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A study of semiquasihomogeneous singularities by using holonomic system (holonomic 系を用いた半擬斉次孤立特異点の考察)

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1 Introduction

In this note, we study the quasihomogeneous singularity and exceptional singularities of modality 1 by using partial differential operators. We examine, in particular, algebraic local cohomology classes attached to these singularities.

Let X be an open neighborhood of the origin O in the n dimensional affine space $\mathbb{C}^n$. Let f be a holomorphic function with an isolated singularity at the origin O. Denote by I the ideal in the sheaf $\mathcal{O}_X$ of holomorphic functions on X generated by the partial derivatives $f_j = \dfrac{\partial f}{\partial z_j}$ $(j = 1, \ldots, n)$ of the function f. We denote by Σ the set of cohomology classes in $\mathcal{H}^n_{[0]}(\mathcal{O}_X)$ annihilated by every element in I. Since the pairing

$$\Omega_{X,O}/I\Omega_{X,O} \times \Sigma \to \mathbb{C} \tag{1.1}$$

defined by the Grothendieck local residue is non-degenerate, Σ becomes the dual space of $\mathcal{O}_{X,O}/I \cong \Omega_{X,O}/I\Omega_{X,O}$ as a vector space where Ω_X is the sheaf of holomorphic differential n-forms on X.

Let σ be an algebraic local cohomology class which generates Σ over $\mathcal{O}_{X,O}$. Since the algebraic local cohomology group $\mathcal{H}^n_{[O]}(\mathcal{O}_X)$ has a structure of $\mathcal{D}_X$ modules, we can consider annihilators of σ in the sheaf $\mathcal{D}_X$ of differential operators on X. In this paper, we consider the ideal, denoted by $Ann_{\leq 1}$, in $\mathcal{D}_X$ generated by annihilators of σ of at most first order.

We give the precise definition of $Ann_{\leq 1}$ and we give an description of the solution space of the holonomic system $\mathcal{D}_X/Ann_{\leq 1}$ in §2.

In §3, we examine $Ann_{\leq 1}$ in the case of quasihomogeneous singularities. We verify that the cohomology class σ attached to a quasihomogeneous singularity can be characterized as the solution of the system of differential equations of at most first order (Theorem 3.1).

For non-quasihomogeneous isolated singularities, we show that the dimension of the solution space of the holonomic system $\mathcal{D}_X/Ann_{\leq 1}$ is greater than or equal to 2. Especially, in the case of exceptional singularities of modality 1, we verify that the dimension of the solution space of $\mathcal{D}_X/Ann_{\leq 1}$ is just equal to 2 and the basis is given by σ and the delta function (Theorem 4.1 in §4). In §4.4, we give results of computations for normal forms of exceptional singularities of modality 1.

2 The first order differential operators acting on Σ

For a holomorphic function $f = f(z_1, \ldots, z_n)$ with an isolated singularity at the origin O, let I be the ideal in $\mathcal{O}_{X,O}$ generated by the partial derivatives $f_j = \dfrac{\partial f}{\partial z_j}$ $(j = 1, \ldots, n)$:

$$I = \langle f_1, \ldots, f_n \rangle_O.$$

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Denote by Σ the set of local cohomology classes annihilated by every element in I:

$$\Sigma = \{\eta \in \mathcal{H}^n_{[O]}(\mathcal{O}_X) \mid g\eta = 0,\ g \in I\}.$$

Let σ be a generator of Σ over $\mathcal{O}_{X,O}$:

$$\Sigma = \mathcal{O}_{X,O}\sigma.$$

Let P be a partial differential operator of first order which annihilates the algebraic local cohomology class σ. Such an operator P has the following property.

Lemma 2.1 *Let σ be a generator of Σ over $\mathcal{O}_{X,O}$. Let P be a linear partial differential operator of first order such that $P\sigma = 0$. Then, we have $P(\Sigma) \subseteq \Sigma$.*

Proof. Since σ generates Σ over $\mathcal{O}_{X,O}$, we can write any $\eta \in \Sigma$ as $\eta = h\sigma$ with some holomorphic function $h \in \mathcal{O}_{X,O}$. Let v_P be the first order part $\sum_{j=1}^n a_j(z)\frac{\partial}{\partial z_j}$ of the annihilator $P = \sum_{j=1}^n a_j(z)\frac{\partial}{\partial z_j} + a_0(z)$. We have

$$\begin{aligned} P(\eta) &= P(h\sigma) \\ &= (Ph - hP)\sigma + hP\sigma \\ &= v_P(h)\sigma \in \Sigma. \end{aligned}$$

Thus we have $P(\Sigma) \subseteq \Sigma$.□

Let $\mathcal{L}$ be the set of linear partial differential operators of at most first order which annihilate σ:

$$\mathcal{L} = \{P = \sum_{j=1}^n a_j(z)\frac{\partial}{\partial z_j} + a_0(z) \mid P\sigma = 0\}.$$

It is obvious from the proof of Lemma 2.1, the condition whether a given first order differential operator R acts on Σ or not depends only on the first order part v_R of R. We denote by $\mathcal{V}$ the set of differential operators of the form $\sum_{j=1}^n a_j(z)\frac{\partial}{\partial z_j}$ acting on Σ. Then, $v = \sum_{j=1}^n a_j(z)\frac{\partial}{\partial z_j}$ is in $\mathcal{V}$ if and only if v satisfies the condition $vg \in I$ for any $g \in I$, i.e.,

$$\mathcal{V} = \{v = \sum_{j=1}^n a_j(z)\frac{\partial}{\partial z_j} \mid vg \in I, {}^{\forall} g \in I\}.$$

Lemma 2.2 *The mapping, from $\mathcal{L}$ to $\mathcal{V}$, which associates the first order part $v_p \in \mathcal{V}$ to $P \in \mathcal{L}$ is a surjective mapping.*

Proof. For any $v \in \mathcal{V}$, there exists a holomorphic function $h \in \mathcal{O}_{X,O}$ such that $v\sigma = h\sigma$. Thun we have an annihilator $P = v - h \in \mathcal{L}$. □

Let us consider the condition that the class $\eta \in \Sigma$ becomes a solution of homogeneous differential equation $P\eta = 0$ for an annihilator P of $\sigma \in \Sigma$. There exists a holomorphic function $h \in \mathcal{O}_{X,O}$ which satisfies $\eta = h\sigma$. Then we have

$$v_P h = \sum_{j=1}^n a_i(z)\frac{\partial h}{\partial z_j} \in I$$

where $v_P \in \mathcal{V}$ is the first order part of the differential operator P. It is obvious that to represent $\eta \in \Sigma$ in the form $\eta = h\sigma$, it suffices to take the modulo class $h \mod I$ of the holomorphic function $h \in \mathcal{O}_{X,O}$. An element $v \in \mathcal{V}$ induces a linear operator acting on $\mathcal{O}_{X,O}/I$ which is also denoted by v:

$$v : \mathcal{O}_{X,O}/I \to \mathcal{O}_{X,O}/I.$$

We can put

$$\mathcal{H} = \{h \in \mathcal{O}_{X,O}/I \mid vh = 0, {}^{\forall} v \in \mathcal{V}\}.$$

Put $\mathcal{A}nn_{\leq 1} = \mathcal{D}_X\mathcal{L}$. $\mathcal{A}nn_{\leq 1}$ defines a left ideal in $\mathcal{D}_X$. We have the next theorem.

Theorem 2.1

$$Hom_{\mathcal{D}_X}(\mathcal{D}_X/Ann_{\leq 1}, \mathcal{H}^n_{[O]}(\mathcal{O}_X)) = \mathrm{Span}\{h\sigma \mid h \in \mathcal{H}\}.$$

Proof. Since $I \subset Ann_{\leq 1}$ as an ideal of multiplicative operators, we have

$$Hom_{\mathcal{D}_X}(\mathcal{D}_X/Ann_{\leq 1}, \mathcal{H}^n_{[O]}(\mathcal{O}_X)) \subset \Sigma.$$

Thus we can write any solution of the holonomic system $\mathcal{D}_X/Ann_{\leq 1}$ as $h\sigma$ for some $h \in \mathcal{O}_{X,O}$. For $P \in \mathcal{L}$, we have

$$P(h\sigma) = v_P(h)\sigma = 0.$$

Thus we have $v_P h = 0$. □

3 The case of quasihomogeneous singularities

Let σ be a generator of Σ over $\mathcal{O}_{X,O}$. Let Ann be a left ideal in $\mathcal{D}_X$ consisting of annihilators of the algebraic local cohomology class σ.

Theorem 3.1 *The following three conditions are equivalent :*

(i) $\mathcal{O}_{X,O}\langle f, f_1, \cdots, f_n\rangle = \mathcal{O}_{X,O}\langle f_1, \cdots, f_n\rangle$, *where* $f_j := \dfrac{\partial f}{\partial z_j}$, $j = 1, \ldots, n$.

(ii) $Ann_{\leq 1} = Ann$.

(iii) $Hom_{\mathcal{D}_X}(\mathcal{D}_X/Ann_{\leq 1}, \mathcal{H}^n_{[O]}(\mathcal{O}_X)) = \mathrm{Span}\{\sigma\}$.

Proof.

(i)⇒(ii) : Suppose that $\mathcal{O}_{X,O}\langle f, f_1, \cdots, f_n\rangle = \mathcal{O}_{X,O}\langle f_1, \cdots, f_n\rangle$ for $f \in \mathcal{O}_{X,O}$. Then the function f can be expressed in terms of the derivatives $f_1, \ldots, f_n$. We have

$$\begin{aligned} f &= a_1 f_1 + \cdots + a_n f_n \\ &= a_1\frac{\partial f}{\partial z_1} + \cdots + a_n\frac{\partial f}{\partial z_n} \\ &= (a_1\frac{\partial}{\partial z_1} + \cdots + a_n\frac{\partial}{\partial z_n})f, \end{aligned} \tag{3.1}$$

with $a_1, \ldots, a_n \in \mathcal{O}_{X,O}$. Assume that $(a_1, \cdots, a_n) \neq (0, \cdots, 0)$. Put $v = a_1\dfrac{\partial}{\partial z_1} + \cdots + a_n\dfrac{\partial}{\partial z_n}$. ¿From (3.1), we have $f_j = (a_{1j}f_1 + \cdots + a_{nj}f_n) + vf_j$ where $a_{kj} = \dfrac{\partial a_k}{\partial z_j}$. As $vf_j = f_j - (a_{1j}f_1 + \cdots + a_{nj}f_n) \in I$, we have $v \in \mathcal{V}$. ¿From Lemma 2.2, we have an annihilator $P = v + a_0$ of the cohomology class σ for some $a_0 \in \mathcal{O}_{X,O}$. We have

$$\langle f_1, \cdots, f_n, P\rangle \subseteq Ann_{\leq 1} \subseteq Ann.$$

It is known in [2] that the Jacobian of $a_1, \ldots, a_n$ is not zero at the origin. This assures that the holonomic system $\mathcal{D}_X/\langle f_1, \cdots, f_n, P\rangle$ becomes simple. Since the holonomic system $\mathcal{D}_X/Ann$ is simple, we have

$$\langle f_1, \cdots, f_n, P\rangle = Ann_{\leq 1} = Ann.$$

(iii)⇒(i) : Assume that $f \notin \mathcal{O}_{X,O}\langle f_1, \cdots, f_n\rangle$. Obviously, we have $f\sigma \neq 0$. Let us denote by $F \in \mathcal{D}_X$ the multiplicative operator defined by $f \in \mathcal{O}_{X,O}$. If the differential operator $P = \sum_{j=1}^n a_j\dfrac{\partial}{\partial z_j} + a_0$ annihilates the cohomology class σ, we have

$$\begin{aligned} P(f\sigma) &= PF\sigma \\ &= (PF - FP)\sigma + FP\sigma \\ &= \sum_{j=1}^n a_j\frac{\partial f}{\partial z_j}\sigma. \end{aligned}$$

Since $\sum_{j=1}^{n} a_j f_j \in I$,

$$P(f\sigma) = 0$$

holds. Thus, there exist at least 2 elements σ and $f\sigma$ in $Hom_{\mathcal{D}_X}(\mathcal{D}_X/Ann_{\leq 1}, \mathcal{H}^n_{[O]}(\mathcal{O}_X))$. As σ and $f\sigma$ are linearly independent elements, we have

$$\dim Hom_{\mathcal{D}_X}(\mathcal{D}_X/Ann_{\leq 1}, \mathcal{H}^n_{[O]}(\mathcal{O}_X)) \geq 2.$$

(ii)$\Rightarrow$(iii) : By assumption, we have

$$Hom_{\mathcal{D}_X}(\mathcal{D}_X/Ann_{\leq 1}, \mathcal{H}^n_{[O]}(\mathcal{O}_X)) = Hom_{\mathcal{D}_X}(\mathcal{D}_X/Ann, \mathcal{H}^n_{[O]}(\mathcal{O}_X))$$

where $Ann = \{P \in \mathcal{D}_X \mid P\sigma = 0\}$. Since $\mathcal{D}_X/Ann$ is simple holonomic system, we have

$$Hom_{\mathcal{D}_X}(\mathcal{D}_X/Ann, \mathcal{H}^n_{[O]}(\mathcal{O}_X)) = \mathrm{Span}\{\sigma\}.$$

□

For the holomorphic function f with an isolated singularity at the origin O, suppose that

$$\mathcal{O}_{X,O}\langle f, f_1, \cdots, f_n\rangle = \mathcal{O}_{X,O}\langle f_1, \cdots, f_n\rangle. \tag{3.2}$$

It is known in [2] there exists some holomorphic coordinate transformation which makes f a quasihomogeneous polynomial. Theorem 3.1 asserts that, it is possible to characterize the algebraic local cohomology class σ attached to a quasihomoegeous singularity as the solution of the system of differential equations of at most first order.

4 The case of exceptional families of singularities of modality 1.

In this Section, we characterize cohomology classes attached to exceptional families of unimodal singularities. Functions having non-degenerate quasihomogeneous principal part of modality 1 can be reduced to three one-parameter families of parabolic singularities and 14 polynomials generating exceptional families. Since the parabolic singularities satisfy (3.2), our objects are the following 14 polynomials.

2variables			3variables		
E_{12}	:	$f(x,y) = x^3 + y^7 + axy^5$			
E_{13}	:	$f(x,y) = x^3 + xy^5 + ay^8$	Q_{10}	:	$f(x,y,z) = x^3 + y^4 + yz^2 + axy^3$
E_{14}	:	$f(x,y) = x^3 + y^8 + axy^6$	Q_{11}	:	$f(x,y,z) = x^3 + y^2z + xz^3 + az^5$
Z_{11}	:	$f(x,y) = x^3y + y^5 + axy^4$	Q_{12}	:	$f(x,y,z) = x^3 + y^5 + yz^2 + axy^4$
Z_{12}	:	$f(x,y) = x^3y + xy^4 + ay^6$	S_{11}	:	$f(x,y,z) = x^4 + y^2z + xz^2 + ay^2x^2$
Z_{13}	:	$f(x,y) = x^3y + y^6 + axy^5$	S_{12}	:	$f(x,y,z) = x^2y + y^2z + xz^3 + az^5$
W_{12}	:	$f(x,y) = x^4 + y^5 + ax^2y^3$	U_{12}	:	$f(x,y,z) = x^3 + y^3 + z^4 + axyz^2$
W_{13}	:	$f(x,y) = x^4 + xy^4 + ay^6$			

These normal forms of quasihomogeneous singularities are given by V.I.Arnold ([1]).

4.1 The quasidegrees of cohomology classes

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a type of quasihomogeneous singularities. A cohomology class $\eta \in \Sigma$ has an expression

$$\eta = \left[\sum_{\mathbf{k}\in E} c_{\mathbf{k}} \frac{1}{z^{\mathbf{k}}}\right]$$

where $c_{\mathbf{k}} \in \mathbb{Q}$ and $z^{\mathbf{k}} = z_1^{k_1} \cdots z_n^{k_n}$ with $\mathbf{k} = (k_1, \ldots, k_n)$ and E is a finite subset of $\mathbf{N}^n$.

Definition 4.1 *A cohomology class* $\left[\frac{1}{z^{\mathbf{k}}}\right]$ *has degree* $-d$ *if*

$$\langle \alpha, \mathbf{k}\rangle = \alpha_1 k_1 + \cdots + \alpha_n k_n = d$$

For a cohomology class $\eta = \left[\sum_{\mathbf{k}\in E_\eta} c_{\mathbf{k}} \frac{1}{z^{\mathbf{k}}}\right]$, we define its degree $d(\eta)$ by the smallest degree of classes $\left[\frac{1}{z^{\mathbf{k}}}\right]$ in η :

$$d(\eta) = \min\{-\langle \alpha, \mathbf{k}\rangle \mid \mathbf{k} \in E_\eta\},$$

where E_η is a set of all exponents $\mathbf{k} = (k_1, \dots, k_n)$ of non-zero term $c_{\mathbf{k}}\frac{1}{z^{\mathbf{k}}}$ in the above expression of the cohomology class η. For both functions and cohomology classes, we denote its degree by $d(\cdot)$.

In the case of semiquasihomogeneous singularities, we have the following result.

Proposition 4.1 *Let f be a semiquasihomogeneous function. For any basis monomial m_j of the vector space $\mathcal{O}_{X,O}/I$, there exists a cohomology class η in the vector space Σ which satisfies following two conditions :*

(i) $m_j\eta = \delta$, *where δ is the delta function with support at the origin.*

(ii) $d(\eta) = -\sum_{j=1}^n \alpha_j - d(m_j)$.

Furthermore, we have the following proposition.

Proposition 4.2 *Let f be a semiquasihomogeneous function. A necessary and sufficient condition for a cohomology class $\sigma \in \Sigma$ to be a generator of Σ over $\mathcal{O}_{X,O}$ is*

$$d(\sigma) = -nd(f) + \sum_{j=1}^n \alpha_j.$$

4.2 Cohomology classes attached to exceptional singularities of modality 1.

Recall that, for a non-quasihomogeneous function f, we have

$$\mathcal{O}_{X,O}\langle f_1, \dots, f_n\rangle \neq \mathcal{O}_{X,O}\langle f, f_1, \dots, f_n\rangle \tag{4.1}$$

and thus

$$\dim Hom_{\mathcal{D}_X}(\mathcal{D}_X/Ann_{\leq 1}, \mathcal{H}^n_{[O]}(\mathcal{O}_X)) \geq 2.$$

We examine the solution space $Hom_{\mathcal{D}_X}(\mathcal{D}_X/Ann_{\leq 1}, \mathcal{H}^n_{[O]}(\mathcal{O}_X))$ of the holonomic system $\mathcal{D}_X/Ann_{\leq 1}$ attached to exceptional singularities of modality 1.

We verify that $\mathcal{H}$ is spanned by 1 and the modulo class of $f(z)$ in $\mathcal{O}_{X,O}/I$. That is, we have the following proposition.

Proposition 4.3 *For a function f defining an exceptional singularity of modality* 1, *we have*

$$\mathcal{H} = \mathrm{Span}\{1, f \mod I\}.$$

Proposition 4.3 is proved by direct computations for each normal form of an exceptional family of singularities of modality 1. Since $z_j f \in I$ $(j = 1, \dots, n)$, we have $f \mod I = c_0 j_F(z) \mod I$ where $j_F(z)$ is the Jacobian $\dfrac{\partial(f_1, \dots, f_n)}{\partial(z_1, \dots, z_n)}$ and c_0 is a non-zero constant. Thus we have the following theorem.

Theorem 4.1 *Let f be a function defining an exceptional singularity of modality* 1. *Then, we have*

$$Hom_{\mathcal{D}_X}(\mathcal{D}_X/Ann_{\leq 1}, \mathcal{H}^n_{[O]}(\mathcal{O}_X)) = \mathrm{Span}\{\sigma, \delta\},$$

where δ is the delta function with support at the origin O.

To give effects of computations, we introduce the following vector spaces :

$$L = \{P = \sum_{j=1}^n a_j(z)\frac{\partial}{\partial z_j} + a_0(z) \mid P\sigma = 0,\ a_j(z) \in \mathcal{O}_{X,O}/I,\ j = 0, \dots, n\},$$

$$V = \{\mathrm{v} = \sum_{j=1}^n a_j(z)\frac{\partial}{\partial z_j} \mid \mathrm{v}g \in I, {}^\forall g \in I,\ a_j(z) \in \mathcal{O}_{X,O}/I,\ j = 1, \dots, n\},$$

$$H = \{h \in \mathcal{O}_{X,O}/I \mid \mathrm{v}h = 0, {}^\forall \mathrm{v} \in V\}.$$

Lemma 4.1 *We have the isomorphism between L and V:*

$$L \cong V.$$

Proof. For any $\mathrm{v} \in V$, there exist $h \in \mathcal{O}_{X,O}$ s.t., $\mathrm{v}\sigma = h\sigma$. By putting $a_0 = -h \mod I$, we have $(\mathrm{v} + a_0)\sigma = 0$. □

4.3 Example : E_{12} singularity.

The quasihomogeneous part of the function $f = x^3 + y^7 + axy^5$ is of type $(7, 3)$ of degree 21. The partial derivatives of f with respect to the variables x and y are $f_x = 3x^2 + ay^5$ and $f_y = 7y^6 + 5axy^4$, respectively. We use the lexicographic order with $x \succ y$ in computations. The standard base of the ideal $I = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle_O$ in $\mathcal{O}_{X,O}$ is

$$\{y^8, 7y^6 + 5ay^4x, 3x^2 + ay^5\}.$$

Basis monomials of the local ring $\mathcal{O}_{X,O}/I$ is given by

$$\begin{array}{cccccccccccc} 1, & y, & y^2, & x, & y^3, & yx, & y^4, & y^2x, & y^5, & y^3x, & y^4x, & y^5x \\ 0, & 3, & 6, & 7, & 9, & 10, & 12, & 13, & 15, & 16, & 19, & 22 \end{array}.$$

The numbers below the monomials are their quasi-degrees.

Any element of Σ is given by linear combination of the next 12 cohomology classes :
$\left[\frac{1}{y^6x^2} + a(-\frac{5}{7}\frac{1}{y^8x} - \frac{1}{3}\frac{1}{yx^4}) + \frac{5}{21}a^2\frac{1}{y^3x^3}\right]$, $\left[\frac{1}{y^5x^2} - \frac{5}{7}a\frac{1}{y^7x} + \frac{21}{5}a^2\frac{1}{y^2x^3}\right]$, $\left[\frac{1}{y^4x^2}\right]$,
$\left[\frac{1}{y^6x} - \frac{1}{3}a\frac{1}{yx^3}\right]$, $\left[\frac{1}{y^3x^2}\right]$, $\left[\frac{1}{y^5x}\right]$, $\left[\frac{1}{y^2x^2}\right]$, $\left[\frac{1}{y^4x}\right]$, $\left[\frac{1}{yx^2}\right]$, $\left[\frac{1}{y^3x}\right]$, $\left[\frac{1}{y^2x}\right]$, $\left[\frac{1}{yx}\right]$.
The first cohomology class generates Σ over $\mathcal{O}_{X,O}$.

The vector space V of differential operators v acting on Σ is generated by the following 14 operators :

$$252yx\partial_x - 30ax\partial_y + 65a^2y^4\partial_x,\ 63y^2\partial_y + 15ax\partial_y + 25a^2y^4\partial_x,$$
$$7y^2x\partial_x - 2ayx\partial_y,\ 7y^3\partial_y + 5ayx\partial_y,\ 6yx\partial_y - 5ay^5\partial_x,$$
$$y^4\partial_y,\ y^3x\partial_x,\ y^2x\partial_y,\ y^5\partial_y,\ y^4x\partial_x,\ y^3x\partial_y,\ y^5x\partial_x,\ y^4x\partial_y,\ y^5x\partial_y\ .$$

The solution space of the simultaneous homogeneous equation $\mathrm{v}h(x, y) = 0$ ($\forall \mathrm{v} \in V$) is

$$H = \mathrm{Span}\{1, y^5x\}.$$

Note that $y^5x = f \mod I$. Since $xf, yf \in I$, we have

$$Hom_{\mathcal{D}_X}(\mathcal{D}_X/Ann_{\leq 1}, \mathcal{H}^n_{[O]}(\mathcal{O}_X)) = \mathrm{Span}\{\sigma, \delta\},$$

where δ is the delta function with support at the origin.

4.4 Computations for normal forms

In this section, we give results of computations for normal forms of singularities of modality 1 listed before. To compute the solution space $Hom_{\mathcal{D}_X}(\mathcal{D}_X/Ann_{\leq 1}, \mathcal{H}^n_{[O]}(\mathcal{O}_X))$, we give

- partial derivatives f_{z_j} of the function $f(z)$,
- the standard base of the ideal I of partial derivatives of $f(z)$ at the origin,
- basis monomials $m_1, \ldots, m_\mu$ of $\mathcal{O}_X/I$ and its degree,
- basis $\sigma_1, \ldots, \sigma_\mu$ of the vector space Σ and its degree,
- basis $\mathrm{v}_1, \ldots, \mathrm{v}_N$ of the vector space V and its degree,

and H as the solution space of the simultaneous homogeneous equations $\mathrm{v}h(z)=0$ for every $\mathrm{v}\in V$. The number below the basis of $\mathcal{O}_{X,O}/I$, Σ, and V is its degree. Here, each z_j has weight α_j and each ∂_j has weight $-\alpha_j$.

We give basis σ_j of Σ which satisfies Proposition 4.1 for every basis monomial $m_{\mu-j+1}$ of $\mathcal{O}_{X,O}/I$, where $\mu=\dim\mathcal{O}_{X,O}/I$ is the Milnor number. That is, $\{\sigma_1,\dots,\sigma_\mu\}$ is the dual base of the monomial base $\{m_\mu,\dots,m_1\}$ of $\mathcal{O}_{X,O}/I$. The cohomology class σ_1 generates Σ over $\mathcal{O}_{X,O}$. Note that in expressions of the basis of Σ, we find the basis of the set of local cohomology classes annihilated by partial derivatives of quasihomogeneous part of the function f if we substitute $a=0$.

We use the standard basis in computations with respect to the lexicographic order with $z_i\succ z_j$ or $z_i\succ z_j\succ z_k$ of $\alpha_i\geq\alpha_j\geq\alpha_k$ where α_i is the weight of the variable z_i (resp. j, k). Therefore, the monomial basis of the local ring $\mathcal{O}_{X,O}/\langle f_1,\dots,f_n\rangle$ of Z_{12}, Q_{10}, S_{11} and S_{12} used in this paper are different from that in [1].

4.4.1 E_{12} : $x^3+y^7+axy^5$

$f=x^3+y^7+axy^5$ (of type (7,3) of degree 21)
Partial derivatives : $f_x=3x^2+ay^5$, $f_y=7y^6+5axy^4$
The standard base of $I=\langle f_x,f_y\rangle_O$: $\{y^8,7y^6+5ay^4x,3x^2+ay^5\}$
Basis of the local ring $\mathcal{O}_{X,O}/I$ and its degrees :

m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}	m_{11}	m_{12}
1	y	y^2	x	y^3	yx	y^4	y^2x	y^5	y^3x	y^4x	y^5x
0	3	6	7	9	10	12	13	15	16	19	22

Basis of Σ :

$$\sigma_1=\left[\frac{1}{y^6x^2}+a(-\frac{5}{7}\frac{1}{y^8x}-\frac{1}{3}\frac{1}{yx^4})+\frac{5}{21}a^2\frac{1}{y^3x^3}\right],\ \sigma_2=\left[\frac{1}{y^5x^2}-\frac{5}{7}a\frac{1}{y^7x})+\frac{21}{5}a^2\frac{1}{y^2x^3}\right],$$

$$\sigma_3=\left[\frac{1}{y^4x^2}\right],\ \sigma_4=\left[\frac{1}{y^6x}-\frac{1}{3}a\frac{1}{yx^3}\right],\ \sigma_5=\left[\frac{1}{y^3x^2}\right],\ \sigma_6=\left[\frac{1}{y^5x}\right],\ \sigma_7=\left[\frac{1}{y^2x^2}\right],\ \sigma_8=\left[\frac{1}{y^4x}\right],$$

$$\sigma_9=\left[\frac{1}{yx^2}\right],\ \sigma_{10}=\left[\frac{1}{y^3x}\right],\ \sigma_{11}=\left[\frac{1}{y^2x}\right],\ \sigma_{12}=\left[\frac{1}{yx}\right]$$

Degrees :

σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}
−32	−29	−26	−25	−23	−22	−20	−19	−17	−16	−13	−10

Basis of V :

$$\mathrm{v}_1=252yx\partial_x-30ax\partial_y+65a^2y^4\partial_x,\ \mathrm{v}_2=63y^2\partial_y+15ax\partial_y+25a^2y^4\partial_x,$$

$$\mathrm{v}_3=7y^2x\partial_x-2ayx\partial_y,\ \mathrm{v}_4=+7y^3\partial_y+5ayx\partial_y,\ \mathrm{v}_5=6yx\partial_y-5ay^5\partial_x,$$

$$\mathrm{v}_6=y^4\partial_y,\ \mathrm{v}_7=y^3x\partial_x,\ \mathrm{v}_8=y^2x\partial_y,\ \mathrm{v}_9=y^5\partial_y,\ \mathrm{v}_{10}=y^4x\partial_x,\ \mathrm{v}_{11}=y^3x\partial_y,$$

$$\mathrm{v}_{12}=y^5x\partial_x,\ \mathrm{v}_{13}=y^4x\partial_y,\ \mathrm{v}_{14}=y^5x\partial_y$$

Degrees :

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}
3	3	6	6	7	9	9	10	12	12	13	15	16	19

Solution space H : $H=\mathrm{Span}\{1,y^5x\}$.

4.4.2 $E_{13}\ :\ x^3+xy^5+ay^8$

$f=x^3+xy^5+ay^8$ (of type $(5,2)$ of degree 15)
Partial derivatives : $f_x=3x^2+y^5$, $f_y=5xy^4+8ay^7$
The standard base of $I=\langle f_x,f_y\rangle_O$: $\{y^9, 5y^4x+8ay^7, 3x^2+y^5\}$
Basis of the local ring $\mathcal{O}_{X,O}/I$ and its degrees :

m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}	m_{11}	m_{12}	m_{13}
1	y	y^2	x	y^3	yx	y^4	y^2x	y^5	y^3x	y^6	y^7	y^8
0	2	4	5	6	7	8	9	10	11	12	14	16

Basis of Σ :

$$\sigma_1=\left[\frac{1}{y^9x}-\frac{1}{3}\frac{1}{y^4x^3}+a(-\frac{8}{5}\frac{1}{y^6x^2}+\frac{8}{15}\frac{1}{yx^4})\right],\ \sigma_2=\left[\frac{1}{y^8x}-\frac{1}{3}\frac{1}{y^3x^3}-\frac{8}{5}a\frac{1}{y^5x^2}\right],$$

$$\sigma_3=\left[\frac{1}{y^7x}-\frac{1}{3}\frac{1}{y^2x^3}\right],\ \sigma_4=\left[\frac{1}{y^4x^2}\right],\ \sigma_5=\left[\frac{1}{y^6x}-\frac{1}{3}\frac{1}{yx^3}\right],\ \sigma_6=\left[\frac{1}{y^3x^2}\right],\ \sigma_7=\left[\frac{1}{y^5x}\right],$$

$$\sigma_8=\left[\frac{1}{y^2x^2}\right],\ \sigma_9=\left[\frac{1}{y^4x}\right],\ \sigma_{10}=\left[\frac{1}{yx^2}\right],\ \sigma_{11}=\left[\frac{1}{y^3x}\right],\ \sigma_{12}=\left[\frac{1}{y^2x}\right],\ \sigma_{13}=\left[\frac{1}{yx}\right]$$

Degrees :

σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}	σ_{13}
−23	−21	−19	−18	−17	−16	−15	−14	−13	−12	−11	−9	−7

Basis of V :

$$v_1=125yx\partial_x+50y^2\partial_y-40ay^4\partial_x+192a^2y^2x\partial_x,\ v_2=20y^4\partial_x+15x\partial_y-156ay^2x\partial_x,$$

$$v_3=5y^2x\partial_x+2y^3\partial_y,\ v_4=yx\partial_y-4ay^3x\partial_x,\ v_5=5y^5\partial_x-24ay^3x\partial_x,$$

$$v_6=5y^3x\partial_x+2y^4\partial_y,\ v_7=y^2x\partial_y,\ v_8=y^6\partial_x,\ v_9=y^5\partial_y,\ v_{10}=y^3x\partial_y,$$

$$v_{11}=y^7\partial_x,\ v_{12}=y^6\partial_y,\ v_{13}=y^8\partial_x,\ v_{14}=y^7\partial_y,\ v_{15}=y^8\partial_y$$

Degrees :

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}
2	3	4	5	5	6	7	7	8	9	9	10	11	12	14

Solution space H : $H=\mathrm{Span}\{1,y^8\}$

4.4.3 $E_{14}\ :\ x^3+y^8+axy^6$

$f=x^3+y^8+axy^6$ (of type $(8,3)$ of degree 24)
Partial derivatives : $f_x=3x^2+ay^6$, $f_y=8y^7+6axy^5$
The standard base of $I=\langle f_x,f_y\rangle_O$: $\{y^9, 4y^7+3ay^5x, 3x^2+ay^6\}$
Basis of the local ring $\mathcal{O}_{X,O}/I$ and its degrees :

m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}	m_{11}	m_{12}	m_{13}	m_{14}
1	y	y^2	x	y^3	yx	y^4	y^2x	y^5	y^3x	y^6	y^4x	y^5x	y^6x
0	3	6	8	9	11	12	14	15	17	18	20	23	26

Basis of Σ :

$$\sigma_1=\left[\frac{1}{y^7x^2}+a(-\frac{3}{4}\frac{1}{y^9x}-\frac{1}{3}\frac{1}{yx^4})+\frac{1}{4}a^2\frac{1}{y^3x^3})\right],\ \sigma_2=\left[\frac{1}{y^6x^2}-\frac{3}{4}a\frac{1}{y^8x}+\frac{1}{4}a^2\frac{1}{y^2x^3}\right],$$

$$\sigma_3=\left[\frac{1}{y^5x^2}\right],\ \sigma_4=\left[\frac{1}{y^7x}-\frac{1}{3}a\frac{1}{yx^3}\right],\ \sigma_5=\left[\frac{1}{y^4x^2}\right],\ \sigma_6=\left[\frac{1}{y^6x}\right],\ \sigma_7=\left[\frac{1}{y^3x^2}\right],\ \sigma_8=\left[\frac{1}{y^5\cdots}\right.$$

$$\sigma_9=\left[\frac{1}{y^2x^2}\right],\ \sigma_{10}=\left[\frac{1}{y^4x}\right],\ \sigma_{11}=\left[\frac{1}{yx^2}\right],\ \sigma_{12}=\left[\frac{1}{y^3x}\right],\ \sigma_{13}=\left[\frac{1}{y^2x}\right],\ \sigma_{14}=\left[\frac{1}{yx}\right]$$

σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}	σ_{13}	σ_{14}
−37	−34	−31	−29	−28	−26	−25	−23	−22	−20	−19	−17	−14	−11

Basis of V :

$$\mathrm{v}_1 = 28yx\partial_x - 3ax\partial_y - 4a^2y^5\partial_x,\ \mathrm{v}_2 = 28y^2\partial_y + 6ax\partial_y + 15a^2y^5\partial_x,$$
$$\mathrm{v}_3 = 4y^2x\partial_x - ayx\partial_y,\ \mathrm{v}_4 = 4y^3\partial_y + 3ayx\partial_y,\ \mathrm{v}_5 = yx\partial_y - ay^6\partial_x,$$
$$\mathrm{v}_6 = y^4\partial_y,\ \mathrm{v}_7 = y^3x\partial_x,\ \mathrm{v}_8 = y^2x\partial_y,\ \mathrm{v}_9 = y^5\partial_y,\ \mathrm{v}_{10} = y^4x\partial_x,\ \mathrm{v}_{11} = y^3x\partial_y,$$
$$\mathrm{v}_{12} = y^6\partial_y,\ \mathrm{v}_{13} = y^5x\partial_x,\ \mathrm{v}_{14} = y^4x\partial_y,\ \mathrm{v}_{15} = y^6x\partial_x,\ \mathrm{v}_{16} = y^5x\partial_y,\ \mathrm{v}_{17} = y^6x\partial_y$$

Degrees :

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}
3	3	6	6	8	9	9	11	12	12	14	15	15	17	18	20	23

Solution space H : $H = \mathrm{Span}\{1, y^6x\}$

4.4.4 Z_{11} : $x^3y + y^5 + axy^4$

$f = x^3y + y^5 + axy^4$ (of type $(4,3)$ of degree 15)
Partial derivatives : $f_x = 3x^2y + ay^4$, $f_y = x^3 + 5y^4 + 4axy^3$
The standard base of $I = \langle f_x, f_y\rangle_O$: $\{y^6, 15y^5 + 11ay^4x, 3yx^2 + ay^4, x^3 + 5y^4 + 4ay^3x\}$
Basis of the local ring $\mathcal{O}_{X,O}/I$ and its degrees :

m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}	m_{11}
1	y	x	y^2	yx	x^2	y^3	y^2x	y^4	y^3x	y^4x
0	3	4	6	7	8	9	10	12	13	16

Basis of Σ :

$$\sigma_1 = \left[\frac{1}{y^5x^2} - 5\frac{1}{yx^5} + a\left(-\frac{11}{15}\frac{1}{y^6x} - \frac{1}{3}\frac{1}{y^2x^4}\right) + \frac{11}{45}a^2\frac{1}{y^3x^3}\right],\ \sigma_2 = \left[\frac{1}{y^4x^2} - \frac{4}{5}a\frac{1}{y^5x} + \frac{4}{15}a^2\frac{1}{y^2x^3}\right],$$
$$\sigma_3 = \left[\frac{1}{y^5x} - 5\frac{1}{yx^4} - \frac{1}{3}a\frac{1}{y^2x^3}\right],\ \sigma_4 = \left[\frac{1}{y^3x^2}\right],\ \sigma_5 = \left[\frac{1}{y^4x}\right],\ \sigma_6 = \left[\frac{1}{yx^3}\right],\ \sigma_7 = \left[\frac{1}{y^2x^2}\right],$$
$$\sigma_8 = \left[\frac{1}{y^3x}\right],\ \sigma_9 = \left[\frac{1}{yx^2}\right],\ \sigma_{10} = \left[\frac{1}{y^2x}\right],\ \sigma_{11} = \left[\frac{1}{yx}\right]$$

σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}
−23	−20	−19	−17	−16	−15	−14	−13	−11	−10	−7

Basis of V :

$$\mathrm{v}_1 = 15yx\partial_x - a(61x^2\partial_x + 48yx\partial_y),\ \mathrm{v}_2 = 15y^2\partial_y + a(108x^2\partial_x + 83yx\partial_y),$$
$$\mathrm{v}_3 = 60x^2\partial_x + 45yx\partial_y + ax^2\partial_y,\ \mathrm{v}_4 = 5y^3\partial_x + 2x^2\partial_y,$$
$$\mathrm{v}_5 = y^3\partial_y,\ \mathrm{v}_6 = y^2x\partial_x,\ \mathrm{v}_7 = y^2x\partial_y,\ \mathrm{v}_8 = y^4\partial_x,\ \mathrm{v}_9 = y^4\partial_y,$$
$$\mathrm{v}_{10} = y^3x\partial_x,\ \mathrm{v}_{11} = y^3x\partial_y,\ \mathrm{v}_{12} = y^4x\partial_x,\ \mathrm{v}_{13} = y^4x\partial_y$$

Degrees :

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}
3	3	4	5	6	6	7	8	9	9	10	12	13

Solution space H : $H = \mathrm{Span}\{1, y^4x\}$

4.4.5 $Z_{12}\,:\,x^3y+xy^4+ay^6$

$f=x^3y+xy^4+ay^6$ (of type $(3,2)$ of degree 11)
Partial derivatives : $f_x=3x^2y+y^4$, $f_y=x^3+4xy^3+6ay^5$
The standard base of $I=\langle f_x,f_y\rangle_O$:

$$\{y^7, 33y^4x-7ay^6, 3yx^2+y^4+2ay^3x, 33x^3+132y^3x-33ay^5-14a^3y^6\}$$

Basis of the local ring $\mathcal{O}_{X,O}/I$ and its degrees :

m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}	m_{11}	m_{12}
1	y	x	y^2	yx	x^2	y^3	y^2x	y^4	y^3x	y^5	y^6
0	2	3	4	5	6	6	7	8	9	10	12

Basis of Σ :

$$\sigma_1=\left[\frac{1}{y^7x}-\frac{1}{3}\frac{1}{y^4x^3}+\frac{4}{3}\frac{1}{yx^5}+a(\frac{6}{11}\frac{1}{y^2x^4}-\frac{18}{11}\frac{1}{y^5x^2})\right],\ \sigma_2=\left[\frac{1}{y^6x}-\frac{1}{3}\frac{1}{y^3x^3}-\frac{3}{2}a\frac{1}{y^4x^2}\right],$$

$$\sigma_3=\left[\frac{1}{y^4x^2}-4\frac{1}{yx^4}\right],\ \sigma_4=\left[\frac{1}{y^5x}-\frac{1}{3}\frac{1}{y^2x^3}\right],\ \sigma_5=\left[\frac{1}{y^3x^2}\right],\ \sigma_6=\left[\frac{1}{y^4x}\right],\ \sigma_7=\left[\frac{1}{yx^3}\right],$$

$$\sigma_8=\left[\frac{1}{y^2x^2}\right],\ \sigma_9=\left[\frac{1}{y^3x}\right],\ \sigma_{10}=\left[\frac{1}{yx^2}\right],\ \sigma_{11}=\left[\frac{1}{y^2x}\right],\ \sigma 12=\left[\frac{1}{yx}\right]$$

Degrees :

σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}
−17	−15	−14	−13	−12	−11	−11	−10	−9	−8	−7	−5

Basis of V :

$$\mathrm{v}_1=1936y^2\partial_x-13068ayx\partial_x+2673a^2x^2\partial_x+4374a^3y^2x\partial_x+1452x\partial_y,$$
$$\mathrm{v}_2=132yx\partial_x+88y^2\partial_y-99ax^2\partial_x-162a^2y^2x\partial_x,\ \mathrm{v}_3=3x^2\partial_x+2yx\partial_y,$$
$$\mathrm{v}_4=4y^3\partial_x+3yx\partial_y-27ay^2x\partial_x,\ \mathrm{v}_5=3y^2x\partial_x+2y^3\partial_y,\ \mathrm{v}_6=8y^3\partial_y-9x^2\partial_y,$$
$$\mathrm{v}_7=y^2x\partial_y,\ \mathrm{v}_8=y^4\partial_x,\ \mathrm{v}_9=y^4\partial_y,\ \mathrm{v}_{10}=y^3x\partial_x,\ \mathrm{v}_{11}=y^3x\partial_y,$$
$$\mathrm{v}_{12}=y^5\partial_x,\ \mathrm{v}_{13}=y^5\partial_y,\ \mathrm{v}_{14}=y^6\partial_x,\ \mathrm{v}_{15}=y^6\partial_y$$

Degrees :

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}
1	2	3	3	4	4	5	5	6	6	7	7	8	9	10

Solution space H : $H=\mathrm{Span}\{1,y^6\}$

4.4.6 $Z_{13}\,:\,x^3y+y^6+axy^5$

$f=x^3y+y^6+axy^5$ (of type $(5,3)$ of degree 18)
Partial derivatives : $f_x=3x^2y+ay^5$, $f_y=x^3+6y^5+5axy^4$
The standard base of $I=\langle f_x,f_y\rangle_O$: $\{y^7, 9y^6+7ay^5x, 3yx^2+ay^5, x^3+6y^5+5ay^4x\}$
Basis of the local ring $\mathcal{O}_{X,O}/I$ and its degrees :

m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}	m_{11}	m_{12}	m_{13}
1	y	x	y^2	yx	y^3	x^2	y^2x	y^4	y^3x	y^5	y^4x	y^5x
0	3	5	6	8	9	10	11	12	14	15	17	20

Basis of Σ :

$$\sigma_1=\left[\frac{1}{y^6x^2}-6\frac{1}{yx^5}+a(-\frac{7}{9}\frac{1}{y^7x}+\frac{7}{27}\frac{1}{y^3x^3}-\frac{1}{3}\frac{1}{y^2x^4})\right],\ \sigma_2=\left[\frac{1}{y^5x^2}-\frac{5}{6}a\frac{1}{y^6x}+\frac{5}{18}a^2\frac{1}{y^2x^3}\right]$$

$$\sigma_3=\left[\frac{1}{y^5x^2}-5a\frac{1}{yx^4}\right],\ \sigma_4=\left[\frac{1}{y^4x^2}\right],\ \sigma_5=\left[\frac{1}{y^5x}\right],\ \sigma_6=\left[\frac{1}{y^3x^2}\right],\ \sigma_7=\left[\frac{1}{yx^3}\right],\ \sigma_8=\left[\frac{1}{y^4x}\right]$$

$$\sigma_9 = \left[\frac{1}{y^2x^2}\right], \sigma_{10} = \left[\frac{1}{y^3x}\right], \sigma_{11} = \left[\frac{1}{yx^2}\right], \sigma_{12} = \left[\frac{1}{y^2x}\right], \sigma_{13} = \left[\frac{1}{yx}\right]$$

Degrees :

σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}	σ_{13}
−28	−25	−23	−22	−20	−19	−18	−17	−16	−14	−13	−11	−8

Basis of V :

$$v_1 = 9yx\partial_x - a(23x^2\partial_x + 15yx\partial_y),\ v_2 = 9y^2\partial_y + a(60x^2\partial_x + 37yx\partial_y),$$
$$v_3 = 45x^2\partial_x + 27yx\partial_y + ax^2\partial_y,\ v_4 = y^3\partial_y,\ v_5 = y^2x\partial_x,\ v_6 = 3y^4\partial_x + x^2\partial_y,$$
$$v_7 = y^2x\partial_y,\ v_8 = y^4\partial_y,\ v_9 = y^3x\partial_x,\ v_{10} = y^5\partial_x,\ v_{11} = y^3x\partial_y,\ v_{12} = y^5\partial_y,$$
$$v_{13} = y^4x\partial_x,\ v_{14} = y^4x\partial_y,\ v_{15} = y^5x\partial_x,\ v_{16} = y^5x\partial_y$$

Degrees :

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}
3	3	5	6	6	7	8	9	9	10	11	12	12	14	15	17

Solution space H : $H = \mathrm{Span}\{1, y^5x\}$

4.4.7 W_{12} : $x^4 + y^5 + ax^2y^3$

$f = x^4 + y^5 + ax^2y^3$ (of type (5, 4) of degree 20)
Partial derivatives : $f_x = 4x^3 + 2axy^3$, $f_y = 5y^4 + 3ax^2y^2$
The standard base of $I = \langle f_x, f_y\rangle_O$: $\{y^6, y^4x, 5y^4 + 3ay^2x^2, 2x^3 + ay^3x\}$
Basis of the local ring $\mathcal{O}_{X,O}/I$ and its degrees :

m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}	m_{11}	m_{12}
1	y	x	y^2	yx	x^2	y^3	y^2x	yx^2	y^3x	y^2x^2	y^3x^2
0	4	5	8	9	10	12	13	14	17	18	22

Basis of Σ :

$$\sigma_1 = \left[\frac{1}{y^4x^3} + a(-\frac{3}{5}\frac{1}{y^6x} - \frac{1}{2}\frac{1}{yx^5})\right], \sigma_2 = \left[\frac{1}{y^3x^3} - \frac{3}{5}a\frac{1}{y^5x}\right], \sigma_3 = \left[\frac{1}{y^4x^2} - \frac{1}{2}a\frac{1}{yx^4}\right],$$

$$\sigma_4 = \left[\frac{1}{y^2x^3}\right], \sigma_5 = \left[\frac{1}{y^3x^2}\right], \sigma_6 = \left[\frac{1}{y^4x}\right], \sigma_7 = \left[\frac{1}{yx^3}\right], \sigma_8 = \left[\frac{1}{y^2x^2}\right], \sigma_9 = \left[\frac{1}{y^3x}\right],$$

$$\sigma_{10} = \left[\frac{1}{yx^2}\right], \sigma_{11} = \left[\frac{1}{y^2x}\right], \sigma_{12} = \left[\frac{1}{yx}\right]$$

Degrees :

σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}
−31	−27	−26	−23	−22	−21	−19	−18	−17	−14	−13	−9

Basis of V :

$$v_1 = 10yx\partial_x - 3ax^2\partial_y,\ v_2 = 10y^2\partial_y + 3ax^2\partial_y,\ v_3 = 3x^2\partial_x + ay^3\partial_x,\ v_4 = 2yx\partial_y - ay^3\partial_x$$
$$v_5 = y^3\partial_y,\ v_6 = y^2x\partial_x,\ v_7 = y^2x\partial_y,\ v_8 = yx^2\partial_x,\ v_9 = yx^2\partial_y,\ v_{10} = y^3x\partial_x,$$
$$v_{11} = y^3x\partial_y,\ v_{12} = y^2x^2\partial_x,\ v_{13} = y^2x^2\partial_y,\ v_{14} = y^3x^2\partial_x,\ v_{15} = y^3x^2\partial_y$$

Degrees :

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}
4	4	5	5	8	8	9	9	10	12	13	13	14	17	18

Solution space H : $H = \mathrm{Span}\{1, y^3x^2\}$

4.4.8 W_{13} : $x^4 + xy^4 + ay^6$

$f = x^4 + xy^4 + ay^6$ (of type (4,3) of degree 16)
Partial derivatives : $f_x = 4x^3 + y^4$, $f_y = 4xy^3 + 6ay^5$
The standard base of $I = \langle f_x, f_y \rangle_O$: $\{y^7, 2y^3x + 3ay^5, 4x^3 + y^4\}$
Basis of the local ring $\mathcal{O}_{X,O}/I$ and its degrees :

m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}	m_{11}	m_{12}	m_{13}
1	y	x	y^2	yx	x^2	y^3	y^2x	yx^2	y^4	y^2x^2	y^5	y^6
0	3	4	6	7	8	9	10	11	12	14	15	18

Basis of Σ :

$$\sigma_1 = \left[\frac{1}{y^7x} - \frac{1}{4}\frac{1}{y^3x^4} + a(-\frac{3}{2}\frac{1}{y^5x^2} + \frac{3}{8}\frac{1}{yx^5})\right], \sigma_2 = \left[\frac{1}{y^6x} - \frac{1}{4}\frac{1}{y^2x^4} - \frac{3}{2}a\frac{1}{y^4x^2}\right], \sigma_3 = \left[\frac{1}{y^3x^3}\right],$$

$$\sigma_4 = \left[\frac{1}{y^5x} - \frac{1}{4}\frac{1}{yx^4}\right], \sigma_5 = \left[\frac{1}{y^2x^3}\right], \sigma_6 = \left[\frac{1}{y^3x^2}\right], \sigma_7 = \left[\frac{1}{y^4x}\right], \sigma_8 = \left[\frac{1}{yx^3}\right],$$

$$\sigma_9 = \left[\frac{1}{y^2x^2}\right], \sigma_{10} = \left[\frac{1}{y^3x}\right], \sigma_{11} = \left[\frac{1}{yx^2}\right], \sigma_{12} = \left[\frac{1}{y^2x}\right], \sigma_{13} = \left[\frac{1}{yx}\right]$$

Degrees :

σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}	σ_{13}
−25	−22	−21	−19	−18	−17	−16	−15	−14	−13	−11	−10	−7

Basis of V :

$$v_1 = 4yx\partial_x + 3y^2\partial_y - 3ay^3\partial_x,\ v_2 = 8x^2\partial_x - 9ay^3\partial_y,\ v_3 = 2yx\partial_y + 3ay^3\partial_y,$$

$$v_4 = 3y^3\partial_x + 4x^2\partial_y,\ v_5 = 3y^3\partial_y + 4y^2x\partial_x,\ v_6 = y^2x\partial_y,\ v_7 = yx^2\partial_x,$$

$$v_8 = yx^2\partial_y,\ v_9 = y^4\partial_x,\ v_{10} = y^4\partial_y,\ v_{11} = y^2x^2\partial_x,\ v_{12} = y^2x^2\partial_y,$$

$$v_{13} = y^5\partial_x,\ v_{14} = y^5\partial_y,\ v_{15} = y^6\partial_x,\ v_{16} = y^6\partial_y$$

Degrees :

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}
3	4	4	5	6	7	7	8	8	9	10	11	11	12	14	15

Solution space H : $H = \mathrm{Span}\{1, y^6\}$

4.4.9 Q_{10} : $x^3 + y^4 + yz^2 + axy^3$

$f = x^3 + y^4 + yz^2 + axy^3$ (of type (8,6,9) of degree 24)
Partial derivatives : $f_x = 3x^2 + ay^3$, $f_y = 4y^3 + z^2 + 3axy^2$, $f_z = 2yz$
The standard base of $I = \langle f_x, f_y \rangle_O$: $\{x^4, 4yx^2 + 3ax^3, 3x^2 + ay^3, zx^2, zy, 12x^2 - az^2 - 3a^2y^2x\}$
Basis of the local ring $\mathcal{O}_{X,O}/I$ and its degrees :

m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}
1	y	x	z	y^2	yx	zx	y^3	y^2x	y^3x
0	6	8	9	12	14	17	18	20	26

Basis of Σ :

$$\sigma_1 = \left[\frac{1}{zy^4x^2} - 4\frac{1}{z^3yx^2} + a(-\frac{3}{4}\frac{1}{zy^5x} - \frac{1}{3}\frac{1}{zyx^4}) + \frac{1}{4}a^2\frac{1}{zy^2x^3}\right], \sigma_2 = \left[\frac{1}{zy^3x^2} - \frac{3}{4}a\frac{1}{zy^4x} - \frac{1}{4}a^2\frac{1}{zyx^3}\right.$$

$$\sigma_3 = \left[\frac{1}{zy^4x} - 4\frac{1}{z^3yx} + \frac{1}{3}a\frac{1}{zyx^3}\right], \sigma_4 = \left[\frac{1}{z^2yx^2}\right], \sigma_5 = \left[\frac{1}{zy^2x^2}\right], \sigma_6 = \left[\frac{1}{zy^3x}\right],$$

$$\sigma_7 = \left[\frac{1}{z^2yx}\right], \sigma_8 = \left[\frac{1}{zyx^2}\right], \sigma_9 = \left[\frac{1}{zy^2x}\right], \sigma_{10} = \left[\frac{1}{zyx}\right]$$

Degrees :

σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}
−49	−43	−41	−40	−37	−35	−32	−31	−29	−23

Basis of V :

$$\mathrm{v}_1 = z\partial_y + 4y^2\partial_z + 3ayx\partial_z,\ \mathrm{v}_2 = 8yx\partial_x - a(4yx\partial_y + 3zx\partial_z),$$
$$\mathrm{v}_3 = 4y^2\partial_y + a(3yx\partial_y + 3zx\partial_z),\ \mathrm{v}_4 = 4yx\partial_y + 6zx\partial_z - 2ay^3\partial_x,$$
$$\mathrm{v}_5 = zx\partial_x,\ \mathrm{v}_6 = 4y^3\partial_z + 3ay^2x\partial_z,\ \mathrm{v}_7 = zx\partial_y + 4y^2x\partial_z,$$
$$\mathrm{v}_8 = y^3\partial_y,\ \mathrm{v}_9 = y^2x\partial_x,\ \mathrm{v}_{10} = y^2x\partial_y,\ \mathrm{v}_{11} = y^3x\partial_z,\ \mathrm{v}_{12} = y^3x\partial_x,\ \mathrm{v}_{13} = y^3x\partial_y$$

Degrees :

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}
3	6	6	8	9	9	11	12	12	14	17	18	20

Solution space H : $H = \mathrm{Span}\{1, y^3x\}$

4.4.10 Q_{11} : $x^3 + y^2z + xz^3 + az^5$

$f = x^3 + y^2z + xz^3 + az^5$ (of type (6, 7, 4) of degree 18)
Partial derivatives : $f_x = 3x^2 + z^3$, $f_y = 2yz$, $f_z = y^2 + 3xz^2 + 5az^4$
The standard base of $I = \langle f_x, f_y\rangle_O$: $\{x^4, yx^2, 9x^3 - 5ay^2x, y^3, zx^3, zy, 3z^2x + y^2 - 15azx^2, 3x^2 + z^3\}$
Basis of the local ring $\mathcal{O}_{X,O}/I$ and its degrees :

m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}	m_{11}
1	z	x	y	z^2	zx	z^3	yx	z^2x	z^4	z^5
0	4	6	7	8	10	12	13	4	16	20

Basis of Σ :

$$\sigma_1 = \left[\frac{1}{z^6yx} + \frac{1}{zy^3x^2} - \frac{1}{3}\frac{1}{z^3yx^3} + a(-\frac{5}{3}\frac{1}{z^4yx^2} + \frac{5}{9}\frac{1}{zyx^4})\right], \sigma_2 = \left[\frac{1}{z^5yx} - \frac{1}{3}\frac{1}{z^2yx^3} - \frac{5}{3}a\frac{1}{z^3yx^2}\right],$$
$$\sigma_3 = \left[\frac{1}{zy^3x} - \frac{1}{3}\frac{1}{z^3yx^2}\right], \sigma_4 = \left[\frac{1}{zy^2x^2}\right], \sigma_5\left[\frac{1}{z^4yx} - \frac{1}{3}\frac{1}{zyx^3}\right], \sigma_6 = \left[\frac{1}{z^2yx^2}\right],$$
$$\sigma_7 = \left[\frac{1}{z^3yx}\right], \sigma_8 = \left[\frac{1}{zy^2x}\right], \sigma_9 = \left[\frac{1}{zyx^2}\right], \sigma_{10} = \left[\frac{1}{z^2yx}\right], \sigma_{11} = \left[\frac{1}{zyx}\right]$$

Degrees :

σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}
−37	−33	−31	−30	−29	−27	−25	−24	−23	−21	−17

Basis of V :

$$\mathrm{v}_1 = 3zx\partial_y + y\partial_z + 5az^3\partial_y,\ \mathrm{v}_2 = 6zx\partial_x + 4z^2\partial_z - 5ayx\partial_y,$$
$$\mathrm{v}_3 = 2yx\partial_y + 2zx\partial_z - 5az^2x\partial_x,\ \mathrm{v}_4 = 6zx\partial_z + 2z^3\partial_x - a(15z^2x\partial_x + 10z^2x\partial_x),$$
$$\mathrm{v}_5 = yx\partial_x,\ \mathrm{v}_6 = 3z^2x\partial_y + 5az^4\partial_y,\ \mathrm{v}_7 = 3z^2x\partial_x + 2z^3\partial_z,\ \mathrm{v}_8 = z^4\partial_y - yx\partial_z,$$
$$\mathrm{v}_9 = z^2x\partial_z,\ \mathrm{v}_{10} = z^4\partial_x,\ \mathrm{v}_{11} = z^4\partial_z,\ \mathrm{v}_{12} = z^5\partial_y,\ \mathrm{v}_{13} = z^5\partial_x,\ \mathrm{v}_{14} = z^5\partial_z$$

Degrees :

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}
3	4	6	6	7	7	8	9	10	10	12	13	14	16

Solution space H : $H = \mathrm{Span}\{1, z^5\}$

4.4.11 Q_{12} : $x^3+y^5+yz^2+axy^4$

$f=x^3+y^5+yz^2+axy^4$ (of type (5,3,6) of degree 15)
Partial derivatives : $f_x=3x^2+ay^4$, $f_y=5y^4+z^2+4axy^3$, $f_z=2yz$
The standard base of $I=\langle f_x,f_y\rangle_O$: $\{x^4, 5yx^2+4ax^3, 3x^2+ay^4, zx^2, zy, 15x^2-az^2-4a^2y^3x\}$
Basis of the local ring $\mathcal{O}_{X,O}/I$ and its degrees :

m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}	m_{11}	m_{12}
1	y	x	z	y^2	yx	y^3	zx	y^2x	y^4	y^3x	y^4x
0	3	5	6	6	8	9	11	11	12	14	17

Basis of Σ :

$$\sigma_1=\left[\frac{1}{zy^5x^2}-5\frac{1}{z^3yx^2}+a(-\frac{4}{5}\frac{1}{zy^6x}-\frac{1}{3}\frac{1}{zyx^4})+\frac{4}{15}a^2\frac{1}{zy^2x^3}\right],\ \sigma_2=\left[\frac{1}{zy^4x^2}-\frac{4}{5}a\frac{1}{zy^5x}+\frac{4}{15}a^2\frac{1}{zyx^3}\right],$$

$$\sigma_3=\left[\frac{1}{zy^5x}-5\frac{1}{z^3yx}-\frac{a}{3}\frac{1}{zyx^3}\right],\ \sigma_4=\left[\frac{1}{zy^3x^2}\right],\ \sigma_5=\left[\frac{1}{z^2yx^2}\right],\ \sigma_6=\left[\frac{1}{zy^4x}\right]\sigma_7=\left[\frac{1}{zy^2x^2}\right],$$

$$\sigma_8=\left[\frac{1}{zy^3x}\right],\ \sigma_9=\left[\frac{1}{z^2yx}\right],\ \sigma_{10}=\left[\frac{1}{zyx^2}\right],\ \sigma 11=\left[\frac{1}{zy^2x}\right],\ \sigma_{12}=\left[\frac{1}{zyx}\right],$$

Degrees :

σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}
−31	−28	−26	−25	−25	−23	−22	−20	−20	−19	−17	−14

Basis of V :

$$\mathrm{v}_1=5yx\partial_x-a(2yx\partial_y+2zx\partial_z),\ \mathrm{v}_2=z\partial_y+5y^3\partial_z+4ay^2x\partial_z,$$
$$\mathrm{v}_3=5y^2\partial_y+a(4yx\partial_y+6zx\partial_z),\ \mathrm{v}_4=3yx\partial_y+6zx\partial_z-2ay^4\partial_x,$$
$$\mathrm{v}_5=y^3\partial_y,\ \mathrm{v}_6=y^2x\partial_x,\ \mathrm{v}_7=zx\partial_x,\ \mathrm{v}_8=5y^4\partial_z+4ay^3x\partial_z,$$
$$\mathrm{v}_9=y^2x\partial_y,\ \mathrm{v}_{10}=zx\partial_y+5y^3x\partial_z,\ \mathrm{v}_{11}=y^4\partial_y,\ \mathrm{v}_{12}=y^3x\partial_x,$$
$$\mathrm{v}_{13}=y^3x\partial_y,\ \mathrm{v}_{14}=y^4x\partial_z,\ \mathrm{v}_{15}=y^4x\partial_x,\ \mathrm{v}_{16}=y^4x\partial_y$$

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}
3	3	3	5	6	6	6	6	8	8	9	9	11	11	12	14

Solution space H : $H=\mathrm{Span}\{1, y^4x\}$

4.4.12 S_{11} : $x^4+y^2z+xz^2+ay^2x^2$

$f=x^4+y^2z+xz^2+ay^2x^2$ (of type (4,5,6) of degree 16)
Partial derivatives : $f_x=4x^3+z^2+2ay^2x$, $f_y=2zy+2ayx^2$, $f_z=y^2+2zx$
The standard base of $I=\langle f_x,f_y\rangle_O$:

$$\{4x^5+5a^2x^6, 4yx^3+5a^2yx^4, 8x^4+5ay^2x^2, y^3-2ayx^3, 2zx+y^2, zy+ayx^2, 4x^3+z^2+2ay^2x\}$$

Basis of the local ring $\mathcal{O}_{X,O}/I$ and its degrees :

m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}	m_{11}
1	x	y	z	x^2	yx	y^2	x^3	yx^2	y^2x	y^2x^2
0	4	5	6	8	9	10	12	13	14	18

Basis of Σ :

$$\sigma_1=\left[\frac{1}{zy^3x^3}+2\frac{1}{z^4yx}-\frac{1}{2}\frac{1}{z^2yx^4}+a(-\frac{1}{z^2y^3x}+\frac{1}{2}\frac{1}{z^3yx^2}-\frac{5}{8}\frac{1}{zyx^5})\right],$$

$$\sigma_2=\left[\frac{1}{zy^3x^2}-\frac{1}{2}\frac{1}{z^2yx^3}-\frac{1}{2}a\frac{1}{zyx^4}\right],\ \sigma_3=\left[\frac{1}{zy^2x^3}-a\frac{1}{z^2y^2x}\right],\ \sigma_4=\left[\frac{1}{z^3yx}-\frac{1}{4}\frac{1}{zyx^4}\right],$$

$$f\sigma_5=\left[-\frac{1}{zy^3x}+\frac{1}{2}\frac{1}{z^2yx^2}\right],\ \sigma_6=\left[\frac{1}{zy^2x^2}\right],\ \sigma_7=\left[\frac{1}{zyx^3}\right],\ \sigma_8=\left[\frac{1}{z^2yx}\right],$$

$$\sigma_9 = \left[\frac{1}{zy^2x}\right], \sigma_{10} = \left[\frac{1}{zyx^2}\right], \sigma_{11} = \left[\frac{1}{zyx}\right]$$

Degrees :

σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}
−33	−29	−28	−27	−25	−24	−23	−21	−20	−19	−15

Basis of V :

$$v_1 = 64x^2\partial_x + 32y^2\partial_z + 144ax^3\partial_z - 65a^2y^2x\partial_z,$$
$$v_2 = 16yx\partial_y - 16y^2\partial_z - 48ax^3\partial_z + 15a^2y^2x\partial_z,\ v_3 = y^2\partial_y,$$
$$v_4 = yx\partial_x,\ v_5 = 4y^2\partial_x + 48x^3\partial_z - 21ay^2x\partial_z,\ v_6 = 2x^3\partial_y + yx^2\partial_z,$$
$$v_7 = 2x^3\partial_x + y^2x\partial_z,\ v_8 = yx^2\partial_y - y^2x\partial_z,\ v_9 = yx^2\partial_x,\ v_{10} = y^2x\partial_y,$$
$$v_{11} = y^2x\partial_x,\ v_{12} = y^2x^2\partial_z,\ v_{13} = y^2x^2\partial_y,\ v_{14} = y^2x^2\partial_x$$

Degrees :

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}
4	4	5	5	6	7	8	8	9	9	10	12	13	14

Solution space H : $H = \mathrm{Span}\{1, y^2x^2\}$

4.4.13 S_{12} : $x^2y + y^2z + xz^3 + az^5$

$f = x^2y + y^2z + xz^3 + az^5$ (of type $(4,5,3)$ of degree 13)
Partial derivatives : $f_x = 2xy + z^3$, $f_y = x^2 + 2yz$, $f_z = y^2 + 3xz^2 + 5az^4$
The standard base of $I = \langle f_x, f_y\rangle_O$:

$$\{x^4, yx^3, y^2x, 39yx^2 - 20ay^3, 13yx^2 - 10azx^3, x^2 + 2zy, 3z^2x + y^2 + 5ax^3, 2yx + z^3\}$$

Basis of the local ring $\mathcal{O}_{X,O}/I$ and its degrees :

m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}	m_{11}	m_{12}
1	z	x	y	z^2	zx	x^2	z^3	z^2x	zx^2	z^4	z^5
0	3	4	5	6	7	8	9	10	11	12	15

Basis of Σ :

$$\sigma_1 = \left[\frac{1}{z^6yx} + \frac{3}{2}\frac{1}{zy^4x} - \frac{1}{2}\frac{1}{z^3y^2x^2} + \frac{1}{z^2yx^4} + a\left(-\frac{5}{13}\frac{1}{z^2y^3x} - \frac{20}{13}\frac{1}{z^4yx^2} + \frac{10}{13}\frac{1}{zy^2x^3}\right)\right],$$

$$\sigma_2 = \left[\frac{1}{z^5yx} - \frac{1}{2}\frac{1}{z^2y^2x^2} + \frac{1}{zyx^4} - \frac{5}{3}a\frac{1}{z^3yx^2}\right], \sigma_3 = \left[\frac{1}{z^2yx^3} - \frac{1}{2}\frac{1}{z^3y^2x}\right], \sigma_4 = \left[\frac{1}{z^3yx^2} - 3\frac{1}{zy^3x}\right],$$

$$\sigma_5 = \left[\frac{1}{z^4yx} - \frac{1}{2}\frac{1}{zy^2x^2}\right], \sigma_6 = \left[\frac{1}{zyx^3} - \frac{1}{2}\frac{1}{z^2y^2x}\right], \sigma_7 = \left[\frac{1}{z^2yx^2}\right], \sigma_8 = \left[\frac{1}{z^3yx}\right],$$

$$\sigma_9 = \left[\frac{1}{zy^2x}\right], \sigma_{10} = \left[\frac{1}{zyx^2}\right], \sigma_{11} = \left[\frac{1}{z^2yx}\right], \sigma_{12} = \left[\frac{1}{zyx}\right]$$

Degrees :

σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}
−27	−24	−23	−22	−21	−20	−19	−18	−17	−16	−15	−12

Basis of V :

$$v_1 = 39z^2\partial_x + 39zx\partial_y + 26y\partial_z + 130az^3\partial_y + 200a^2zx^2\partial_y,$$
$$v_2 = 104zx\partial_x - 65x^2\partial_y + 78z^2\partial_z + 260az^2x\partial_y + 400a^2z^4\partial_y,$$
$$v_3 = 13z^3\partial_y + 26zx\partial_z + 80azx^2\partial_y,\ v_4 = 13x^2\partial_x - 13z^3\partial_y - 30azx^2\partial_y,$$
$$v_5 = 78z^2x\partial_y + 13x^2\partial_z + 110az^4\partial_y,\ v_6 = 26z^3\partial_x - 13x^2\partial_z + 50az^4\partial_y,$$
$$v_7 = 3zx^2\partial_y - 2z^3\partial_z,\ v_8 = 2z^2x\partial_x + zx^2\partial_y,\ v_9 = z^4\partial_y + 2z^2x\partial_z,\ v_{10} = zx^2\partial_x - z^4\partial_y,$$
$$v_{11} = zx^2\partial_z,\ v_{12} = z^4\partial_x,\ v_{13} = z^4\partial_z,\ v_{14} = z^5\partial_y,\ v_{15} = z^5\partial_x,\ v_{16} = z^5\partial_z$$

Degrees :

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}
2	3	4	4	5	5	6	6	7	7	8	8	8	10	11	12

Solution space H : $H = \mathrm{Span}\{1, z^5\}$

4.4.14 U_{12} : $x^3+y^3+z^4+axyz^2$

$f = x^3+y^3+z^4+axyz^2$ (of type $(4,4,3)$ of degree 12)
Partial derivatives : $f_x = 3x^2+ayz^2$, $f_y = 3y^2+axz^2$, $f_z = 4z^3+2axyz$
The standard base of $I = \langle f_x, f_y\rangle_O$: $\{x^4, yx^2, y^2x, x^3-y^3, zx^2, zy^2, 3y^2+az^2x, 3x^2+az^2y, 2z^3+azyx\}$
Basis of the local ring $\mathcal{O}_{X,O}/I$ and its degrees :

m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}	m_{11}	m_{12}
1	z	y	x	z^2	zy	zx	yx	z^2y	z^2x	zyx	z^2yx
0	3	4	4	6	7	7	8	10	10	11	14

Basis of Σ :

$$\sigma_1 = \left[\frac{1}{z^3y^2x^2} - \frac{1}{3}a\frac{1}{zyx^4} + a(-\frac{1}{2}\frac{1}{z^5yx} - \frac{1}{3}\frac{1}{zy^4x})\right], \sigma_2 = \left[\frac{1}{z^2y^2x^2} - \frac{1}{2}a\frac{1}{z^4yx}\right],$$

$$\sigma_3 = \left[\frac{1}{z^3y^2x} - \frac{1}{3}a\frac{1}{zyx^3}\right], \sigma_4 = \left[\frac{1}{z^3yx^2} - \frac{1}{3}a\frac{1}{zy^3x}\right], \sigma_5 = \left[\frac{1}{zy^2x^2}\right], \sigma_6 = \left[\frac{1}{z^2yx^2}\right],$$

$$\sigma_7 = \left[\frac{1}{z^2y^2x}\right], \sigma_8 = \left[\frac{1}{z^3yx}\right], \sigma_9 = \left[\frac{1}{zyx^2}\right], \sigma_{10} = \left[\frac{1}{zy^2x}\right], \sigma_{11} = \left[\frac{1}{z^2yx}\right], \sigma_{12} = \left[\frac{1}{zyx}\right]$$

Degrees :

σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}
−25	−22	−21	−21	−19	−18	−18	−17	−15	−15	−14	−11

Basis of V :

$$\mathrm{v}_1 = 6zx\partial_x - ayx\partial_z,\ \mathrm{v}_2 = 6zy\partial_y - ayx\partial_z,\ \mathrm{v}_3 = -3z^2\partial_z - ayx\partial_z,\ \mathrm{v}_4 = 6yx\partial_x - az^2x\partial_y,$$

$$\mathrm{v}_5 = -3zx\partial_z + az^2y\partial_x,\ \mathrm{v}_6 = 3zy\partial_z - az^2x\partial_y,\ \mathrm{v}_7 = -6yx\partial_y + az^2y\partial_x,$$

$$\mathrm{v}_8 = z^2y\partial_y,\ \mathrm{v}_9 = z^2x\partial_x,\ \mathrm{v}_{10} = z^2y\partial_z,\ \mathrm{v}_{11} = z^2x\partial_z,\ \mathrm{v}_{12} = zyx\partial_y,\ \mathrm{v}_{13} = zyx\partial_x,$$

$$\mathrm{v}_{14} = zyx\partial_z,\ \mathrm{v}_{15} = z^2yx\partial_y,\ \mathrm{v}_{16} = z^2yx\partial_x,\ \mathrm{v}_{17} = z^2yx\partial_z$$

Degrees :

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}
3	3	3	4	4	4	4	6	6	7	7	7	7	8	10	10	11

Solution space H : $H = \mathrm{Span}\{1, z^2yx\}$

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