^{\otimes m} \cong \mathcal{O}_M$, because $m\Theta$ is the principal divisor defined by the holomorphic function π. We set $F_s := \pi^{-1}(s)$ (so $F_0 = m\Theta$). Then L has the following property: (1) For $s \neq 0$, the restriction $L|_{F_s}$ is a trivial bundle on F_s, and (2) the restriction $L|_\Theta$ is a line bundle on Θ such that $(L|_\Theta)^{\otimes m} \cong \mathcal{O}_\Theta$.

Next, we take an open covering $M = \bigcup_\alpha U_\alpha$, and let $U_\alpha \times \mathbb{C}$ be local trivializations $U_\alpha \times \mathbb{C}$ of L, with coordinates $(z_\alpha, \zeta_\alpha) \in U_\alpha \times \mathbb{C}$. We take a non-vanishing holomorphic section $\tau = \{\tau_\alpha\}$ of $L^{\otimes(-m)} \cong \mathcal{O}_M$. Equations[6] $\tau_\alpha(z_\alpha)\zeta_\alpha^m + 1 = 0$ define a smooth hypersurface $\widetilde{M}$ in L. The map $f : \widetilde{M} \to M$ given by $f(z_\alpha, \zeta_\alpha) = z_\alpha$ is an unramified cyclic m-covering. From the property of the line bundle L, (1) for $s \neq 0$, $f^{-1}(F_s)$ has m connected components such that each connected component is diffeomorphic to F_s, and (2) $\widetilde{\Theta} := f^{-1}(\Theta)$ is connected, and $f|_{\widetilde{\Theta}} : \widetilde{\Theta} \to \Theta$ is an unramified cyclic m-covering.

In order to show that $\pi : M \to \Delta$ is atomic, we shall prove that for an arbitrary deformation $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ of π, $\pi_t : M_t \to \Delta_t$ has a unique singular fiber, and it is of the form $m\Theta_t$, where Θ_t is diffeomorphic to Θ. For this purpose, we first construct an unramified cyclic covering of $\mathcal{M}$; notice that $\mathcal{M}$ is diffeomorphic to $M \times \Delta^\dagger$, and the map $\widetilde{M} \times \Delta^\dagger \longrightarrow M \times \Delta^\dagger$, $(x, t) \mapsto (f(x), t)$ is an unramified cyclic m-covering. Thus we have an unramified cyclic m-covering $\rho : \widetilde{\mathcal{M}} \to \mathcal{M}$, where we give the complex structure on $\widetilde{\mathcal{M}}$ induced from that on $\mathcal{M}$ by ρ. (This is possible, because ρ is unramified.) By construction, setting $\widetilde{M}_t := \rho^{-1}(M_t)$, the restriction $p_t : \widetilde{M}_t \to M_t$ of ρ to $\widetilde{M}_t$ is also an unramified cyclic m-covering. Applying the Stein factorization to the map $\pi_t \circ p_t : \widetilde{M}_t \to \Delta$, we obtain a commutative diagram

$$
\begin{tikzcd}
\widetilde{M}_t \arrow[r, "p_t"] \arrow[d, "\widetilde{\pi}_t"'] & M_t \arrow[d, "\pi_t"] \\
\widetilde{\Delta}_t \arrow[r, "\overline{p}_t"] & \Delta_t,
\end{tikzcd}
\tag{3.1.1}
$$

where (1) $\widetilde{\Delta}_t$ is a smooth[7] curve, and $\overline{p}_t : \widetilde{\Delta}_t \to \Delta$ is an m-covering, and (2) $\widetilde{\pi}_t : \widetilde{M}_t \to \widetilde{\Delta}_t$ is a proper surjective map such that all fibers are (topologically) connected. We notice that since p_t is a cyclic covering, from the commutativity of the above diagram, it is easy to check that $\overline{p}_t$ is also a cyclic covering.

3.2 The proof of Theorem 2.0.2 for the type (II)

After the above preparation, we prove Theorem 2.0.2 for the type (II). The key ingredients of the proof are the following two claims, which together imply that the Stein factorization (3.1.1) is nothing but the stable reduction of $\pi_t : M_t \to \Delta_t$. In what follows, we always assume that $|t|$ is sufficiently small.

Claim A $\widetilde{\pi}_t : \widetilde{M}_t \to \widetilde{\Delta}_t$ is a smooth family, i.e. all fibers of $\widetilde{\pi}_t$ are smooth.

[6]These equations are compatible with the transition functions of L.

[7]The Stein Factorization Theorem implies that since $\widetilde{M}_t$ is normal, $\widetilde{\Delta}_t$ is also normal. As is well known, a normal curve is smooth.

Claim B $\widetilde{\Delta}_t$ is an open disk.

Assuming Claims A and B for a moment, we will verify that $\pi_t : M_t \to \Delta_t$ has only one singular fiber, and it is of the form $m\Theta_t$. First, we note the following.

Lemma 3.2.1 *Suppose that $p : \widetilde{\Delta} \to \Delta$ is a cyclic m-covering, where $\widetilde{\Delta}$ and Δ are open unit[8] disks. Then the covering transformation group fixes exactly one point in $\widetilde{\Delta}$, and p is given by the map $z \mapsto z^m$ possibly after coordinate change.*

Proof. Let $\gamma : \widetilde{\Delta} \to \widetilde{\Delta}$ be a generator of the covering transformation group. Then γ is an element of $\mathrm{Aut}(\widetilde{\Delta})$, which is isomorphic to the fractional linear transformation group $\mathrm{PSL}_2(\mathbb{R})$ of the unit disk (Poincaré disk). From $\gamma^m = 1$, the transformation γ is an elliptic element. Thus it fixes exactly one point in $\widetilde{\Delta}$, and γ is of the form $z \mapsto e^{2\pi i/m} z$ possibly after coordinate change. Thus $p : \widetilde{\Delta} \to \Delta$ is given by $z \mapsto z^m$. □

Now we complete the proof of the theorem. By Claim A, $\widetilde{\pi}_t : \widetilde{M}_t \to \widetilde{\Delta}_t$ is a smooth family. Let $\widetilde{\gamma}_t$ be a generator of the covering transformation group of $\widetilde{M}_t \to \widetilde{M}_t$. By the construction of the Stein factorization of $\pi_t \circ p_t$, the transformation $\widetilde{\gamma}_t$ determines a generator γ_t of the covering transformation group of $\widetilde{\Delta}_t \to \Delta_t$ such that the following diagram commutes.

(3.2.1)
$$\begin{tikzcd} \widetilde{M}_t \arrow[r, "\widetilde{\gamma}_t"] \arrow[d, "\widetilde{\pi}_t"'] & \widetilde{M}_t \arrow[d, "\widetilde{\pi}_t"] \\ \widetilde{\Delta}_t \arrow[r, "\gamma_t"] & \widetilde{\Delta}_t. \end{tikzcd}$$

Namely, the pair $(\widetilde{\gamma}_t, \gamma_t)$ generates an equivariant $\mathbb{Z}_m$-action on $\widetilde{\pi}_t : \widetilde{M}_t \to \widetilde{\Delta}_t$, and $\pi_t : M_t \to \Delta_t$ is the quotient of $\widetilde{\pi}_t : \widetilde{M}_t \to \widetilde{\Delta}_t$ by this action. Recall that Δ_t is a disk, while by Claim B, $\widetilde{\Delta}_t$ is also a disk. Applying Lemma 3.2.1 to the cyclic m-covering $\widetilde{\Delta}_t \to \Delta_t$, we see that γ_t fixes exactly one point, say $\widetilde{x}_t$ on $\widetilde{\Delta}_t$. From the commutativity of the diagram (3.2.1), we have

Lemma 3.2.2 *The $\widetilde{\gamma}_t$-action on $\widetilde{M}_t$ stabilizes precisely one fiber $\widetilde{\Theta}_t := \widetilde{\pi}_t^{-1}(\widetilde{x}_t)$ and except this fiber this action cyclically permutes the m fibers in each orbit.*

$$\begin{tikzcd} \widetilde{\Theta}_t \subset \widetilde{M}_t \arrow[r, "\widetilde{\gamma}_t"] \arrow[d, "\widetilde{\pi}_t"'] & \widetilde{\Theta}_t \subset \widetilde{M}_t \arrow[d, "\widetilde{\pi}_t"] \\ \widetilde{x}_t \in \widetilde{\Delta}_t \arrow[r, "\gamma_t"] & \widetilde{x}_t \in \widetilde{\Delta}_t. \end{tikzcd}$$

As $\pi_t : M_t \to \Delta_t$ is the quotient of the smooth family $\widetilde{\pi}_t : \widetilde{M}_t \to \widetilde{\Delta}_t$ by the equivariant $\mathbb{Z}_m$-action, it follows from Lemma 3.2.2 that $\pi_t : M_t \to \Delta_t$ has a unique

[8]This is not restrictive at all; any open disk is biholomorphic to the unit one.

singular fiber over the point $x_t := \overline{p}_t(\widetilde{x}_t)$. This fiber is a multiple of a smooth curve, because $\widetilde{M}_t \to M_t$ is unramified cyclic, so in particular, the $\mathbb{Z}_m$-action on $\widetilde{\Theta}_t$ is unramified cyclic action. Namely, the singular fiber is $m\Theta_t$, where Θ_t is the image of $\widetilde{\Theta}_t$ under the quotient map (the multiplicity equals the order m of the $\widetilde{\gamma}_t$-action on $\widetilde{\Theta}_t$). Finally, we claim that Θ_t diffeomorphic to Θ. In fact, the restriction of Ψ to $\bigcup_t \Theta_t$ is a smooth family over the reduced part $\mathsf{D}_{\mathrm{red}}$ of discriminant of Ψ. (Note that $\mathsf{D}_{\mathrm{red}}$ is a disk. See Remark 3.3.3 below.) By Ehresmann's Theorem, any fiber Θ_t is diffeomorphic to $\Theta_0 = \Theta$. Thus, assuming Claims A and B, we proved Theorem 2.0.2, and so it remains to demonstrate these claims.

3.3 Proof of Claim A

We will show that $\widetilde{\pi}_t$ is a smooth family, i.e. any fiber of $\widetilde{\pi}_t$ is smooth. This is a crucial step in the proof of the theorem.

Step 1. Preparation

Let $X_1, X_2, \ldots, X_{\mathsf{d}}$ be the singular fibers of $\pi_t : M_t \to \Delta_t$, and set $x_i := \pi_t(X_i)$. We need to introduce notation associated to the basic diagram:

(3.3.1)
$$\begin{array}{ccc} \widetilde{M}_t & \xrightarrow{p_t} & M_t \\ {\scriptstyle \widetilde{\pi}_t}\downarrow & & \downarrow{\scriptstyle \pi_t} \\ \widetilde{\Delta}_t & \xrightarrow{\overline{p}_t} & \Delta_t \end{array}$$

We set $\overline{p}_t^{-1}(x_i) := \{\, \widetilde{x}_i^{(1)}, \widetilde{x}_i^{(2)}, \cdots, \widetilde{x}_i^{(N_i)} \,\}$, and let r_i be the ramification index[9] of $\widetilde{x}_i^{(j)}$ (so $\overline{p}_t : z \mapsto z^{r_i}$ around $\widetilde{x}_i^{(j)}$). Since the covering degree of $\overline{p}_t : \widetilde{\Delta}_t \to \Delta_t$ is m, we have

(3.3.2)
$$m = r_i \cdot \#(\overline{p}_t^{-1}(x_i)) = r_i N_i.$$

We write $\widetilde{X}_i^{(j)} = \widetilde{a}_i \widetilde{Y}_i^{(j)}$, where $\widetilde{a}_i$ is a positive integer and $\widetilde{Y}_i^{(j)}$ is not a multiple divisor, i.e. $\gcd\{\text{coefficients of } \widetilde{Y}_i^{(j)}\} = 1$. (Note that $\widetilde{a}_i$ does not depend on j, because $\overline{p}_t : \widetilde{\Delta}_t \to \Delta_t$ is a cyclic covering.) Next, recalling that X_i is a singular fiber of $\pi_t : M_t \to \Delta_t$, we write $X_i = a_i Y_i$, where a_i is a positive integer and Y_i is not a multiple divisor. Notice that

(3.3.3)
$$(\overline{p}_t \circ \widetilde{\pi}_t)^{-1}(x_i) = r_i \widetilde{a}_i \widetilde{Y}_i^{(j)},$$

where r_i is the ramification index of $\overline{p}_t$ at $\widetilde{x}_i^{(j)}$. As p_t is unramified, the fiber[10] of $\pi_t \circ p_t : \widetilde{M}_t \to \Delta_t$ over the point x_i is a multiple fiber of multiplicity a_i. Thus from the commutativity of the diagram (3.3.1), together with (3.3.3), we have

(3.3.4)
$$a_i = r_i \widetilde{a}_i.$$

We notice

[9] r_i does not depend on j, because $\overline{p}_t : \widetilde{\Delta}_t \to \Delta_t$ is a cyclic covering.

[10] The fiber $(\pi_t \circ \overline{p}_t)^{-1}(x_i)$ is not connected; there are N_i connected components.

Lemma 3.3.1 $\quad m\,\widetilde{a}_i = N_i a_i.$

Proof. Indeed, $m\,\widetilde{a}_i = r_i N_i \widetilde{a}_i = a_i N_i$, where the first and second equalities follows from (3.3.2) and (3.3.4) respectively. □

Next, we note that if there is a singular fiber of $\widetilde{\pi}_t$, then it is a fiber over some $\widetilde{x}_i^{(j)}$. In fact, if $\widetilde{X}$ is a singular fiber of $\widetilde{\pi}_t$, then the image $p_t(\widetilde{X})$ is a singular fiber of π_t. Therefore, to prove Claim A, it is enough to demonstrate that for any $\widetilde{x}_i^{(j)}$, the fiber $\widetilde{X}_i^{(j)} = \widetilde{\pi}_t^{-1}(\widetilde{x}_i^{(j)})$ is smooth.

Step 2. All $\widetilde{X}_i^{(j)}$ are smooth

Now we shall show that all $\widetilde{X}_i^{(j)}$ are smooth. Although the proof is involved, the essential part of the idea is to relate the singular fibers of $\pi_t \circ p_t$ and the singular fiber of $\pi_0 \circ p_0$. Namely, using the diagram[11]

$$\widetilde{\mathcal{M}} \xrightarrow{\rho} \mathcal{M} \xrightarrow{\Psi} \Delta \times \Delta^{\dagger},$$

we relates the singular fibers of the following two diagrams ('embedded' in the above diagram) by taking the limit $t \to 0$:

$$\widetilde{M}_t \xrightarrow{p_t} M_t \xrightarrow{\pi_t} \Delta_t \qquad \text{and} \qquad \widetilde{M}_0 \xrightarrow{p_0} M_0 \xrightarrow{\pi_t} \Delta_0.$$

Step 2.1 We consider the discriminant $\mathsf{D} \subset \Delta \times \Delta^{\dagger}$ of Ψ; it is a complex subspace (plane curve) of $\Delta \times \Delta^{\dagger}$ through $(0,0)$, and defined by the locus where the rank of $d\Psi$ is not maximal. Roughly, D is $\{\,(s,t) \in \Delta \times \Delta^{\dagger} \;:\; \Psi^{-1}(s,t) \text{ is singular}\,\}$, but possibly non-reduced. For our discussion, we rather use the reduced part $\mathsf{D}_{\mathsf{red}}$ of D. By the Weierstrass Preparation Theorem, the reduced plane curve $\mathsf{D}_{\mathsf{red}}$ is defined by a Weierstrass polynomial

$$s^{\mathsf{n}} + c_{\mathsf{n}-1}(t)\,s^{\mathsf{n}-1} + c_{\mathsf{n}-2}(t)\,s^{\mathsf{n}-2} + \cdots + c_0(t) = 0, \tag{3.3.5}$$

where $c_i(t)$ is a holomorphic function with $c_i(0) = 0$. By the definition of the reduced part, this equation contains no multiple root, in other words, the discriminant $\boldsymbol{\Delta}(t)$ of the above Weierstrass polynomial does not vanish identically (but possibly vanishes for some t). Now we claim that $\mathsf{n} = \mathsf{d}$, where d is the number of the singular fibers in $\pi_t : M_t \to \Delta_t$. Indeed, when $t = 0$, (3.3.5) is $s^{\mathsf{n}} = 0$, which clearly has a multiple root, so $\boldsymbol{\Delta}(0) = 0$. Since zeroes of the holomorphic function $\boldsymbol{\Delta}(t)$ are isolated, $\boldsymbol{\Delta}(t)$ does not vanish for sufficiently small t ($t \neq 0$). Consequently, (3.3.5) has n distinct roots, and so π_t has precisely n singular fibers, implying that $\mathsf{n} = \mathsf{d}$. This verifies the claim, and we have

$$\mathsf{D}_{\mathsf{red}} = \{\, s^{\mathsf{d}} + c_{\mathsf{d}-1}(t)\,s^{\mathsf{d}-1} + c_{\mathsf{d}-2}(t)\,s^{\mathsf{d}-2} + \cdots + c_0(t) = 0\,\}. \tag{3.3.6}$$

[11] We do not use the Stein factorization of the map $\Psi \circ \rho$, but it is worth while pointing out that it factors through a normal surface S, which possibly has a singularity. In contrast, the Stein factorization for the map with a one-dimensional base factors through a *smooth* curve.

Next, we define a ramified d-fold $\phi : \mathsf{D}_{\mathrm{red}} \to \Delta^\dagger$ by $(s,t) \mapsto t$. Then

$$\phi^{-1}(t) = \begin{cases} \text{d distinct points} & \text{for } t \neq 0 \\ \text{a multiple point } s^{\mathsf{d}} = 0 & \text{for } t = 0. \end{cases}$$

Step 2.2 To relate the singular fibers of $\pi_t \circ p_t$ and $\pi_0 \circ p_0$, we consider the hypersurface $\widetilde{\mathcal{H}} := (\Psi \circ \rho)^{-1}(\mathsf{D}_{\mathrm{red}})$ in the complex 3-manifold $\widetilde{\mathcal{M}}$. For the remainder of the proof, to emphasize the parameter t, we use 'precise' notation $\widetilde{X}_{i,t}^{(j)}$ instead of $\widetilde{X}_i^{(j)}$ etc. Notice that

$$\mathcal{H} \cap \widetilde{M}_t = \begin{cases} \text{the disjoint union of all } \widetilde{X}_{i,t}^{(j)} & \text{for } t \neq 0 \\ \mathsf{d}m\widetilde{\Theta} & \text{for } t = 0, \end{cases} \tag{3.3.7}$$

where we can see $\mathcal{H} \cap \widetilde{M}_0 = \mathsf{d}m\widetilde{\Theta}$ as follows. Since $\pi_0^{-1}(0) = m\Theta$ and p_0 is unramified (locally biholomorphic), we have $(\pi_0 \circ p_0)^{-1}(0) = m\widetilde{\Theta}$, hence the fiber of $\pi_0 \circ p_0$ over the multiple point $s^{\mathsf{d}} = 0$ is $\mathsf{d}m\widetilde{\Theta}$, so $\mathcal{H} \cap \widetilde{M}_0 = \mathsf{d}m\widetilde{\Theta}$.

By the first equation of (3.3.7), our goal is to show that $\mathcal{H} \cap \widetilde{M}_t$ is smooth for all $t \neq 0$. To demonstrate this, fixing an arbitrary point $y \in \widetilde{\Theta}\,(= p_0^{-1}(\Theta))$, we take a local coordinate[12] (z_1, z_2, t) around y in $\widetilde{M}$, such that $z_1 = 0$, $t = 0$ locally defines $\widetilde{\Theta}$. Let $f(z_1, z_2, t) = 0$ be a defining equation of $\widetilde{\mathcal{H}}$ around y in $\widetilde{\mathcal{M}}$. For later discussion, we use the notation $f_t(z_1, z_2)$ instead of $f(z_1, z_2, t)$. By the first equation of (3.3.7), $\widetilde{\mathcal{H}} \cap M_t = \amalg_{i=1}^{\mathsf{d}} (\amalg_{j=1}^{N_i} \widetilde{X}_{i,t}^{(j)})$ (disjoint union) and $\widetilde{X}_{i,t}^{(j)} = a_i \widetilde{Y}_{i,t}^{(j)}$, so we can write

$$f_t = \prod_{i=1}^{\mathsf{d}} f_{i,t}^{a_i}, \qquad \text{where } f_{i,t} = \prod_{j=1}^{N_i} g_{i,t}^{(j)}, \tag{3.3.8}$$

and $g_{i,t}^{(j)} = 0$ defines $\widetilde{Y}_{i,t}^{(j)}$ locally. By the second equation of (3.3.7), $f_0(z_1, z_2) = z_1^{\mathsf{d}m}$, hence setting[13] $t = 0$ in (3.3.8), we have

$$z_1^{\mathsf{d}m} = f_0 = \prod_{i=1}^{\mathsf{d}} f_{i,0}^{a_i}, \tag{3.3.9}$$

and so we may express $g_{i,0}^{(j)}(z_1, z_2) = z_1^{d_i^{(j)}} \cdot u_i^{(j)}(z_1, z_2)$, where $d_i^{(j)}$ is a positive integer, and $u_i^{(j)}$ is a non-vanishing holomorphic function. By the comparison of the degrees of z_1 in (3.3.9), we have

$$\mathsf{d}m = \sum_{i=1}^{\mathsf{d}} a_i (d_i^{(1)} + d_i^{(2)} + \cdots + d_i^{(N_i)}). \tag{3.3.10}$$

Now we show the key lemma.

---

[12]By the definition of deformations, $\mathrm{pr}_2 \circ \Psi : \mathcal{M} \to \Delta \times \Delta^\dagger \to \Delta^\dagger$ is a submersion. Since ρ is unramified, $\mathrm{pr}_2 \circ \Psi \circ \rho : \widetilde{\mathcal{M}} \to \Delta^\dagger$ is also a submersion. By the Implicit Function Theorem, we may 'lift' $t \in \Delta^\dagger$ to a coordinate of $\widetilde{\mathcal{M}}$.

[13]We take the limit $t \to 0$ along a path l in $\mathsf{D}_{\mathrm{red}}$ such that l is homeomorphically mapped to a path in $\Delta^\dagger$ under the ramified covering $\mathsf{D}_{\mathrm{red}} \to \Delta^\dagger$, $(s,t) \mapsto t$. For example, in $s^2 - t^3 = (s - t^{3/2})(s + t^{3/2})$, two factors are multi-valued on $\mathsf{D}_{\mathrm{red}}$, so taking $t \to 0$, we must choose a path l on which each factor is single-valued.

Lemma 3.3.2 $\widetilde{a}_i = d_i^{(1)} = d_i^{(2)} = \cdots = d_i^{(N_i)} = 1$ *for* $i = 1, 2, \ldots, \mathsf{d}$.

Proof. First, we note

$$\begin{aligned} \mathsf{d}m &= \sum_{i=1}^{\mathsf{d}} a_i(d_i^{(1)} + d_i^{(2)} + \cdots + d_i^{(N_i)}) && \text{by (3.3.10)} \\ &\geq \sum_{i=1}^{\mathsf{d}} a_i N_i && \text{by } d_i^{(1)}, d_i^{(2)}, \ldots, d_i^{(N_i)} \geq 1 \qquad (3.3.11) \\ &= \sum_{i=1}^{\mathsf{d}} \widetilde{a}_i m && \text{by Lemma 3.3.1.} \end{aligned}$$

Thus we have $\mathsf{d}m \geq \sum_{i=1}^{\mathsf{d}} \widetilde{a}_i m$, which implies that $\widetilde{a}_1 = \widetilde{a}_2 = \cdots = \widetilde{a}_{\mathsf{d}} = 1$, and this inequality is an equality. In particular, (3.3.11) is also an equality, and so $d_i^{(1)} = d_i^{(2)} = \cdots = d_i^{(N_i)} = 1$. This complete the proof. □

Now, it is immediate to complete the proof of Claim A. From $\widetilde{a}_i = 1$, we have $\widetilde{X}_{i,t}^{(j)} = \widetilde{Y}_{i,t}^{(j)}$. On the other hand, from $\widetilde{d}_i = 1$, $\widetilde{Y}_{i,0}^{(j)}$ is smooth, because it is locally defined by $z_1 \cdot u_i^{(j)}(z_1, z_2) = 0$. Thus for sufficiently small t, $\widetilde{Y}_{i,t}^{(j)}$ is smooth, and so $\widetilde{X}_{i,t}^{(j)} = \widetilde{Y}_{i,t}^{(j)}$ is smooth. This completes the proof of Claim A.

Remark 3.3.3 If $\mathsf{d} = 1$, i.e. $\pi_t : M_t \to \Delta_t$ has only one singular fiber, then $\mathsf{D}_{\mathrm{red}} = \{s + c_0(t) = 0\}$ (see (3.3.6)) is a disk in $\Delta \times \Delta^\dagger$.

3.4 Proof of Claim B

We shall show Claim B which asserts that $\widetilde{\Delta}_t$ is a disk. The proof below is based on a topological argument, and by shrinking M_t, $\widetilde{M}_t$, Δ_t and $\widetilde{\Delta}_t$, we regard them with closed manifolds with boundary. We first take diffeomorphisms[14] $\phi_t : M_0 \to M_t$ and $\overline{\phi}_t : \partial\Delta_0 \to \partial\Delta_t$ which make the following diagram commute:

$$\begin{array}{ccc} \partial M_0 & \xrightarrow{\phi_t} & \partial M_t \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \pi_t} \\ \partial\Delta_0 & \xrightarrow{\overline{\phi}_t} & \partial\Delta_t. \end{array}$$

(Namely, the restriction of $\overline{\phi}_t$ to the boundary ∂M_0 is fiber-preserving.) Recall that we constructed $p_t : \widetilde{M}_t \to M_t$ from $p_0 : \widetilde{M}_0 \to M_0$ via the diffeomorphism $\phi_t : M_0 \to M_t$. Hence there is a natural diffeomorphism $\Phi_t : \widetilde{M}_0 \to \widetilde{M}_t$, which is a lifting of ϕ_t (that is, $\Phi_t \circ p_t = p_0 \circ \phi_t$), and the restriction of Φ_t to $\partial\widetilde{M}_0$ is

[14]For the existence of ϕ_t, see Lemma 3.5.1 in §3.5 Supplement below.

fiber-preserving, i.e. the following diagram commutes

$$\begin{tikzcd} \partial\widetilde{M}_0 \arrow[r, "\Phi_t"] \arrow[d, "\widetilde{\pi}_0"'] & \partial\widetilde{M}_t \arrow[d, "\widetilde{\pi}_t"] \\ \partial\widetilde{\Delta}_0 \arrow[r, "\overline{\Phi}_t"] & \partial\widetilde{\Delta}_t, \end{tikzcd}$$

where $\overline{\Phi}_t$ is a diffeomorphism. Now we fix a fiber $C_0 := \widetilde{\pi}_0^{-1}(y_0)$, where $y_0 \in \partial\widetilde{\Delta}_0$, and let $\iota_0 : C_0 \hookrightarrow \widetilde{M}_0$ be the natural embedding. Then $C_t := \Phi_t(C_0)$ is a fiber of $\widetilde{\pi}_t$ over $y_t := \overline{\Phi}_t(y_0) \in \partial\widetilde{\Delta}_t$, and let $\iota_t : C_t \hookrightarrow \widetilde{M}_t$ be the natural embedding.

$$\begin{tikzcd} C_0 \subset \partial\widetilde{M}_0 \arrow[r, "\Phi_t"] \arrow[d, "\widetilde{\pi}_0"'] & C_t \subset \partial\widetilde{M}_t \arrow[d, "\widetilde{\pi}_t"] \\ y_0 \in \partial\widetilde{\Delta}_t \arrow[r, "\overline{\Phi}_t"] & y_t \in \partial\widetilde{\Delta}_t. \end{tikzcd}$$

After this preparation, we can demonstrate that $\widetilde{\Delta}_t$ is a disk. Note that $\widetilde{\Delta}_t$ is a *real* compact surface with a connected[15] boundary (which is isomorphic to S^1). Thus if the genus of $\widetilde{\Delta}_t$ is g, then $\widetilde{\Delta}_t$ is homotopically equivalent to the bouquet $S^1 \vee S^1 \vee \cdots \vee S^1$ of $2g$ circles, and so

$$\pi_1(\widetilde{\Delta}_t) = \underbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}_{2g}, \quad \text{the free group of rank } 2g.$$

Hence it suffices to show that $\pi_1(\widetilde{\Delta}_t) = 1$. For this, we first take the homotopy exact sequence associated to the differentiable fiber bundle[16] $\widetilde{\pi}_0 : \widetilde{M}_0 \to \widetilde{\Delta}_0$.

$$\pi_2(\widetilde{\Delta}_0) \longrightarrow \pi_1(C_0) \xrightarrow{\iota_{0*}} \pi_1(\widetilde{M}_0) \longrightarrow \pi_1(\widetilde{\Delta}_0) \longrightarrow 1 \tag{3.4.1}$$

Next, noting that from Claim A, $\widetilde{\pi}_t : \widetilde{M}_t \to \widetilde{\Delta}_t$ is a differentiable fiber bundle, so we may take the homotopy exact sequence associated to it.

$$\pi_2(\widetilde{\Delta}_t) \longrightarrow \pi_1(C_t) \xrightarrow{\iota_{t*}} \pi_1(\widetilde{M}_t) \longrightarrow \pi_1(\widetilde{\Delta}_t) \longrightarrow 1 \tag{3.4.2}$$

The following commutative diagram relates (3.4.1) and (3.4.2):

$$\begin{tikzcd} \pi_2(\widetilde{\Delta}_0) \arrow[r] & \pi_1(C_0) \arrow[r, "\iota_{0*}"] \arrow[d] & \pi_1(\widetilde{M}_0) \arrow[r] \arrow[d] & \pi_1(\widetilde{\Delta}_0) \arrow[r] & 1 \\ \pi_2(\widetilde{\Delta}_t) \arrow[r] & \pi_1(C_t) \arrow[r, "\iota_{t*}"] & \pi_1(\widetilde{M}_t) \arrow[r] & \pi_1(\widetilde{\Delta}_t) \arrow[r] & 1, \end{tikzcd} \tag{3.4.3}$$

where the vertical arrows are induced by Φ_t. Since $\widetilde{\Delta}_0$ is a disk, we have $\pi_1(\widetilde{\Delta}_0) = \pi_2(\widetilde{\Delta}_0) = 1$, and so ι_{0*} is an isomorphism. Two vertical arrows are also isomorphisms, because they are induced by the diffeomorphism Φ_t. From the commutativity of the diagram (3.4.3), we see that ι_{t*} is an isomorphism. Then the exactness of (3.4.1) implies that $\pi_1(\widetilde{\Delta}_t) = 1$ and so $\widetilde{\Delta}_t$ is a disk.

[15] By the construction of $\widetilde{M}_t$, the boundary $\partial\widetilde{M}_t$ is connected, and so $\partial\widetilde{\Delta}_t$ is connected.

[16] By Ehresmann's Theorem, a smooth family is a fiber bundle in the differentiable category.

3.5 Supplement: Construction of diffeomorphisms

Suppose that $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a deformation of $\pi : M \to \Delta$. Note that the restriction $\pi_t|_{\partial M_t} : \partial M_t \to \partial\Delta_t$ is a fiber bundle[17]. The following lemma may be known to the geometers, but for the convenience of the reader, we include the proof. (Hereafter, for consistency, we denote $\pi_0 : M_0 \to \Delta_0$ instead of $\pi : M \to \Delta$)

Lemma 3.5.1 *There exists a diffeomorphism $\phi_t : M_0 \to M_t$ such that the restriction $\phi_t|_{\partial M_0}$ preserves fibers, that is, there exists a diffeomorphism $\overline{\phi}_t : \partial\Delta_0 \to \Delta_t$ which makes the following diagram commute:*

$$\begin{array}{ccc} \partial M_0 & \xrightarrow{\phi_t} & \partial M_t \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \pi_t} \\ \partial\Delta_0 & \xrightarrow{\overline{\phi}_t} & \partial\Delta_t. \end{array}$$

Warning: Although the *restriction* of ϕ_t to the boundary ∂M_0 commutes with maps π_0 and π_t, this is not case for ϕ_t itself.

Proof. For simplicity, we assume that Δ is the unit disk. We choose $r_1, r_2 \in \mathbb{R}$ so that $0 < r_2 < r_1 < 1$, and define an open covering $\Delta \times \Delta^\dagger = U_{\text{in}} \cup U_{\text{out}}$, where

$$U_{\text{in}} := \{(s,t) \in \Delta \times \Delta^\dagger : |s| < r_1\}, \qquad U_{\text{out}} := \{(s,t) \in \Delta \times \Delta^\dagger : |s| > r_2\}.$$

We then take an open covering $\mathcal{M} = \mathcal{M}_{\text{in}} \cup \mathcal{M}_{\text{out}}$, where $\mathcal{M}_{\text{in}} := \Psi^{-1}(U_{\text{in}})$ and $\mathcal{M}_{\text{out}} := \Psi^{-1}(U_{\text{out}})$. Taking r_1 sufficiently close to 1, we assume that $\mathcal{M}_{\text{out}}$ contains no singular fiber, i.e. the restriction $\Psi_{\text{out}} := \Psi|_{\mathcal{M}_{\text{out}}}$ is a fiber bundle. In particular, Ψ_{out} is a submersion. Hence there exists a vector field v_{out} on $\mathcal{M}_{\text{out}}$ such that

$$d\Psi_{\text{out}}(v_{\text{out}}) = \frac{\partial}{\partial t}. \tag{3.5.1}$$

Similarly, we set $\Psi_{\text{out}} := \Psi|_{\mathcal{M}_{\text{out}}}$. By the definition of deformations, the composite map $\mathrm{pr}_2 \circ \Psi_{\text{in}} : \mathcal{M}_{\text{out}} \to \Delta^\dagger$ is a fiber bundle with smooth complex surfaces as fibers, and so a submersion[18]. Thus there exists a vector field v_{in} on $\mathcal{M}_{\text{in}}$ such that

$$d(\mathrm{pr}_2 \circ \Psi_{\text{in}})(v_{\text{in}}) = \frac{\partial}{\partial t}. \tag{3.5.2}$$

Notice that in (3.5.1), $\frac{\partial}{\partial t}$ is a vector field on $\Delta \times \Delta^\dagger$, while in (3.5.2), it is a vector field on $\Delta^\dagger$. We shall 'patch' two vector fields v_{in} and v_{out} by a partition of unity, and define a vector field v on $\mathcal{M}$; we first define open subsets $U'_{\text{in}} \subset U_{\text{in}}$ (resp. $U'_{\text{out}} \subset U_{\text{out}}$) as follows. Take $r'_1, r'_2 \in \mathbb{R}$ satisfying $0 < r'_1 < r_2 < r_1 < r'_2 < 1$, and set

$$U'_{\text{in}} := \{(s,t) \in \Delta \times \Delta^\dagger : |s| < r'_1\}, \qquad U'_{\text{out}} := \{(s,t) \in \Delta \times \Delta^\dagger : |s| > r'_2\}.$$

[17]In this subsection, by a fiber bundle we always mean a differentiable one.

[18]$\Psi_{\text{in}} : \mathcal{M}_{\text{in}} \to \Delta \times \Delta^\dagger$ has a singular fiber, and so it is *not* a fiber bundle.

Notice that $U'_{\text{in}} \cap U'_{\text{out}} = \emptyset$. Now we put $\mathcal{M}'_{\text{in}} := \Psi^{-1}(U'_{\text{in}})$ and $\mathcal{M}'_{\text{out}} := \Psi^{-1}(U'_{\text{out}})$. Then $\mathcal{M}'_{\text{in}} \cap \mathcal{M}'_{\text{out}} = \emptyset$. Using a partition of unity, we can construct a vector field v on $\mathcal{M}$ such that

$$v = \begin{cases} v_{\text{in}} & \text{on } \mathcal{M}'_{\text{in}} \\ v_{\text{out}} & \text{on } \mathcal{M}'_{\text{out}} \end{cases}$$

Finally, we integrate the vector field v on $\mathcal{M}$ to obtain a one-parameter family of diffeomorphisms $\phi_t : M_0 \to M_t$ with the desired property. $\square$

4 Topological monodromies and singular fibers

Before we proceed to state splitting criteria, we briefly review the relation between topological monodromies and configurations of singular fibers (see [MM2] and [Ta,II] for details). First, we recall the topological monodromy of a degeneration $\pi : M \to \Delta$. For this purpose, it is convenient to consider M and Δ as manifolds with boundary, so Δ is the closed unit disk. We write $\partial\Delta = \{e^{i\theta} : 0 \leq \theta \leq 2\pi\}$, and set $C_\theta := \pi^{-1}(e^{i\theta})$. Using a partition of unity, we construct a vector field v on ∂M such that $d\pi(v) = \partial/\partial\theta$. Then the integration of v yields a one-parameter family of diffeomorphisms $h_\theta : C_0 \to C_\theta$ (see Figure 1). In particular, $h_{2\pi}$ is a self-homeomorphism of C_0. Setting $h := h_{2\pi}$, we refer to h as the *topological monodromy* of $\pi : M \to \Delta$.

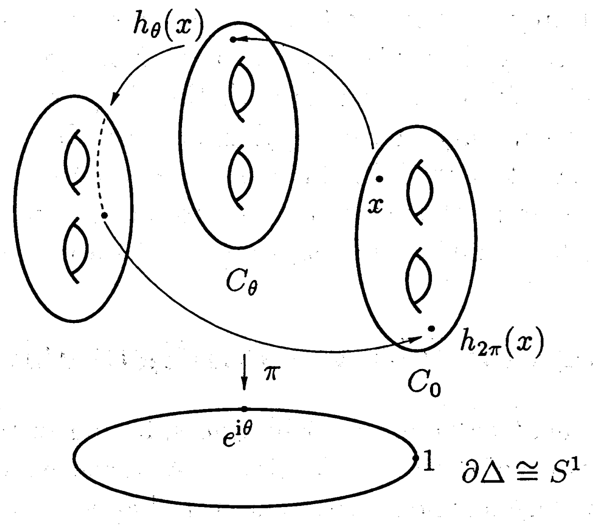

Figure 1:

Topological monodromies are very special homeomorphisms; they are either periodic or pseudo-periodic (see [MM2], and also [ES, Im, ST]). Recall that a homeomorphism h of a curve C is (1) *periodic* if for some positive integer m, h^m is isotopic to the identity, and (2) *pseudo-periodic* if for some loops $l_1, l_2, \ldots, l_n$ on C, the restriction h on $C \setminus \{l_1, l_2, \ldots, l_n\}$ is periodic. (In [MM2], periodic homeomorphisms are considered to be special cases of pseudo-periodic homeomorphisms by taking $\{l_1, l_2, \ldots, l_n\} = \emptyset$. However for our discussion it is convenient to distinguish periodic homeomorphisms with pseudo-periodic ones.) According to whether the topological monodromy is periodic or pseudo-periodic, the singular fiber is star-shaped or non-star shaped. In some sense, a non-star-shaped singular fiber is obtained by 'bonding' star-shaped ones (see [MM2] and [Ta,II]).

Remark 4.0.2 Based on a topological argument, Matsumoto and Montesinos [MM2] showed that the configuration of the singular fiber of a degeneration is completely determined by its topological monodromy. In [Ta,II], we gave an algebro-geometric proof for their results, and clarified the relation between topological monodromies and quotient singularities.

Now the followings are the simplest examples for periodic and pseudo-periodic homeomorphisms respectively:

Example 4.0.3 (Periodic) h is an unramified periodic homeomorphism, that is, the quotient map $C \to C/\langle h \rangle$ is a unramified cyclic covering.

Example 4.0.4 (Pseudo-periodic) h is a right Dehn twist along one loop l on C, so the restriction of h to $C \setminus l$ is isotopic to the identity.

A degeneration with the topological monodromy in Example 4.0.3 has a singular fiber $m\Theta$, where m is the order of h, and Θ is a smooth curve which is the quotient of C by the action of h. On the other hand, the singular fiber of a degeneration with the topological monodromy in Example 4.0.3 is a reduced curve with one node (this node is obtained by 'pinching' l on C). By Theorem 2.0.2, both of these degenerations are atomic. Namely, all degenerations with the simplest topological monodromies are atomic. To the contrary, if the topological monodromy is 'complicated', what can we say about splittability? In this case, the singular fiber is also complicated, so the reader may imagine that they are not atomic (complicated objects should not be atoms!). In the later half of this paper, we will show that this intuition is true.

5 Splitting criteria via configurations, I

In this and subsequent sections, we will give splitting criteria of degenerations in terms of configurations of their singular fibers. As a consequence of these criteria, we will see that many degenerations with non-star-shaped singular fibers always admit splitting deformations. We point out that these criteria are powerful for determining atomic degenerations by induction with respect to genus g (see §6.3 for details).

In the discussion below, we often use the realization of M as a graph of π; for a degeneration $\pi : M \to \Delta$, the graph of π is defined by

$$\mathrm{Graph}(\pi) = \{(x, s) \in M \times \Delta \, : \, \pi(x) - s = 0\}.$$

Of course, $\mathrm{Graph}(\pi)$ is a smooth hypersurface in $M \times \Delta$, and M is canonically isomorphic to $\mathrm{Graph}(\pi)$ by $x \in M \mapsto (x, \pi(x)) \in M \times \Delta$. Under this isomorphism, the map $\pi : M \to \Delta$ corresponds to the projection $(x, s) \in \text{Graph}(\pi) \mapsto s \in \Delta$. In the discussion below, we identify $\text{Graph}(\pi)$ with M via the canonical isomorphism, and we write M instead of $\text{Graph}(\pi)$.

5.1 Criterion in terms of nodes

In this subsection, we shall provide splitting criteria in terms of some singularity on the singular fiber. We start with a definition. Consider a singularity

$$V_m := \{(x, y) \in \mathbb{C}^2 : x^m y^m = 0\},$$

where m is a positive integer. We say that V_m is a *multiple node of multiplicity* m. Note that when $m \geq 2$, V_m is non-reduced. By abuse of terminology, we also say that the origin of V_m is a multiple node.

We consider a hypersurface $\mathcal{M} := \{(x, y, s, t) \in \mathbb{C}^4 : (xy + t)^m - s = 0\}$ in $\mathbb{C}^4$, and define a holomorphic map $\Psi : \mathcal{M} \to \mathbb{C}^2$ by $(x, y, s, t) \mapsto (s, t)$. Clearly, $\Psi^{-1}(0, 0) = V_m$, and so Ψ is a two-parameter deformation of V_m. Next, we shall compute the discriminant of Ψ. Since

$$\frac{\partial \Psi}{\partial x} = mx(xy + t)^{m-1}, \qquad \frac{\partial \Psi}{\partial y} = my(xy + t)^{m-1},$$

we have $\partial\Psi/\partial x = \partial\Psi/\partial y = 0$ if and only if either (1) $x = y = 0$ or (2) $xy + t = 0$. We note that $t^m - s = 0$ for (1), and $s = 0$ for (2).

Lemma 5.1.1 *The discriminant of Ψ consists of curves $s = t^m$ and $s = 0$ in $\mathbb{C}^2$. To be explicit, for $t \neq 0$,*

(1) *$\Psi^{-1}(t^m, t)$ is a disjoint union of $m - 1$ annuli and a node,*

(2) *$\Psi^{-1}(0, t)$ is a multiple of an annulus of multiplicity m.*

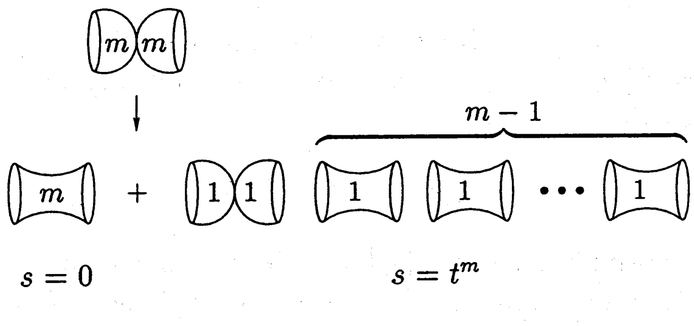

Figure 2:

Proof. The fiber $\Psi^{-1}(t^m, t)$ $(t \neq 0)$ is defined by

$$xy[\,(xy)^{m-1} + {}_m\mathrm{C}_1(xy)^{m-2}t + \cdots + {}_m\mathrm{C}_i(xy)^{m-i-1}t^i + \cdots + {}_m\mathrm{C}_1 t^{m-1}\,] = 0.$$

This equation factorizes as $xy\prod_{i=1}^{m-1}(xy+\alpha_i t)=0$, where $\alpha_i \in \mathbb{C}$ $(i=1,2,\ldots,m-1)$ are the solutions of $X^{m-1}+{}_m\mathrm{C}_1X^{m-2}+\cdots+{}_m\mathrm{C}_iX^{m-i-1}+\cdots+{}_m\mathrm{C}_1=0$. Hence $\Psi^{-1}(t^m,t)$ $(t\neq 0)$ is a disjoint union of a node $xy=0$ and $m-1$ annuli $xy+\alpha_i=0$ $(i=1,2,\ldots,m-1)$. On the other hand, $\Psi^{-1}(0,t)=\{(xy+t)^m=0\}$ is a multiple annulus of multiplicity m. □

Now we can show the following.

Criterion 5.1.2 *Let $\pi: M \to \Delta$ be normally minimal such that the singular fiber X has a multiple node p of multiplicity at least 2. Then there exists a splitting deformation of $\pi: M \to \Delta$, which splits X into X_1 and X_2, where X_1 is a reduced curve with one node and X_2 is obtained from X by replacing the multiple node p by a multiple annulus* (see Figure 4 for example).

Proof. Take an open covering $M=M_0\cup M_1$, such that (1) M_0 is an open ball around p (hence $M_0\cap X$ is the multiple node), and (2) $M_1\cap X$ is 'outside' the multiple node (see Figure 3). We take local coordinates $(z_\beta,\zeta_\beta)\in M_0$ around p,

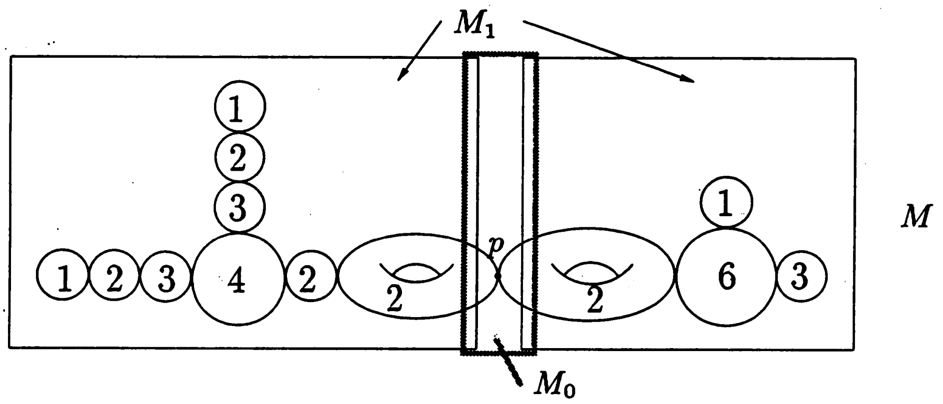

Figure 3:

then we have $\pi(z_\beta,\zeta_\beta)=z_\beta^m\zeta_\beta^m$. Next, we take local coordinates $(z_\alpha,\zeta_\alpha)\in M_1$ near p. Then $\pi(z_\alpha,\zeta_\alpha)=\zeta_\alpha^m f_\alpha(z_\alpha,\zeta_\alpha)$, where f_α is a non-vanishing holomorphic function. As $\pi(z_\alpha,\zeta_\alpha)=\pi(z_\beta,\zeta_\beta)$, we have

$$\zeta_\alpha^m f_\alpha(z_\alpha,\zeta_\alpha)=z_\beta^m\zeta_\beta^m.$$

Note that the holomorphic function $z_\beta^m\zeta_\beta^m$ on the right has an m-th root $z_\beta\zeta_\beta$, which is a single-valued function. Thus $\zeta_\alpha^m f_\alpha$ also has a *single valued* m-th root function $\zeta_\alpha f_\alpha^{1/m}$ such that $\zeta_\alpha f_\alpha^{1/m}=z_\beta\zeta_\beta$. Rewriting $\zeta_\alpha f_\alpha^{1/m}$ by ζ_α, the gluing map of M_0 and M_1 is of the form

$$z_\alpha=\phi_{\alpha\beta}(z_\beta,\zeta_\beta),\quad \zeta_\alpha=z_\beta\zeta_\beta \quad \text{around } p,$$

where $\phi_{\alpha\beta}$ is holomorphic.

Now we consider a smooth hypersurface $\mathcal{M}_0$ in $M_0 \times \Delta \times \Delta^\dagger$ given by

$$\{(z_\beta, \zeta_\beta, s, t) \in M_0 \times \Delta \times \Delta^\dagger \,:\, (z_\beta\zeta_\beta + t)^m - s = 0\}.$$

We also define a smooth hypersurface $\mathcal{M}_1$ in $M_1 \times \Delta \times \Delta^\dagger$ by

$$\{(x, s, t) \in M_1 \times \Delta \times \Delta^\dagger \,:\, \pi(x) - s = 0\}.$$

Let $\Psi_i : \mathcal{M}_i \to \Delta \times \Delta^\dagger$ $(i = 0, 1)$ be the natural projection. From Lemma 5.1.1, for $t \neq 0$,

$$(5.1.1) \qquad \Psi_0^{-1}(s,t) = \begin{cases} \text{disjoint union of } m-1 \text{ annuli and a node}, & s = t^m, \\ \text{a multiple annulus of multiplicity } m, & s = 0. \end{cases}$$

On the other hand, we have

$$(5.1.2) \qquad \Psi_1^{-1}(s,t) = \begin{cases} X \cap M_1, & s = 0, \\ \text{smooth}, & \text{otherwise.} \end{cases}$$

Now we glue $\mathcal{M}_0$ with $\mathcal{M}_1$ by

$$z_\alpha = \phi_{\alpha\beta}(z_\beta, \zeta_\beta), \quad \zeta_\alpha = z_\beta\zeta_\beta + t.$$

Note that this map transforms the defining equation of $\mathcal{M}_0$ near p to that of $\mathcal{M}_1$. Then we obtain a complex 3-manifold $\mathcal{M}$. Letting $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be the natural projection, we consider two fibers:

$$X_1 = \Psi^{-1}(t^m, t), \quad X_2 = \Psi^{-1}(0, t).$$

(X_1 and X_2 are fibers of $\pi_t : M_t \to \Delta_t$.) From (5.1.1) and (5.1.2), X_1 is a reduced curve with one node, and X_2 is obtained from X by replacing the multiple node by a multiple annulus, and no other singular fibers. As both of X_1 and X_2 are normally minimal, it follows from Lemma 1.0.1 that $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a splitting deformation, which splits X into X_1 and X_2. □

The above construction of $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ also works for the case where p is a multiple node of multiplicity 1. But $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is *not* necessarily a splitting deformation of $\pi : M \to \Delta$. This is exactly the case when $X \setminus \{p\}$ is smooth, i.e. X is a reduced curve with one node. In which case, $X_2 = \Psi^{-1}(0, t)$ is a smooth fiber (in fact, π is atomic by Theorem 2.0.2). Except this case, $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a splitting deformation of $\pi : M \to \Delta$, which splits X into X_1 and X_2, where X_1 is a reduced curve with one node, and X_2 is obtained from X by replacing the reduced node by an annulus. Combined this result with Criterion 5.1.2, we have the following criterion.

Criterion 5.1.3 *Let $\pi : M \to \Delta$ is normally minimal such that the singular fiber X contains a multiple node (of multiplicity $m \geq 1$). Then $\pi : M \to \Delta$ is atomic if and only if X is a reduced curve with one node.*

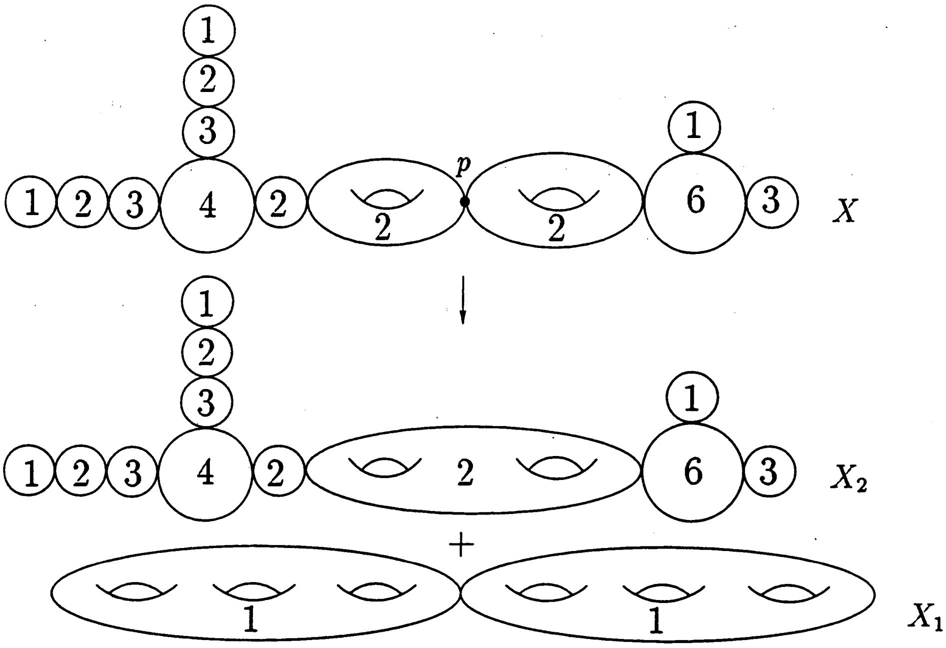

Figure 4: An example for Criterion 5.1.2

We digress to give a topological remark. Taking a real number ε $(0 < \varepsilon < 1)$, we consider a germ $\{(x,y) \in \mathbb{C}^2 : |x^m y^m| \leq \varepsilon\}$ of the multiple node of multiplicity m. Its boundary is a *real* 3-manifold, which is a disjoint union of two solid tori $T_x := \{|x| = 1\,,\ |y| \leq \varepsilon^{1/m}\}$ and $T_y := \{|y| = 1\,,\ |x| \leq \varepsilon^{1/m}\}$. In Figure 5, T_x and T_y are respectively described by the gray and black bold lines (in the real 2-dimensional figure, two gray lines are disconnected, but they are in fact connected; the same for two black lines).

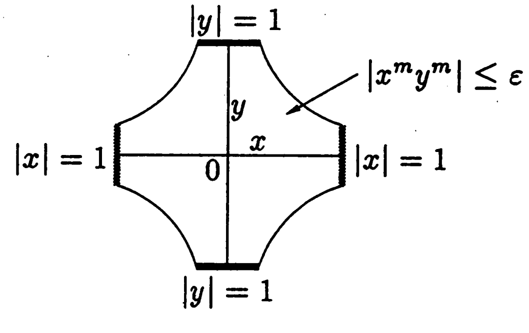

Figure 5:

Remark 5.1.4 In the construction of Ψ in Criterion 5.1.3, we only used one multiple node. When X has n multiple nodes p_i $(i = 1, 2, \ldots, n)$ of multiplicity m_i, we can generalize the construction in Criterion 5.1.3 to construct a splitting deformation of $\pi : M \to \Delta$, such that $\pi_t : M_t \to \Delta_t$ contains singular fibers X_i $(i = 1, 2, \ldots, n)$, which is obtained from X by replacing the multiple node p_i by the multiple annulus of multiplicity m_i.

5.2 Criterion in terms of plane curve singularities

In this subsection, we always suppose that $\pi : M \to \Delta$ is relatively minimal (not necessarily normally minimal). We will exhibit a splitting criterion in terms of plane curve singularities on X. We begin by introducing some terminology. Assume that the origin of $V := \{(x,y) \in \mathbb{C}^2 \ : \ F(x,y) = 0\}$ is a plane curve singularity. (In this paper, *a plane curve singularity always means a reduced one.*) For a positive integer m, setting

$$V_m := \{(x,y) \in \mathbb{C}^2 \ : \ F(x,y)^m = 0\},$$

we say that V_m is a *multiple plane curve singularity* of *multiplicity* m. (We also use the notation mV for V_m.)

Proposition 5.2.1 *Suppose that there exists a point $p \in X$ such that a germ of p in X is a multiple of a plane curve singularity and the multiplicity m is at least 2. Then $\pi : M \to \Delta$ admits a splitting deformation.*

Proof. We choose an open covering $M = M_0 \cup M_1$, where (1) $M_0 \cap X$ is a germ of the multiple plane curve singularity mV and (2) $M_1 \cap X$ is 'outside' mV. (See Figure 6.) We take local coordinates $(z_\beta, \zeta_\beta) \in M_0$. Then $\pi(z_\beta, \zeta_\beta) = F(z_\beta, \zeta_\beta)^m$,

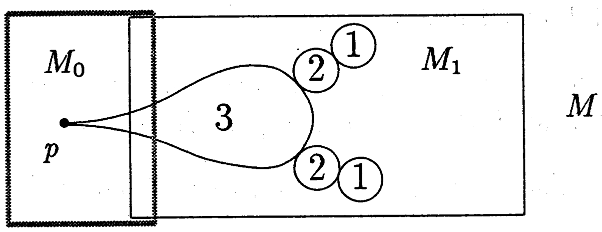

Figure 6:

where $F(z_\beta, \zeta_\beta) = 0$ defines the plane curve singularity V. Next, we take local coordinates $(z_\alpha, \zeta_\alpha) \in M_1$ near p, then $\pi(z_\alpha, \zeta_\alpha) = \zeta_\alpha^m u_\alpha(z_\alpha, \zeta_\alpha)^m$ for some non-vanishing holomorphic function u_α. Rewriting $\zeta_\alpha u_\alpha$ by ζ_α, we have $\pi(z_\alpha, \zeta_\alpha) = \zeta_\alpha$. Since $\pi(z_\alpha, \zeta_\alpha) = \pi(z_\beta, \zeta_\beta)$, we have $\zeta_\alpha^m = F(z_\beta, \zeta_\beta)^m$. As in the proof of Criterion 5.1.2, possibly after coordinate change, we have $\zeta_\alpha = F(z_\beta, \zeta_\beta)$. So the gluing map of M_0 and M_1 is of the form

$$z_\alpha = \phi_{\alpha\beta}(z_\beta, \zeta_\beta), \qquad \zeta_\alpha = F(z_\beta, \zeta_\beta) \qquad \text{near } p,$$

where $\phi_{\alpha\beta}$ is holomorphic. Next, we take a non-equisingular deformation of V:

$$V_t : \ F(z_\beta, \zeta_\beta) + G(z_\beta, \zeta_\beta, t) = 0, \quad \text{where } G \text{ is holomorphic and } G(z_\beta, \zeta_\beta, 0) = 0.$$

For example, if V is a node (A_1-singularity), take $G(z_\beta, \zeta_\beta, t) := t$, and otherwise take a *Morsification*[19] of V, i.e. V_t ($t \neq 0$) has only nodes (A_1-singularities). Next, we define a smooth hypersurface $\mathcal{M}_0$ in $M_0 \times \Delta \times \Delta^\dagger$, by

$$\{(z_\beta, \zeta_\beta, s, t) \in M_0 \times \Delta \times \Delta^\dagger \ : \ (F(z_\beta, \zeta_\beta) + G(z_\beta, \zeta_\beta, t))^m - s = 0\}.$$

[19] An isolated hypersurface singularity always admits a Morsification. See, for example Dimca

Similarly, we define a smooth hypersurface $\mathcal{M}_1$ in $M_1 \times \Delta \times \Delta^\dagger$, by

$$\{(x,s,t) \in M_1 \times \Delta \times \Delta^\dagger \,:\, \pi(x) - s = 0\}.$$

We glue $\mathcal{M}_0$ with $\mathcal{M}_1$ by

$$z_\alpha = \phi_{\alpha\beta}(z_\beta, \zeta_\beta), \qquad \zeta_\alpha = F(z_\beta,\zeta_\beta) + G(z_\beta,\zeta_\beta,t) \qquad \text{near } p,$$

which yields a complex 3-manifold $\mathcal{M}$. Letting $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be the natural projection, the fiber $X_1 := \Psi^{-1}(0,t)$ is a singular fiber, which is obtained from X by replacing the multiple plane curve singularity mV with mV_t. (To describe other singular fibers, it is necessary to compute the discriminant of $(F(z_\beta,\zeta_\beta) + G(z_\beta,\zeta_\beta,t))^m - s = 0$.) Since $\pi : M \to \Delta$ is relatively minimal, $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a splitting deformation. $\square$

In the assumption of the above proposition, if we replace $m \geq 2$ by $m = 1$, what can we say about the splittability of $\pi : M \to \Delta$? Also in this case, the above construction works, and we obtain a splitting deformation, except the case where p is a node and $X \setminus p$ is smooth (this is an atomic case). Combined with Proposition 5.2.1, we have the following results.

Criterion 5.2.2 *Let $\pi : M \to \Delta$ be relatively minimal. Suppose that the singular fiber X has a point p, such that a germ of p in X is either*

(1) *a multiple of a plane curve singularity of multiplicity at least* 2, *or*

(2) *a plane curve singularity such that if it is a node, then $X \setminus p$ is not smooth.*

Then $\pi : M \to \Delta$ admits a splitting deformation.

6 Splitting criteria via configurations, II

In this section, we shall present another type of splitting criteria in terms of existence of an irreducible component of multiplicity 1 satisfying a certain property.

6.1 Criterion in terms of connected components

Criterion 6.1.1 *Let $\pi : M \to \Delta$ be normally minimal. Suppose that the singular fiber X contains an irreducible component Θ_0 of multiplicity* 1 *such that $X \setminus \Theta_0$ is (topologically) disconnected. Denote by $Y_1, Y_2, \ldots, Y_l$ $(l \geq 2)$ all connected components of $X \setminus \Theta_0$. Then $\pi : M \to \Delta$ admits a splitting deformation which splits X into $X_1, X_2, \ldots, X_l$, where X_i $(i = 1, 2, \ldots, l)$ is obtained from X by 'smoothing' $Y_1, Y_2, \ldots, \check{Y}_i, \ldots, Y_l$* (see Figure 7 for example). *Here $\check{Y}_k$ is the omission of Y_i.*

Proof. To avoid complicated notation, we only show the statement for the case where Y_i and Θ_0 intersects only at one point p_i. (The construction below works for the general case.) We take an open covering $M = M_0 \cup M_1 \cup \cdots \cup M_l$, such that

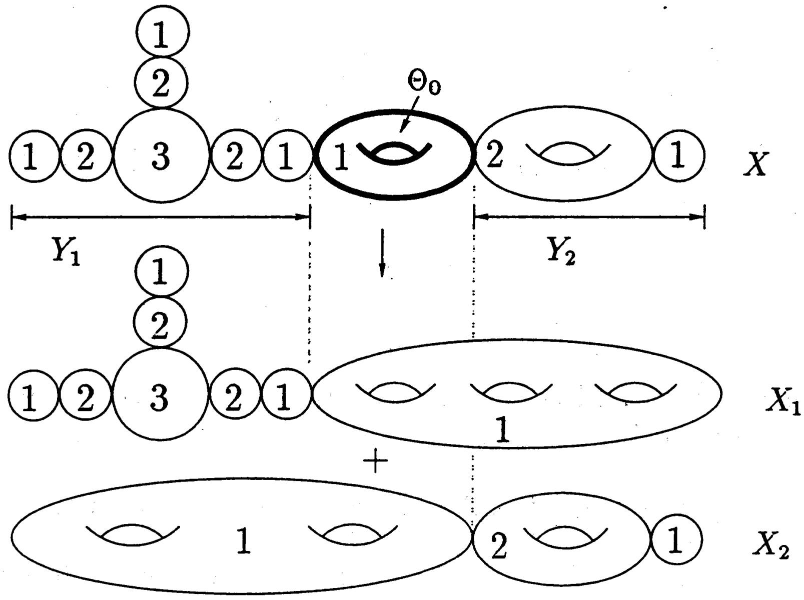

Figure 7: An example for Criterion 6.1.1

(1) $M_i \cap X = Y_i \cup D_i$, where $D_i \subset \Theta_0$ is a disk around p_i,

(2) $M_0 \cap X = \Theta_0 \setminus \{D'_1 \cup D'_2 \cup \cdots \cup D'_l\}$, where D'_i is a disk satisfying $p_i \in D'_i \subset D_i$. (See Figure 8.)

Here, we choose M_i so that D_i (and so D'_i) are sufficiently small. For simplicity, we

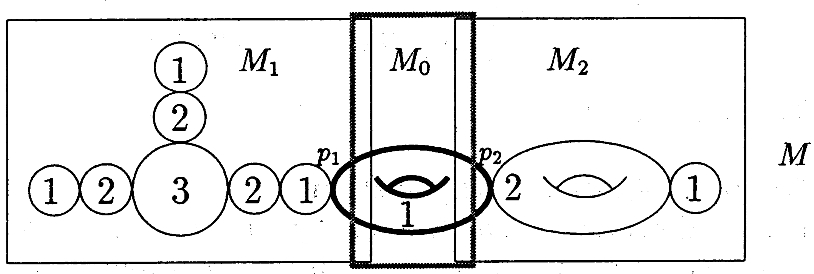

Figure 8:

set $Y_i^+ := Y_i \cup D_i$ and $\Theta_0^- := \Theta_0 \setminus \{D'_1 \cup D'_2 \cup \cdots \cup D'_l\}$. See Figure 9.

Now we shall construct a splitting deformation of π in the following steps: First, construct complex 3-manifolds $\mathcal{M}_i$ $(i = 0, 1, \ldots, l)$ with proper holomorphic maps Ψ_i on $\mathcal{M}_i$. Secondly, glue $\mathcal{M}_i$ together to construct a complex 3-manifold $\mathcal{M}$ so that Ψ_i $(i = 0, 1, \ldots, l)$ determine a holomorphic map Ψ on $\mathcal{M}$. Finally, we show that $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a splitting deformation of π.

Step 1. Construction of complex 3-manifolds $\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_l$

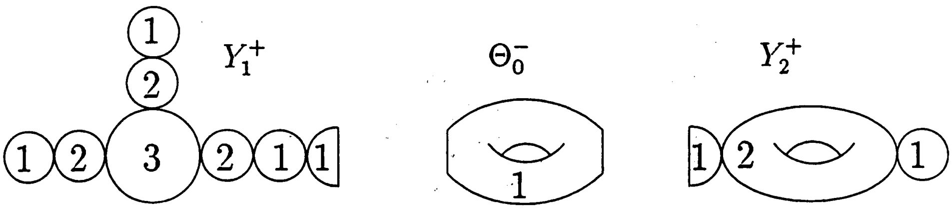

Figure 9:

We put $\mu := e^{2\pi i/l}$, and consider a smooth hypersurface $\mathcal{M}_i$ in $M_i \times \Delta \times \Delta^\dagger$ $(i = 1, 2, \ldots, l)$ defined by

$$\{(x, s, t) \in M_i \times \Delta \times \Delta^\dagger \;:\; \pi(x) - s + \mu^i t = 0\}. \tag{6.1.1}$$

Let $\Psi_i : \mathcal{M}_i \to \Delta \times \Delta^\dagger$ be the natural projection. Then for $t \neq 0$, we have

$$\Psi_i^{-1}(s, t) = \begin{cases} Y_i^+, & s = \mu^i t, \\ \text{smooth}, & \text{otherwise}. \end{cases} \tag{6.1.2}$$

Next, we consider a smooth hypersurface $\mathcal{M}_0$ in $M_0 \times \Delta \times \Delta^\dagger$ defined by

$$\{(x, s, t) \in M_0 \times \Delta \times \Delta^\dagger \;:\; \pi(x) - s = 0\}.$$

Let $\Psi_0 : \mathcal{M}_0 \to \Delta \times \Delta^\dagger$ be the natural projection. Then for $t \neq 0$, we have

$$\Psi_0^{-1}(s, t) = \begin{cases} \Theta_0^-, & s = 0, \\ \text{smooth}, & \text{otherwise}. \end{cases} \tag{6.1.3}$$

(Note that Θ_0^- is also smooth!)

Step 2. Gluing $\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_l$ together

Now we take local coordinates of M around p_i. Let $(z_\alpha, \zeta_\alpha) \in M_0$ and $(z_\beta, \zeta_\beta) \in M_i$ be local coordinates around p_i. Denote by m_i the multiplicity of the irreducible component intersecting Θ_0 at p_i. Then we have

$$\pi(z_\alpha, \zeta_\alpha) = \zeta_\alpha f_\alpha(z_\alpha, \zeta_\alpha), \qquad \pi(z_\beta, \zeta_\beta) = z_\beta^{m_i} \zeta_\beta\, g_\beta(z_\beta, \zeta_\beta),$$

where f_α and g_β are non-vanishing holomorphic functions. We shall change coordinates. Rewriting $\zeta_\alpha f_\alpha$ by ζ_α, we have $\pi(z_\alpha, \zeta_\alpha) = \zeta_\alpha$. Likewise, rewriting $\zeta_\beta g_\beta$ by ζ_β, we have $\pi(z_\beta, \zeta_\beta) = z_\beta^{m_i}\zeta_\beta$. Since $\pi(z_\alpha, \zeta_\alpha) = \pi(z_\beta, \zeta_\beta)$, we obtain a relation $\zeta_\alpha = z_\beta^{m_i}\zeta_\beta$. Hence the gluing map of M_0 and M_i around p_i is of the form

$$z_\alpha = \phi_{\alpha\beta}(z_\beta, \zeta_\beta), \qquad \zeta_\alpha = z_\beta^{m_i}\zeta_\beta,$$

where $\psi_{\alpha\beta}$ is holomorphic. Next, we glue $\mathcal{M}_0$ with $\mathcal{M}_i$ $(i = 1, 2, \ldots, l)$ around p_i by

$$z_\alpha = \phi_{\alpha\beta}(z_\beta, \zeta_\beta), \qquad \zeta_\alpha = z_\beta^{m_i}\zeta_\beta + \mu^i t \quad \text{around } p_i,$$

which yields a complex 3-manifold $\mathcal{M}$. Note that the above map transforms the defining equation of $\mathcal{M}_i$ near p_i to that of $\mathcal{M}_0$. Let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$, $(x, s, t) \mapsto (s, t)$, be the natural projection. From (6.1.2) and (6.1.3), for $t \neq 0$,

$$\Psi^{-1}(s,t) = \begin{cases} X_i, & s = \mu^i t, \\ \text{smooth}, & \text{otherwise}, \end{cases}$$

where X_i is obtained from X by smoothing $Y_1^+, Y_2^+, \ldots, \check{Y}_i^+, \ldots, Y_l^+$. As X_i is normally minimal, it follows from Lemma 1.0.1 that $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a splitting deformation which splits X into $X_1, X_2, \ldots, X_l$. This verifies our assertion. (Note: the discriminant of $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is $\prod_{i=1}^{l}(s - \mu^i t) = 0$.) □

From the above construction, we can deduce some property of topological monodromies. Let γ be the topological monodromy of $\pi : M \to \Delta$, and γ_i be the topological monodromy around X_i in $\pi_t : M_t \to \Delta_t$. Then we have a relation $\gamma = \gamma_1 \gamma_2 \cdots \gamma_l$. Moreover, the following holds.

Proposition 6.1.2 *The topological monodromies $\gamma_1, \gamma_2, \ldots, \gamma_l$ commute.*

Proof. We slightly modify the above construction of $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$; let σ be an arbitrary permutation of the set $\{1, 2, \ldots, l\}$. Instead of $\mathcal{M}_i$, we define $\mathcal{M}_{\sigma,i}$ as follows (cf. (6.1.1)): $\mathcal{M}_{\sigma,i} := \{(x, s, t) \in M_i \times \Delta \times \Delta^\dagger \ : \ \pi(x) - s + \mu^{\sigma(i)} t = 0\}$, while we take $\mathcal{M}_0$ as in the above construction: $\{(x, s, t) \in M_0 \times \Delta \times \Delta^\dagger \ : \ \pi(x) - s = 0\}$. Then we glue $\mathcal{M}_0$ with $\mathcal{M}_{\sigma,i}$ $(i = 1, 2, \ldots, l)$ by $z_\alpha = \phi_{\alpha\beta}(z_\beta, \zeta_\beta)$ and $\zeta_\alpha = z_\beta^{m_i} \zeta_\beta + \mu^{\sigma(i)} t$, and obtain a complex 3-manifold $\mathcal{M}_\sigma$. The natural projection $\Psi_\sigma : \mathcal{M}_\sigma \to \Delta \times \Delta^\dagger$ is also splitting deformation which splits X into $X_1, X_2, \ldots, X_l$. But $X_1, X_2, \ldots, X_l$ appears in the order $X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(l)}$, hence we have a relation $\gamma = \gamma_{\sigma(1)} \gamma_{\sigma(2)} \cdots \gamma_{\sigma(l)}$. Since σ is an arbitrary permutation, it follows that $\gamma_1, \gamma_2, \ldots, \gamma_l$ commute. □

Remark 6.1.3 In the construction of Ψ in Criterion 6.1.1, we used only one irreducible component of multiplicity 1. As is clear from the construction, we can similarly construct a splitting deformation by using several irreducible component $\Theta_0^{(1)}, \Theta_0^{(2)}, \ldots, \Theta_0^{(n)}$ of multiplicity 1 *simultaneously*, provided that $X \setminus \{\Theta_0^{(1)} \cup \Theta_0^{(2)} \cup \cdots \cup \Theta_0^{(n)}\}$ is disconnected. More generally, in some cases, we can construct a splitting deformation, by 'mixing up' all constructions in this paper.

6.2 Inductive criterion

Let $\pi : M \to \Delta$ be normally minimal, such that its singular fiber X contains an irreducible component Θ_0 of multiplicity 1. We suppose that $X \setminus \Theta_0$ is connected. Also in this case, we have some splitting criterion. To state our results, we need to introduce some notation. Let $Y := X \setminus \Theta_0$, and $p_1, p_2, \ldots, p_n$ be the intersection points of Θ_0 with other irreducible components of X. Take an open covering $M = M_0 \cup M_1$, such that

(1) $M_1 \cap X = Y \cup D_1 \cup D_2 \cup \cdots \cup D_n$, where $D_i \subset \Theta_0$ is a disk around p_i,

(2) $M_0 \cap X = \Theta_0 \setminus \{D'_1 \cup D'_2 \cup \cdots \cup D'_n\}$, where D'_i is a disk satisfying $p_i \in D'_i \subset D_i$. (See Figure 10.)

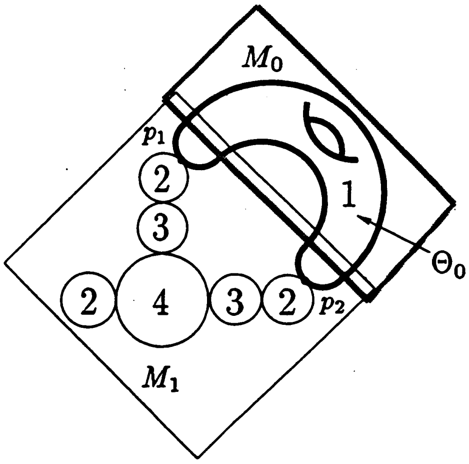

Figure 10:

Here, we choose M_1 so that D_i (and so D'_i) are sufficiently small. For simplicity, we set

$$Y^+ := Y \cup D_1 \cup D_2 \cup \cdots \cup D_n, \quad \Theta_0^- := \Theta_0 \setminus \{D'_1 \cup D'_2 \cup \cdots \cup D'_n\} \quad \text{(Figure 11)}.$$

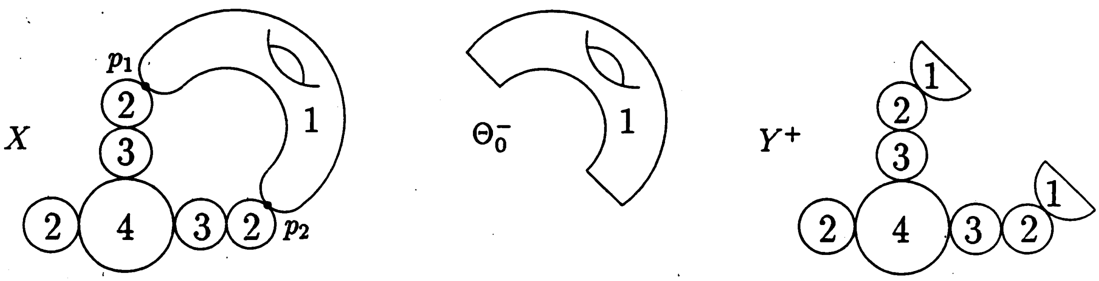

Figure 11:

Criterion 6.2.1 *Let $\pi : M \to \Delta$ be normally minimal such that the singular fiber X contains an irreducible component Θ_0 of multiplicity 1. Let $\pi_1 : M_1 \to \Delta$ be the restriction of π to a tubular neighborhood M_1 of $X \setminus \Theta_0$ in M. Suppose that $\pi_1 : M_1 \to \Delta$ admits a splitting deformation Ψ_1 which splits Y^+ into $Y_1^+, Y_2^+, \ldots, Y_l^+$. Then $\pi : M \to \Delta$ admits a splitting deformation Ψ which splits X into $X_1, X_2, \ldots, X_l$, where X_i is obtained from Y_i^+ by gluing Θ_0^- along the boundary.*

Note: We note that $\pi_1 : M_1 \to \Delta$ is a degeneration of curves with boundary, for which we may also define the notion of splitting deformations in the same way as for degenerations of compact curves.

Proof. As in the proof of Criterion 6.1.1, we take local coordinates $(z_\alpha, \zeta_\alpha) \in M_0$ near p_i with $\pi(z_\alpha, \zeta_\alpha) = \zeta_\alpha$, and local coordinates $(z_\beta, \zeta_\beta) \in M_1$ near p_i with $\pi(z_\beta, \zeta_\beta) = z_\beta^{m_i}\zeta_\beta$ such that the gluing map of M_0 and M_i around p_i is of the form

$$z_\alpha = \phi_{\alpha\beta}(z_\beta, \zeta_\beta), \qquad \zeta_\alpha = z_\beta^{m_i}\zeta_\beta,$$

where $\phi_{\alpha\beta}$ is holomorphic. Now, letting $\Psi_1 : \mathcal{M}_1 \to \Delta \times \Delta^\dagger$ be the splitting deformation of π_1 in the assumption, we consider a map $\widetilde{\pi}_1 := \mathrm{pr}_1 \circ \Psi_1 : \mathcal{M} \to \Delta$, and then realize $\mathcal{M}_1$ as the graph of $\widetilde{\pi}_1$:

$$\mathcal{M}_1 = \{(x, s, t) \in M_1 \times \Delta \times \Delta^\dagger \,:\, \widetilde{\pi}_1(x,t) - s = 0\}.$$

Notice that $\widetilde{\pi}_1(x, 0) = \pi_1(x)$, hence we may express $\widetilde{\pi}_1(x,t) = \pi_1(x) + h_1(x,t)$, where h_1 is a holomorphic function satisfying $h_1(x, 0) = 0$. Next, we define a smooth hypersurface $\mathcal{M}_0$ in $M_0 \times \Delta \times \Delta^\dagger$ by

$$\mathcal{M}_0 = \{(x, s, t) \in M_0 \times \Delta \times \Delta^\dagger \,:\, \pi(x) - s = 0\}.$$

Finally, we glue $\mathcal{M}_0$ with $\mathcal{M}_1$ around p_i by

$$z_\alpha = \phi_{\alpha\beta}(z_\beta, \zeta_\beta), \qquad \zeta_\alpha = z_\beta^{m_i}\zeta_\beta + h_1(z_\beta, \zeta_\beta),$$

and we obtain a complex 3-manifold $\mathcal{M}$. Then the natural projection $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a splitting deformation of π. In fact, assuming that the fiber Y_k^+ of Ψ_1 over the point $x_k \in \Delta_t$ is singular, by construction, $\Psi^{-1}(x_k)$ is obtained by gluing Y_k^+ with Θ_0^- along the boundary. □

From $\pi_1 : M_1 \to \Delta$ in Criterion 6.2.1, we shall construct a degeneration $\pi' : M' \to \Delta$ of *compact* curves, whose singular fiber X' is obtained by replacing the disk D_i $(i = 1, 2, \ldots, n)$ by a projective line (see Figure 12), after that, we will restate Criterion 6.2.1 in terms of this degeneration. First, we glue M_1 with $D_i \times \Delta$ by

$$z_\alpha = \phi_{\alpha\beta}(z_\beta, \zeta_\beta), \quad \zeta_\alpha = z_\beta^{m_i}\zeta_\beta,$$

where $(z_\alpha, \zeta_\alpha) \in M_1$ is coordinates near p_i, and $(z_\beta, \zeta_\beta) \in D_i \times \Delta$. Then we obtain a complex surface M'. Define a map $\pi' : M' \to \Delta$ by $\pi'|_{M_1} = \pi$, and $\pi'|_{D_i \times \Delta}(z_\beta, \zeta_\beta) = \zeta_\beta$. By construction, the singular fiber of π' is obtained by replacing D_i $(i = 1, 2, \ldots, n)$ by a projective line.
Then Criterion 6.2.1 is restated as follows:

Criterion 6.2.1′ *If $\pi' : M' \to \Delta$ admits a splitting deformation, then $\pi : M \to \Delta$ also admits a splitting deformation.* (Note: By construction, the converse is true.)

Let g (resp. g') be the genus of a smooth fiber of $\pi : M \to \Delta$ (resp. $\pi' : M' \to \Delta$). Except the case where Θ_0 is a projective line and intersects other irreducible components at only one point, we have $g' < g$, and so $\pi' : M' \to \Delta$ is a degeneration of curves of lower genus. Indeed, let Θ_0 intersect other irreducible components at n points. By a topological consideration, it is easy to see that

$$g = g' + (n-1) + \mathrm{genus}(\Theta_0). \tag{6.2.1}$$

Hence we have $g' < g$, unless Θ_0 is a projective line and $n = 1$.

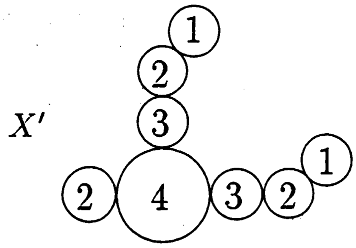

Figure 12:

6.3 Consequence of splitting criteria

As before, in this subsection, we assume that any degeneration is normally minimal. The splitting criteria obtained in this paper altogether imply that if the singular fiber X is not star-shaped, then in many cases, $\pi : M \to \Delta$ admits a splitting deformation. Taking into account Theorem 2.0.2, it is interesting to know whether the following conjecture[20] is true or not (cf. Conjecture 6.3.1′ below):

Conjecture 6.3.1 *A degeneration is atomic if and only if its singular fiber is either a reduced curve with one node, or a multiple of a smooth curve.*

See [Ta,III], [Ta] for results on this conjecture. Next, we deduce a useful theorem from our splitting criteria. Let Λ_g be a set of degenerations $\pi : M \to \Delta$ of curves of genus g such that

(1) the singular fiber X has a multiple node (here we exclude the case where X is a reduced curve with only one node), or

(2) X contains an irreducible component Θ_0 of multiplicity 1 satisfying the following condition[21]: if $X \setminus \Theta_0$ is connected, then either genus$(\Theta_0) \geq 1$, or Θ_0 is a projective line intersecting other irreducible components at at least two points.

As a consequence of our splitting criteria, we obtain the following.

Theorem 6.3.2 *Suppose that Conjecture 6.3.1 is valid for genus $\leq g-1$. If $\pi : M \to \Delta$ is a degeneration in Λ_g, then π is not atomic.*

Proof. First, by Criterion 5.1.3, if the singular fiber contains a multiple node, then π admits a splitting deformation. Next, suppose that X contains an irreducible component Θ_0 of multiplicity 1. if $X \setminus \Theta_0$ is not connected, then $\pi : M \to \Delta$ has a splitting deformation (Criterion 6.1.1). On the other hand, if $X \setminus \Theta_0$ is connected, then under the assumption of this theorem, we can apply Criterion 6.2.1′, and see that $\pi : M \to \Delta$ admits a splitting deformation, except the case where Θ_0 is a

[20]This conjecture is valid for the genus 1 and 2 cases: for the genus 1 case, any atomic fiber is either a rational curve with one node, or a multiple of a smooth elliptic curves by [Mo], and for the genus 2 case, any atomic fiber is a reduced curve with one node by [Ho].

[21]If $X \setminus \Theta_0$ is not connected, we pose no condition.

projective line, and Θ_0 intersects other irreducible components at only one point (cf. (6.2.1)). Hence the assertion follows. □

Thus if the assumption of this theorem is fulfilled (for example, $g = 3$), to determine atomic degenerations of curves of genus g, it is enough to investigate the splittability for degenerations $\pi : M \to \Delta$ such that either

(A) $X = \pi^{-1}(0)$ is star-shaped, or

(B) X is not star-shaped and (B.1) X has no multiple node and (B.2) if X has an irreducible component Θ_0 of multiplicity 1, then Θ_0 is a projective line, and intersects other irreducible components of X only at one point.

In the terminology of [Ta,II], the singular fibers of a degeneration in (B) is obtained by 'bonding' star-shaped singular fibers such that any bonding of two branches is either (-1)-bonding, or 0-bonding of two branches with the same multiplicity at least 2. See [Ta,II] and also [MM2]. For these cases, we can apply another method (construction of splitting deformations via barkable sub-divisors), which is developed in [Ta,III].

Discussion and open problems

For higher genus cases, Conjecture 6.3.1 seems too optimistic. It is more reasonable to replace 'atomic' with 'absolutely atomic', where a degeneration $\pi : M \to \Delta$ is called *absolutely atomic* if all degenerations with the same topological type as $\pi : M \to \Delta$ are atomic (for example, when X is a reduced curve with one node or a multiple of a smooth curve. See Theorem 2.0.2).

Conjecture 6.3.1′ *A degeneration is absolutely atomic if and only if its singular fiber is either a reduced curve with one node, or a multiple of a smooth curve.*

Accordingly, we can show an analogous statement to Theorem 6.3.2 by the same argument.

Theorem 6.3.2′ *Suppose that Conjecture 6.3.1′ is valid for genus $\leq g-1$. If $\pi : M \to \Delta$ is a degeneration in Λ_g, then π is not absolutely atomic.*

It is plausible that for higher genus cases, there may be an atomic degeneration which is *not* absolutely atomic. However, no examples are known, and so we ask

Problem 6.3.3 *Do there exist two degenerations $\pi_1 : M_1 \to \Delta$ and $\pi_2 : M_2 \to \Delta$ with the same topological type such that π_1 is atomic while π_2 is not?*

Note that for the genus ≥ 2 case, there are degenerations with the same singular fiber, but with different topological types [MM2]. Taking this into account, it is natural ask the following problem analogous to Problem 6.3.3.

Problem 6.3.4 *Do there exist two degenerations* $\pi_1 : M_1 \to \Delta$ *and* $\pi_2 : M_2 \to \Delta$ *with the same singular fiber but with different topological types such that* π_1 *is atomic while* π_2 *is not?*

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