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Surgery along a projective plane in a 4-manifold and D_4-singularity

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1 Introduction

The Price surgery has been defined in [P, KSTY, Y3] as a cut and paste of a 4-manifold N_2 in the 4-sphere S^4 and also in general 4-manifolds, where N_2 is defined as a total space of a non-orientable D^2-bundle over a projective plane with normal Euler number 2 (see [M1, M2, L1, Y1]). It may be expected to make a *fake pair* of 4-manifolds, which means a pair that are homotopy equivalent but non-diffeomorphic to each other, but such a trial seems not to be succeeded yet except the non-orientable example [A1, A2] (see also [KSTY]: Gluck surgery ([Gl]) is realized by Price surgery).

The 4-manifold N_2 is represented by the framed link (see [Ki, GoS]) in Figure 1(1). The boundary ∂N_2 is homeomorphic to the quaternion space Q, which is the quotient space of the unit sphere S^3 of the quaternion field $\mathbf{H} \cong \mathbf{R}^4$ by the quaternion group of order 8. This space Q is also homeomorphic to the linking 3-manifold of D_4-singularity: $S^5 \cap \{f^{-1}(0)\}$, where

$$\begin{array}{cccc} f: & \mathbf{C}^3 & \to & \mathbf{C} \\ & (x,y,z) & \mapsto & x^2+y^3+z^3, \end{array}$$

and we regard S^5 as the unit sphere (the boundary of the unit disk D^6) in $\mathbf{C}^3 \cong \mathbf{R}^6$. Throughout the paper, by the notation D_4 we denote the compact 4-manifold obtained from $D^6 \cap \{f^{-1}(0)\}$ by resolve the singularity minimally, which is represented by the framed link in Figure 1(2).

[0]2000 *Mathematics Subject Classification* : Primary 57Q45, Secondary 57R65, 57N13. Title of the author's talk on 31 May was slightly different from that of this report.

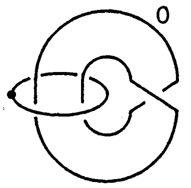

Figure 1(1) : N_2

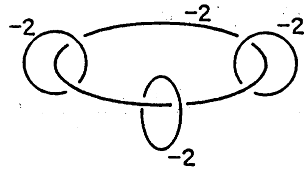

Figure 1(2) : D_4

The boundaries of the two 4-manifolds N_2 and D_4 are homeomorphic to each other and also to Q, thus we can define

Operation : "Cut D_4 off and paste N_2 on" a 4-manifold,

but the resulting 4-manifold is not well-defined because of the ambiguity of the gluing map (self-homeomorphisms on Q). Thus, for a given D_4 in an original 4-manifold M, we study the set $\boldsymbol{\Omega}_M(D_4)$ (consisting of at most three elements, see Section 2) of diffeomorphic class of resulting 4-manifolds.

This operation changes some topological invariants of the ambient 4-manifold: it decreases the Euler characteristic number χ by 4, the negative second Betti number β_2^- by 4 and do not change the positive second Betti number β_2^+, thus increases the signature σ of the 4-manifold by 4.

In this paper, we will report two lemmas related to the operation. One is Lemma 3.1 in § 3, which says that a certain operation consisting of four blowing up's and the operation above is reduced to Price surgery. The other is Lemma 4.1 in § 4 on the resulting manifolds of the operation on the simple elliptic surfaces. Before stating the results, in the next section, we will recall some facts on Price surgery in general 4-manifolds. In §5, we will show some key lemmas by "relative Kirby calculus" (see Section 5.5 in [GoS]), but we do not give the complete proof.

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この研究のきっかけは 1999 年夏 和歌山での研究集会「いろいろなカテゴリーでの多様体のトポロジーと特異点」で奥間智弘氏に初歩的な質問をしてみたことでした。また, 今回の講演後には何人かの初対面の先生方に複素曲面や特異点の構成について教えていただきました。この場をお借りして感謝致します。ありがとうございました。

2 Price surgery

We recall notations and facts on Price surgery from [KSTY, Y3].

(1) We denote by N_2 the total space of a non-orientable D^2-bundle over a projective plane with normal Euler number 2, which is a compact oriented 4-manifold with a boundary, and which is described by the Kirby diagram in Figure 1(1). Note that N_2 has a handlebody decomposition with one 0-handle, one 1-handle and one 2-handle.

(2) The boundary ∂N_2 is diffeomorphic to the quaternion space Q, which admits a Seifert fibered structure whose Seifert invariants in the sense of [O, §5.2] are given by $\{-1; (o_1, 0);\ (2,1), (2,1),\ (2,1)\}$. We call the three singular fibers c_{-1}, c_0, c_1.

(3) In [P], Price has investigated the self-diffeomorphisms of the quaternion space Q and has shown that the mapping class group $\mathcal{M}(Q)$ (the group of isotopy classes of orientation preserving self-diffeomorphisms) is isomorphic to $\mathcal{S}_3$, the symmetric group on three letters $\{-1, 0, 1\}$. For each element σ in $\mathcal{S}_3$, there is a self-diffeomorphism f_σ of Q which preserves the Seifert fibered structure and satisfies $f_\sigma(c_i) = c_{\sigma(i)}$. Each map f_σ represents the class of $\mathcal{M}(Q)$.

(4) Price has also shown that there is a self-diffeomorphism g (g_1 in [P, p.116]) of $Q = \partial N_2$ whose order is two in $\mathcal{M}(Q)$ and that can extend over N_2 as a self-diffeomorphism. (In fact, g is a bundle isomorphism "$-$" : $N_2 \to N_2$ which maps each vector $\vec{v}$ to $-\vec{v}$.) Thus, for a given oriented 4-manifold E whose boundary is $-Q$, we have at most only three 4-manifolds up to diffeomorphism $E \cup_{i\circ\varphi} N_2$ obtained by gluing N_2 to E along the boundary. where we use the compositions of a fixed orientation reversing map i from ∂N_2 to ∂E and an orientation preserving self-diffeomorphism φ on Q as the gluing map. The three 4-manifolds correspond to the classes of φ in the right coset $\mathcal{M}(Q)/\{1, g\}$, which consists of three elements.

3 Equivalence of two operations

Let M be a closed oriented 4-manifold and K a smoothly embedded 2-sphere in M whose normal bundle is trivial. We define two operations **A** and **B** on M along K.

Operation A: Taking a pairwise connected sum of (M, K) with the (positive) standard projective plane (S^4, P_0) (see [PR], [L1], [Y1]), we have an embedded projective plane $(M, K\sharp P_0)$ in M whose normal Euler number 2. The tubular neighborhood $N(K\sharp P_0)$ is diffeomorphic to N_2. Let $\mathbf{\Pi}_M(K\sharp P_0)$ be the set of diffeomorphic class of 4-manifolds obtained by pasting N_2 to the exterior $M\backslash \text{int}N(K\sharp P_0)$ along the boundary. The original manifold M itself and the Gluck surgery $\Sigma_M(K)$ of M along K, by Theorem 4.1 in [KSTY], are contained in the set $\mathbf{\Pi}_M(K\sharp P_0)$. By (4) in Section 2, $\mathbf{\Pi}_M(K\sharp P_0)$ consists of at most three elements.

Operation B: This operation consists of five steps, see Figure 2: (1) Blow up at a point in K. (2) Blow up at the intersection point of the proper lift of K and the exceptional curve. (3) Blow up at a point on the newest exceptional curves. (4) Blow up at a point on the newest exceptional curves again. After this step, we have a D_4 in the ambient 4-manifold $M\sharp 4\overline{\mathbf{C}P^2}$. (5) Do the operation "Cut D_4 off and paste N_2 on" the 4-manifold. By $\mathbf{\Omega}_M(D_4(K))$, we denote the set of the of diffeomorphic class of the resulting 4-manifolds.

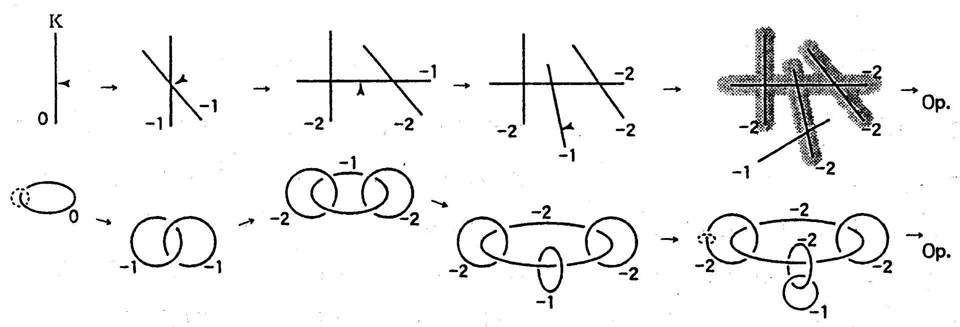

Figure 2

Lemma 3.1 *Two operations* **A** *and* **B** *along* K *on* M *are equivalent, i.e., it holds that* $\mathbf{\Pi}_M(K\sharp P_0) = \mathbf{\Omega}_M(D_4(K))$ *as sets.*

4 Operation on elliptic surfaces

Let $E(n)$ be the simply connected elliptic surface (with section) whose Euler characteristic is $12n$, ($E(1) \cong \mathbf{C}P^2 \sharp 9\overline{\mathbf{C}P^2}$. $E(n)$ is the fiber sum of n copies of $E(1)$. $E(2) \cong$ "the $K3$ surface", ...). In [BGo], using a method "relative Kirby diagram" (see Section 5.5 in [GoS]), a decomposition of $E(n)$ as a union of $n+1$ pieces $N_n \cup W_n \cup W_{n-1} \cup \cdots W_1$ has been shown, where N_n is the *nuclei* of $E(n)$ ([Go]) and W_1 is the E_8-plumbing. Each $W_j (j \geq 2)$ is a cobordism represented by the relative Kirby diagram in Figure 3 (modified from Figure 27 in [BGo]), which clearly contains one E_8-plumbing. An E_8-plumbing contains an obvious D_4. Thus we can do the operation "Cut D_4 off and paste N_2 on" $E(n)$ at most n times. To study the resulting 4-manifolds, we do the operation on W_j. For W_1, see Lemma 5.2.

Lemma 4.1 *The resulting 4-manifold of the operation "Cut D_4 off and paste N_2 on" W_j ($j \geq 2$) does not depend on the gluing map of ∂N_2 and is diffeomorphic to $\mathcal{W}_j \sharp 4\overline{\mathbf{C}P^2}$, where $\mathcal{W}_j$ is the 4-manifold represented by the relative Kirby diagram in Figure 4.*

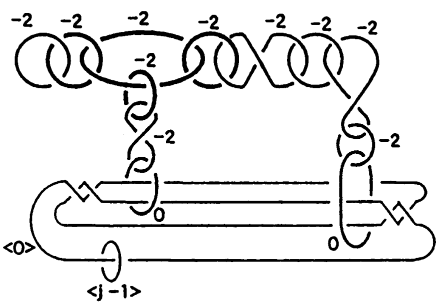

Figure 3 : W_j

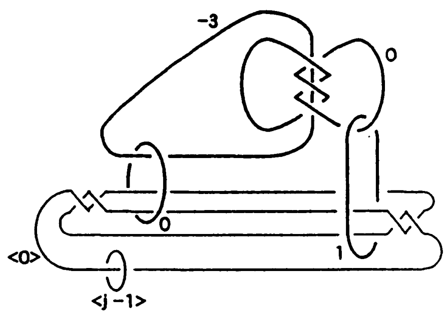

Figure 4 : $\mathcal{W}_j$

Note that W_j is, thus $\mathcal{W}_j$ is also a cobordism from the Seifert homology 3-sphere $-\Sigma(2,3,6(j-1)-1)$ to $\Sigma(2,3,6j-1)$ for $j \geq 2$. We conjecture that all the resulting 4-manifolds, the (non-trivial) union of possible W_j's and $\mathcal{W}_j$'s capped by N_n and their "logarithmic transformation" (as a 4-manifold, not as a complex surface) in N_n are all diffeomorphic to $\beta_2^+(\mathbf{C}P^2) \sharp \beta_2^-(\overline{\mathbf{C}P^2})$.

5 Key of the proof

We show some key lemmas for Lemma 3.1 and give a proof of Lemma 4.1. They are shown by (ordinary) Kirby calculus and relative Kirby calculus (see Section 5.5 in [GoS]).

Lemma 5.1 *See the Kirby calculus from the diagram (A) to (B) of 3-manifolds in Figure 5. It corresponds to a homeomorphism φ from the boundary ∂D_4 of D_4 to ∂N_2. Calculating the curves c_i's with 0-framing in (A) during the process of the Kirby calculus, we get the curves c_i's with framings $(\cdot)$ in (B). They are $\varphi(c_i)$'s in ∂N_2. Thus (under some conditions) the local change from (A) to (B) in a Kirby diagram of a 4-manifold M corresponds to (one of) the operation "Cut D_4 off and paste N_2 on" M.*

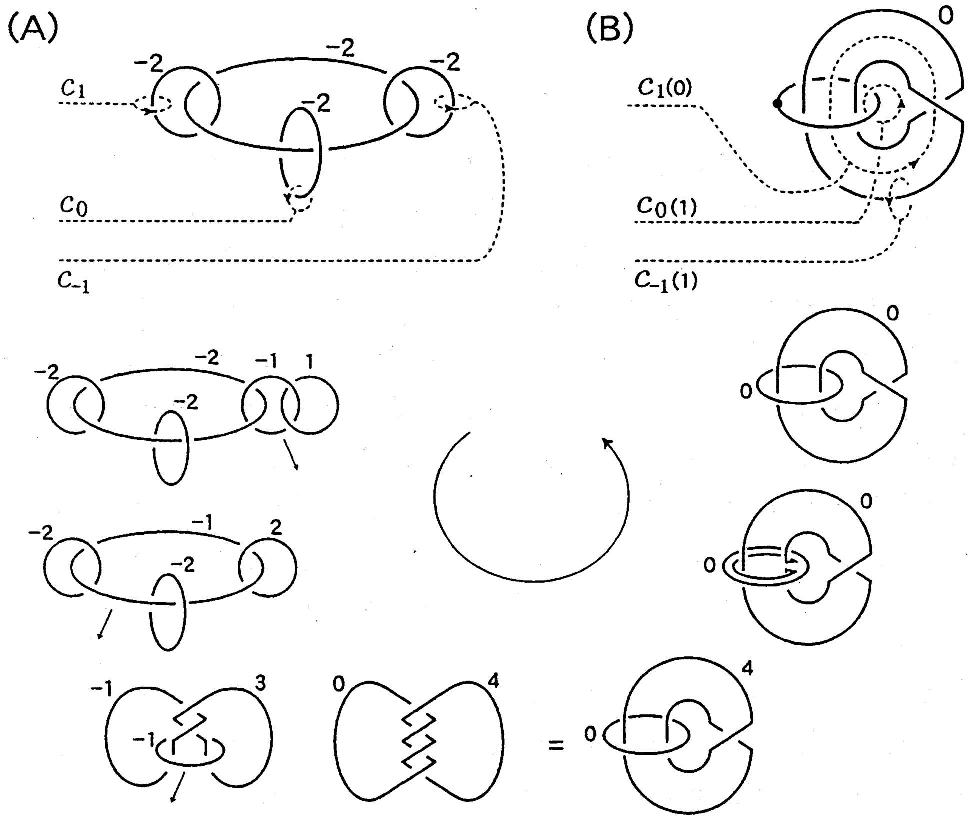

Figure 5

Of course, another Kirby calculus from the diagram (A) to (B) corresponds to another homeomorphism from ∂D_4 to ∂N_2. To prove Lemma 4.1 completely, we need every (six or three) calculus from the diagram (A) to (B) for each element of the mapping class group $\mathcal{M}(Q)$ of order six, but in this paper, we omit the other calculus.

Now we use Lemma 5.1 to study the resulting 4-manifold of the operation "Cut D_4 off and paste N_2 on" the obvious D_4 in the E_8-plumbing W_1.

Lemma 5.2 *The resulting 4-manifold is diffeomorphic to $\mathcal{W}_1\sharp 3\overline{\mathbb{C}P^2}$, where $\mathcal{W}_1$ is the 4-manifold represented by the final Kirby diagram (−1-framed left-hand trefoil) in Figure 6.*

Proof. See the Kirby calculus in Figure 6. □

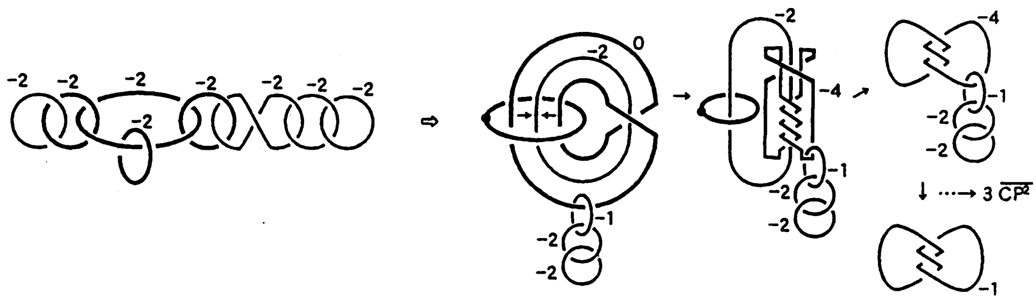

Figure 6

Lemma 4.1 is shown by application of such method.

Note that the action of $\mathcal{M}(Q)$ ($\cong \mathcal{S}_3$) on ∂D_4 is obvious. Thus we can calculate every resulting 4-manifold of the operation on D_4 in the E_8-plumbing for each choice of the gluing map in $\mathcal{M}(Q)$. For a smoothly embedded 2-sphere K in S^4, we can also study the resulting 4-manifold of the operation cut the D_4 and paste an exterior $-X(P_0 \sharp K)$ of a projective plane $P_0 \sharp K$ in S^4 instead of N_2 ($N_2 \cong -X(P_0)$, see [PR, P, L1, L2, Y1, Y2]) by the method "circle with a dot and with a symbol K" in Kirby diagrams introduced in Appendix of [KSTY]. They are all diffeomorphic to $\mathcal{W}_1 \sharp 3\overline{CP^2}$. Note that the Gluck surgery $\Sigma(K)$ along any K in S^4 satisfies that $\Sigma(K) \sharp \overline{CP^2} \cong \overline{CP^2}$.

Outline of the proof of Lemma 3.1: See the Kirby calculus in Figure 2 again. It describes the process of operation **B** near the 2-sphere K, but we have not done the final step yet. Doing the change in Lemma 5.1 to the final diagram, we finish the operation **B** and get the first diagram in Figure 7 (The dotted circle corresponds to a meridian to K in M. The thin circle corresponds to the boundary of a co-core of the 2-handle h. Once ignore them). The diagram describes a 4-manifold obtained by attaching a 2-handle h to N_2. All we have to do is to verify that $(M \backslash \text{int} N(K)) \cup h^\perp \cong M \backslash \text{int} N(K \sharp P_0)$, where we use the notation $h^\perp$ for the piece h since we switch the core and the co-core. See Figure 4(1) and the proof of Theorem 4.1 in [KSTY] for the goal.

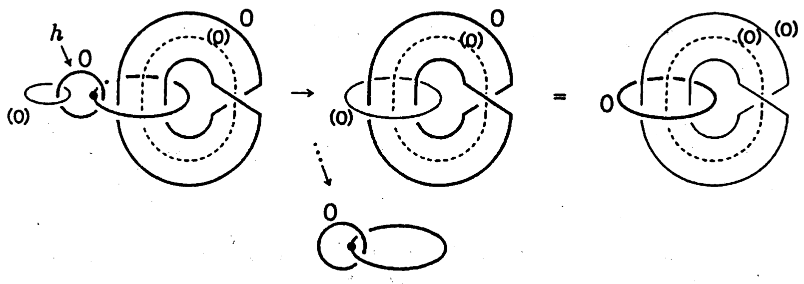

Figure 7

By the calculus in Figure 7, we have the attachinig circle of $h^{\perp}$ in $\partial(M\backslash \mathrm{int}N(K)) \cong S^1 \times S^2$ and the framing: it is the thin circle in the diagram. (If one care orientation of the diagram, it would be better take the mirror image.) We have the lemma. □

References

[A1] S. Akbulut, *On fake* $S^3 \tilde{\times} S^1 \sharp S^2 \times S^2$, in "Combinatorial methods in topology and algebraic geometry", Contemporary Math. **44**, Amer. Math. Soc., 1985, pp.281–286.

[A2] ________, *Constructing a fake* 4*-manifold by Gluck construction to a standard* 4*-manifold*, Topology **27** (1988), 239–243.

[BGo] Ž. Bižaca &R. E. Gompf, *Elliptic surfaces and some simple exotic* $\mathbf{R}^4$*'s*, J. Diff. Geom. **43** (1996), 458–504.

[Gl] H. Gluck, *The embedding of two-spheres in the four-sphere*, Trans. Amer. Math. Soc. **104** (1962), 308–333.

[Go] R. E. Gompf, *Nuclei of elliptic surfaces*, Topology **30** (1991), 479–511.

[GoS] R. E. Gompf and A. I. Stipsicz, *4-Manifolds and Kirby Calculus*, Graduate Studies in Math., Vol.20, Amer. Math. Soc., (1999).

[KSTY] A. Katanaga, O. Saeki, M. Teragaito and Y. Yamada, *Gluck Surgery along a 2-Sphere in a 4-Manifold is Realized by Surgery along a Projective Plane*, Michigan Math. J. **46** No.3 (1999), 555–571.

[Ki] R. C. Kirby, *The topology of* 4*-manifolds*, Lecture Notes in Math., vol.1374, Springer, 1989.

[L1] T. Lawson, *Splitting* S^4 *on* $\mathbf{R}P^2$ *via the branched cover of* $\mathbf{C}P^2$ *over* S^4, Proc. Amer. Math. Soc. **86** (1982), 328–330.

[L2] ________, *Detecting the standard embedding of* $\mathbb{R}P^2$ *in* S^4, Math. Ann. **267** (1984), 439–448.

[M1] W. S. Massey, *Proof of a conjecture of Whitney*, Pacific J. Math. **31** (1969), 143–156.

[M2] ________, *Imbeddings of projective planes and related manifolds in spheres*, Indiana Univ. Math. J. **23** (1973), 791–812.

[O] P. Orlik, *Seifert manifolds*, Lecture Notes in Math., vol.291, Springer, 1972.

[P] T. Price, *Homeomorphisms of quaternion space and projective planes in four space*, J. Austral. Math. Soc. **23** (1977), 112–128.

[PR] T. Price and D. Roseman, *Embeddings of the projective plane in four space*, preprint, 1975.

[Y1] Y. Yamada, *Decomposition of* S^4 *as a twisted double of a certain manifold*, Tokyo J. Math. **20** (1997), 23–33.

[Y2] ________, *Decomposition of the four sphere as a union of* $\mathbb{R}P^2$*-knot exteriors*, Proceedings of Applied Mathematics Workshop vol.8, (G. T. Jin and K. H. Ko eds.) The fifth Korea-Japan School of Knots and Links, 1997, Center for Applied Math., KAIST, Taejon, Korea (1998), 345–351.

[Y3] ________, *Surgery along a band sum of a projective plane and a 2-sphere in 4-manifolds*, Preprint (2001).

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