数理解析研究所講究録 1234 巻 2001 年 179-194

Strichartz inequality and smoothing property for Schrödinger equations with potential superquadratic at infinity *

Kenji Yajima[1] and Guoping Zhang[2]
Department of Mathematical Sciences, University of Tokyo
3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan

1 Introduction, Theorems

In this talk we are concerned with Strichartz inequality and the local smoothing property for Schrödinger equations $i\partial_t u = -(1/2)\Delta u + V(x)u$ on $\mathbb{R}^n$ when the potential $V(x)$ grows at infinity super-quadratically, $V(x) \geq C\langle x\rangle^{2+\varepsilon}$, $\varepsilon > 0$.

1.1 Free Schrödinger equations

We begin with briefly reviewing the results for the free Schrödinger equation:

$$i\frac{\partial u}{\partial t} = -(1/2)\Delta u, \quad x \in \mathbb{R}^n, \ t \in \mathbb{R}; \quad u(0,x) = u_0(x), \quad x \in \mathbb{R}^n. \qquad (1.1)$$

It has been long known that, although the solution of (1.1) is given by $u(t,x) = U(t)u_0$ in terms of the unitary group $U(t) = e^{-itH_0}$ and $U(t)$ is

*Partly supported by the Grant-in-Aid for Scientific Research, The Ministry of Education, Science, Sports and Culture, Japan Grant Nr. 11304006
[1]Partly supported by the Grant-in-Aid for Scientific Research, The Ministry of Education, Science, Sports and Culture, Japan Grant Nr. 11304006
[2]Partly supported by the TonenGeneral International Scholarship Foundation

an isomorphism of $L^2(\mathbb{R}^n)$for every t, any solution $u(t,x)$, or the trajectory $u(t,\cdot) = U(t)u_0$ of the group, belongs to a proper subspace $X \cap L^2(\mathbb{R}^n)$ of $L^2(\mathbb{R}^n)$ for almost all t. We call this remarkable property the smoothing property of the equation. The property is specifically represented by the following two kinds of inequalities which have many applications, e.g. to nonlinear Schrödinger equations ([K3], [KPV]) and to the convergence problem ([V]).

(1) Strichartz ineqaulity: Let $2 \le p, \theta$ be such that $\dfrac{2}{\theta} = n\left(\dfrac{1}{2} - \dfrac{1}{p}\right)$ and $p \neq \infty$ if $n = 2$. Then, there exists a constant $C > 0$ such that

$$\left(\int_0^\infty \|e^{-itH_0}u_0\|_p^\theta dt\right)^{\frac{1}{\theta}} \le C\|u_0\|_2, \quad u \in L^2(\mathbb{R}^n). \tag{1.2}$$

(2) Local smoothing property: For any $T > 0$ and $\Psi \in C_0^\infty(\mathbb{R}^n)$, there exists a constant $C > 0$ such that

$$\left(\int_0^T \|\Psi(x)\langle D\rangle^{\frac{1}{2}} e^{-itH_0}u_0\|_2^2 dt\right)^{\frac{1}{2}} \le C\|u_0\|, \quad u \in L^2(\mathbb{R}^n), \tag{1.3}$$

where T can be set $T = \infty$ if $n \ge 3$. Here and hereafter, $\langle A\rangle = (1 + |A|^2)^{\frac{1}{2}}$ for a self-adjoint operator A and $D = (D_1, \dots, D_n)$, $D_j = -i\partial/\partial x_j$.

The smoothing property of Schrödinger equations was first observed by Kato [K1] in a form slightly different from (1.2): If $n \ge 3$ and $A \in L^{n-\varepsilon} \cap L^{n+\varepsilon}(\mathbb{R}^n)$, $\varepsilon > 0$, then

$$\int_0^\infty \|Ae^{-itH_0}u_0\|_2^2 dt \le C\|u_0\|_2^2, \quad u_0 \in L^2(\mathbb{R}^n).$$

The estimate (1.2) was subsequently obtained by Strichartz [St] for special p and θ and generalized to the form as it is by several authors, we mention [GV], [Y1] among earlier works, and [KT] who recently proved the "end-point" cases. The estimate (1.3) can be considered as a statement of scattering theory that $\Psi(x)\langle D\rangle^{1/2}$ is H_0-smooth in the sense of Kato [K1] and it can be safely said that it had been long known at least implicitly before it was rediscovered by Sjölin [Sj], however, (1.3) had not been considered as an inequality which had expressed a smoothing property of Schrödinger equation before [Sj]. These inequalities are subsequently generalized to the case with potentials which decay at inifinity(see e.g. [CS], [KY], [BAD] and [Y1]).

Before proceeding further, we present here the outlines of the "standard" proof of (1.2) for non-end point cases and the proof of (1.3) which expresses the "physical content" of the estimate. For $1 \le p \le \infty$, p' denotes its dual exponent $1/p + 1/p' = 1$.

Proof of (1.2): Since e^{-itH_0} is unitary, we have $\|e^{-itH_0}u\|_2 = \|u\|_2$ and, since $|e^{-itH_0}(x,y)| \le C|t|^{-n/2}$, we have $\|e^{-itH}u\|_\infty \le C|t|^{-n/2}\|u\|_1$. It follows by interpolation that, for $p \ge 2$,

$$\|e^{-itH_0}u\|_p \le C|t|^{-n(1/2-1/p)}\|u\|_{p'}. \tag{1.4}$$

Then, for p and θ as above, Hardy-Littlewood-Sobolev inequality implies,

$$\begin{aligned} \left\|\int_{\mathbb{R}} e^{-itH_0} f(t)dt\right\|_2^2 &= \int_{\mathbb{R}}\int_{\mathbb{R}} (e^{-i(t-s)H_0} f(t), f(s))dsdt \\ &\le C\int_{\mathbb{R}}\int_{\mathbb{R}} |t-s|^{-n(1/2-1/p)}\|f(t)\|_{p'}\|f(s)\|_{p'}dsdt \le C\|f\|^2_{L^{\theta'}(\mathbb{R},L^{p'}(\mathbb{R}^n))}, \end{aligned}$$

which implies (1.2) by duality.

Proof of (1.3): We have

$$\begin{aligned} \int_0^\infty \|\langle D\rangle^{1/2}\Phi(x)e^{-itH_0}u\|_2^2 dt &= \int_0^\infty (e^{itH_0}\Phi(x)\langle D\rangle\Phi(x)e^{-itH_0}u, u)dt \\ &\sim \int_0^\infty (\langle D\rangle e^{itH_0}\Phi^2(x)e^{-itH_0}u, u)dt \\ &= \left(\langle D\rangle \cdot \left\{\int_0^\infty \Phi^2(x+tD)dt\right\}u, u\right), \end{aligned} \tag{1.5}$$

where we used the formula $e^{itH_0}xe^{-itH_0} = x + tD$. Here we have

$$\int_0^\infty \Phi^2(x+t\xi)dt \sim |\xi|^{-1} \tag{1.6}$$

and $\int_0^\infty \Phi^2(x+tD)dt$ is a pseudodifferential operator of order -1. Hence the right hand side of (1.5) is bounded by $C\|u\|^2$. We note that (1.6) is a results of the obvious fact that the free particle of velocity v can stay in a compact set for the time $\sim v^{-1}$ and we may consider (1.3) its mathematical expression.

1.2 The case $|V(x)| \le C\langle x\rangle^2$

We are still reviewing known results. The Strichartz inequality (1.2) and the local smoothing property (1.3) have been subsequently generalized by [K3] and [Y2] to Schrödinger equations

$$\begin{cases} i\dfrac{\partial u}{\partial t} = -(1/2)\Delta u + V(x)u, & x \in \mathbb{R}^n,\ t \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.7}$$

with potentials $V(x)$ which grow at most quadratically at infinity in the sense

$$|\partial_x^\alpha V(x)| \le C_\alpha, \quad 2 \le |\alpha| \le C_n, \tag{1.8}$$

C_n being a certain constant determined by n. Under the condition (1.8), it is well known that $L : u \mapsto -(1/2)\Delta u + V(x)u$ defined on $C_0^\infty(\mathbb{R}^n)$ is essentially selfadjoint in $L^2(\mathbb{R}^n)$ and the problem (1.7) has a unique solution given by $u(t,x) = e^{-itH}u_0(x)$, where H is the unique selfadjoint extension of L. The critical issue here is that Fujiwara [F] has proven that the fundamental solution, i.e. the distribution kernel $E(t,x,y)$ of the propagator e^{-itH} has the following structure at least for small $0 < |t| < \delta$: Let $(x(t,y,k), p(t,y,k))$ be the solution of Newton's equations corresponding to (1.7):

$$\begin{gathered} \dot{x}(t) = p(t), \quad \dot{p}(t) = -\nabla_x V(x), \\ x(0) = y, \quad p(0) = k. \end{gathered} \tag{1.9}$$

Then, the map $\mathbb{R}^n \ni k \to x(t,y,k) \in \mathbb{R}^n$ is a global diffeo for every $0 < |t| < \delta$ and $y \in \mathbb{R}^n$ and, for any given pair $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ and $0 < t < \delta$, there exists a unique solution of (1.9) such that $x(t) = x$ and $x(0) = y$. Let

$$S(t,x,y) = \int_0^t \{(1/2)\dot{x}(s)^2 - V(x(s))\}ds \tag{1.10}$$

be the action integral of this trajectory. Then, $S(t,x,y)$ satisfies

$$\left|\partial_x^\alpha \partial_y^\beta \left(S(t,x,y) - \frac{(x-y)^2}{2t}\right)\right| \le C_{\alpha\beta}|t|, \quad |\alpha+\beta| \ge 2, \tag{1.11}$$

and the fundamental solution may be written in the form

$$E(t,x,y) = \frac{1}{(2\pi i t)^{n/2}} e^{iS(t,x,y)} a(t,x,y) \tag{1.12}$$

where $a(t,x,y)$ satisfies

$$|\partial_x^\alpha \partial_y^\beta (a(t,x,y)-1)| \le C_{\alpha\beta}|t|, \quad |\alpha+\beta| \ge 0. \tag{1.13}$$

The fact that $S(t,x,y)$ in (1.12) is given as the action integral is particularly important as it connects classical mechanics (1.9) and the Schrödinger equation (1.7). It will become important in the next section that that δ and the constants $C_{\alpha\beta}$ of (1.11) and (1.13) depend only on C_α in (1.8) and not on the specific form of V.

In particular, $E(t,x,y)$ satisfies $|E(t,x,y)| \le C|t|^{-n/2}$ for $|t| \le \delta$. It follows that the unitary group e^{-itH} satisfies also the $L^1 - L^\infty$ estimate: $\|e^{-itH}u_0\|_\infty \le C|t|^{-n/2}\|u\|_1$ and hence (1.4) for $|t| \le \delta$. Then, the same argument used for the free Schrödinger equation and the unitarity of the propagator e^{-itH} yield the time local Strichartz inequality: For any $T>0$,

$$\left(\int^T \|e^{-itH}u_0\|_p^\theta dt\right)^{\frac{1}{\theta}} \le C_T\|u_0\|_2. \tag{1.14}$$

Of course, the time global estimate like (1.2) cannot hold in general because of the existence of the bound states of H.

The proof of the local smoothing property for the free Schrödinger equation can also be generalized to the case that V satisfies (1.8). We note that the classical particle of the large velocity in the potential fields as in (1.8) behaves like a free particle in any compact set K and the re-entrance to K is permitted only after certain time T which is independent of the energy of the particle. Guided by this observation, we have shown in [Y2] by using the structure formula (1.12) that $\int_0^\delta e^{itH}\Phi(x)e^{-itH}dt$ is again a psuedo-differential operator of order -1 and (1.3) holds for finite T with H in place of H_0.

1.3 Theorems

We now turn to our problem here and assume that V grows faster than any quadratic functions at infinity:

Assumption 1.1 *The potential $V(x) > 0$ is real valued and of C^∞-class. There exists $R > 0$ such that V satisfies the following properties for $|x| \ge R$:*

(1) *For $m > 2$, $D_1\langle x\rangle^m \le V(x) \le D_2\langle x\rangle^m$, where $0 < D_1 \le D_2 < \infty$.*

(2) *For $|\alpha| \ge 2$, $|\partial_x^\alpha V(x)| \le C_\alpha \langle x\rangle^{m-|\alpha|}$ for some constants C_α.*

The operator $L : u \mapsto -(1/2)\Delta u + V(x)u$ on $C_0^\infty(\mathbb{R}^n)$ is again essentially selfadjoint in $L^2(\mathbb{R}^n)$ and the solution of (1.7) is given by $u(t,\cdot) = e^{-itH}u_0$ via the unitary group generated by the unique selfadjoint extension H of L. The operator H has only pure point spectrum $\lambda_1 < \lambda_2 \le \ldots \to \infty$.

The behavior of the fundamental solution of (1.7) with superquadratic potentials is very different from that with potentials growing at most quadratically at infinity: $E(t,x,y)$ is nowhere C^1 and is not in general bounded at infinity [Y4], [MY]. Actually, the motivation to this work was to understand how this property of $E(t,x,y)$ is reflected in the local smoothing property of (1.7). We prove the following theorems.

Theorem 1.2 *Let V satisfy* Assumption 1.1. *Let $T > 0$ and $\Psi \in C_0^\infty(\mathbb{R}^n)$. Then, there exists a constant $C > 0$ such that*

$$\left(\int_{-T}^{T} \|\Psi(x)\langle H\rangle^{\frac{1}{2m}} e^{-itH}u_0\|_2^2 dt\right)^{\frac{1}{2}} \le C\|u_0\|, \quad u_0 \in L^2(\mathbb{R}^n). \tag{1.15}$$

We remark that Theorem 1.2 can also be explained in terms of the sojourn time in compact sets of a classical particle of large velocity. Suppose $n = 1$. Then, the particle is subject to periodic motion. Let $K \subset \mathbb{R}$ be compact and let v be its velocity in K. Then the energy λ is $\sim v^2$ and the period is roughly

$$\int_{-v^{2/m}}^{v^{2/m}} \frac{dx}{\sqrt{v^2 - |x|^m}} \sim Cv^{-1+2/m}.$$

Since the particle of velocity v can stay in K for $\sim 1/v$, the fraction of time to find it in K is $\sim v^{-2/m}$ and we expect e^{-itH} improves the differentiablity by the order $-1/m$ at almost all t. Notice that we can find the fraction $v^{-2/m}$ by observing the motion only for one period which is $\sim v^{-1+2/m} \sim \lambda^{-(\frac{1}{2}-\frac{1}{m})}$ if the energy is λ. The proof of Theorem 1.2 and Theorem 1.3 given below is actually guided by this observation.

As for the Strichartz ineqaulity, we show the following theorem.

Theorem 1.3 *Let V satisfy* Assumption 1.1. *Let $T > 0$ and let $2 \le p, \theta$ be such that $\frac{2}{\theta} = n\left(\frac{1}{2} - \frac{1}{p}\right)$ and $p \ne \infty$ if $n = 2$. Then, there exists a constant $C > 0$ such that*

$$\left(\int_{-T}^{T} \|e^{-itH}u_0\|_p^\theta dt\right)^{\frac{1}{\theta}} \le C\|\langle H\rangle^{\frac{1}{\theta}\left(\frac{1}{2}-\frac{1}{m}\right)_+}u_0\|, \quad u_0 \in L^2(\mathbb{R}^n), \tag{1.16}$$

where a_+ denotes any number $> a$.

Theorem 1.2 is sharp as the following one dimensional result shows, however, we believe that Theorem 1.3 is much weaker than best possible. In one dimension, we have the following sharp result which, however, is of a form slightly different from (1.16).

Assumption 1.4 *$V(x)$ is real valued and of C^3-class on $\mathbb{R}^1$. There exists a constant $R > 0$ such that the following conditions are satisfied for $|x| \ge R$:*

(1) *$V(x)$ is convex.*

(2) *For $j = 1, 2, 3$, $|V^{(j)}(x)| \le C_j\langle x\rangle^{-1}|V^{(j-1)}(x)|$ for some constants C_j.*

(3) *For $m > 2$, $D_1\langle x\rangle^m \le V(x) \le D_2\langle x\rangle^m$, where $0 < D_1 \le D_2 < \infty$.*

We define $\theta(m,p)$ as follows, for $2 \le p \le \infty$ and $2 < m < \infty$:

$$\theta(m,p) = \begin{cases} \frac{1}{m}\left(\frac{1}{2} - \frac{1}{p}\right), & \text{if} \quad 2 \le p < 4; \\ \left(\frac{1}{4m}\right)_-, & \text{if} \quad p = 4; \\ \frac{1}{4} - \frac{1}{3}\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{m}\right), & \text{if} \quad 4 < p \le \infty, \end{cases}$$

where a_- denotes any number $< a$.

Theorem 1.5 *Let V satisfy* Assumption 1.4 *and let $2 \le p \le \infty$. Let $T > 0$ and $K \subset \mathbb{R}$ be compact. Then, there exists a constant $C > 0$ such that*

$$\|\langle H\rangle^{\theta(m,p)}e^{-itH}u_0(x)\|_{L^p(\mathbb{R}_x, L^2([-T,T]_t))} \le C_T\|u_0\|_{L^2(\mathbb{R}_x)}, \tag{1.17}$$

$$\sup_{x \in K}\|\langle H\rangle^{\frac{1}{2m}}e^{-itH}u_0(x)\|_{L^2([-T,T])} \le C_T\|u_0\|_{L^2(\mathbb{R}_x)} \tag{1.18}$$

We have the following sharp estimate of the normalized eigenfunction of the one dimensional Schrödinger operator and we see that (1.17) and (1.18) are sharp in the sense that $\theta(m,p)$ and $1/2m$ cannot be replaced by any larger numbers by inserting $u_0(x) = \psi(x,E)$ and letting $E \to \infty$.

Theorem 1.6 *Let* Assumption 1.4 *be satisfied. Let $\psi(x,E)$ be the normalized eigenfunction of $H = -(1/2)\Delta + V(x)$ with the eigenvalue E. Then:*
(1) *For $1 \le p \le \infty$, we have*

$$\|\psi(x,E)\|_{L^p} \sim \begin{cases} C_p E^{-\theta(m,p)}, & \text{if} \quad p \neq 4; \\ C E^{-\frac{1}{4m}} (\log E)^{\frac{1}{4}}, & \text{if} \quad p = 4, \end{cases} \tag{1.19}$$

for large E, where C_p can be taken independent of p, $p \notin (4-\varepsilon, 4+\varepsilon)$, $\varepsilon > 0$.
(2) *For compact interval $K \subset \mathbb{R}$, $\sup_{x\in K} |\psi(x,E)| \sim E^{-\frac{1}{2m}}$ for large E.*

2 Outline of Proofs

We outline the proof of Theorem 1.2 and Theorem 1.3. We refer the reader to [Y5] for the proof of one dimensional results Theorem 1.5 and Theorem 1.6, which heavily depends upon the spectral property of H. Hinted by the observation stated after Theorem 1.2, we decompose the solution $u(t) = e^{-itH}u_0$ into the sum of components $u_j(t)$ which are spectrally concentrated in $(2^{j-1}, 2^{j+1})$ with respect to H:

$$u(t) = \sum_{j=0}^{\infty} u_j(t) = \sum_{j=0}^{\infty} e^{-itH} u_{0j}, \tag{2.20}$$

and analyse each component $u_j(t)$ separately by splitting the time interval $[0,T]$ into subintervals of length $\sim 2^{-j(\frac{1}{2}-\frac{1}{m})}$. Thus, we choose $\psi_0 \in C_0^\infty(\mathbb{R})$ and $\psi \in C_0^\infty(\mathbb{R}^+)$ such that $\operatorname{supp} \psi \subset (2^{-1}, 2)$ and

$$\psi_0(x) + \sum_{j=1}^{\infty} \psi(x/2^j) = 1 \quad \text{for } x \in [0,\infty),$$

and define $u_{0j} = \psi_j(H)u_0$ and $u_j(t) = \psi_j(H)u(t) = e^{-itH}u_{0j}$, $j = 0, 1, \ldots$, where $\psi_j(x) = \psi(x/2^j)$, $j = 1, 2, \ldots$.

2.1 Lemmas

We denote $a(x,\xi) = (1/2)\xi^2 + V(x)$. The first lemma states that the energy cut off can be approximated by a certain pseudo-differential operator which is easier to handle.

Lemma 2.1 *Let $\psi \in C_0^\infty([0,\infty))$ and $\phi \in C_0^\infty(\mathbb{R})$ be such that*

$$\psi(t) = \begin{cases} 1, & 2^{-1} < t < 2^1, \\ 0, & t \notin [2^{-2}, 2^2] \end{cases}, \qquad \phi(t) = \begin{cases} 1, & 2^{-4} < t < 2^4, \\ 0, & t \notin [2^{-5}, 2^5] \end{cases}.$$

Define $\Phi_\lambda(x,\xi) = \phi(a/\lambda)$. Then for any N, there exists C_N such that

$$\|\langle H\rangle^N (1 - \Phi_\lambda(x,D))\psi(H/\lambda)\langle H\rangle^N\| \le C_N \lambda^{-N}, \tag{2.21}$$

where the constant C_N is independent of $\lambda \ge 1$.

To prove Lemma 2.1, we write $\psi(H/\lambda)$ in the form

$$\psi(H/\lambda) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial\tilde{\psi}_\lambda}{\partial\bar{z}}(z)(H-z)^{-1} dz \wedge d\bar{z}, \tag{2.22}$$

where $\tilde{\psi}_\lambda(z) = \tilde{\psi}(z/\lambda)$ and $\tilde{\psi}(z)$ is an almost analytic extension of $\psi(t)$ such that $\tilde{\psi}(z) = 0$ outside $2^{-2} < |z| < 2^2$. We construct the parametrix via the standard pseudo-differential calculus to find

$$(1 - \Phi_\lambda(x,D))(H-z)^{-1} = \sum_{j=0}^{N} Q_j(z,x,D) + R_{\lambda N}(z,x,D)(H-z)^{-1}. \tag{2.23}$$

Here the symbols $Q_j(z,x,\xi)$ are of the form $\sum_{k=j+1}^{2j+1} a_{jk}(x,\xi)(a(x,\xi)-z)^{-k}$ and $\{R_{\lambda N}(z,x,\xi) : z \in \Omega_\lambda, \lambda \ge 1\}$ is bounded in $S(\langle x\rangle^{-(N+1)}\langle\xi\rangle^{-(N+1)}, g)$, where $g = |x|^{-2}dx^2 + |\xi|^{-2}d\xi^2$ and $S(m,g)$ is Hörmander's symbol class [Ho]. We multiply (2.22) by $(1 - \Phi_\lambda(x,D))$ from the left and insert (2.23) in the right of the resulting equation. Then, the contributions from Q_j vanish by Cauchy's formula and that of the reaminder $R_{\lambda N}(z,x,D)(H-z)^{-1}$ is of order $O(\lambda^{-N})$.

Lemma 2.1 allows us to study $e^{-itH}\psi(H/\lambda)u_0$ via $e^{-itH}\Phi_\lambda(x,D)$. We next approximate the propagator $e^{-itH}\Phi_\lambda(x,D)$ by a more tractable one.

Observe that classical particles of energy λ cannot not enter the domain where $V(x) > \lambda$. As $\Phi_\lambda(x, D)$ projects u_0 into states with energy $\sim \lambda$, the dynamics $e^{-itH}\Phi_\lambda(x, D)u_0$ should be well approximated by $e^{-it\tilde{H}}\Phi_\lambda(x, D)u_0$ generated by the Hamiltonian $\tilde{H} = -(1/2)\Delta + \tilde{V}(x)$, where $\tilde{V}(x)$ is the part of V where $V(x) < C\lambda$. We show this is indeed the case in the next lemma for $|t| \le \varepsilon\lambda^{-(\frac{1}{2}-\frac{1}{m})}$, $\varepsilon > 0$ being a small number, which is a fraction of the period of the classical particle of energy $\sim \lambda$. To state and prove this fact, we find it convenient to change the scale of time and convert the equations into the semi-classical form. We introduce the following notation. If we set

$$v(t, x) = u(ht, x), \qquad h = \lambda^{-(\frac{1}{2}-\frac{1}{m})},$$

then $v(t, x)$ satisfies the semi-classical Schrödinger equation

$$ih\frac{\partial v}{\partial t} = \frac{-h^2}{2}\Delta v + V_h v, \qquad V_h(x) = \lambda^{-2(\frac{1}{2}-\frac{1}{m})}V(x). \tag{2.24}$$

We take $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\chi(x) = 1$ for $|x| \le 1$ and $\chi(x) = 0$ for $|x| \ge 2$ and define

$$\tilde{V}_h(x) = V_h(x)\chi(x/C_1\lambda^{\frac{1}{m}}),$$

where $C_1 >> 1$ is taken such that $V(x) \ge 2^5\lambda$ when $|x| \ge C_1\lambda^{\frac{1}{m}}$. We then define the approximation to (2.24) by

$$ih\frac{\partial v}{\partial t} = \frac{-h^2}{2}\Delta v + \tilde{V}_h v = \tilde{H}^h v. \tag{2.25}$$

The point is that $\tilde{V}_h$ satisfies the estimate

$$|\partial_x^\alpha \tilde{V}_h(x)| \le C_\alpha, \quad |\alpha| \ge 2, \tag{2.26}$$

where C_α is independent of $\lambda > 1$. Hence, as was remarked after (1.13), the fundamental solution $E^h(t, x, y)$ of (2.25) has the following structure:

$$E^h(t, x, y) = \frac{1}{(2\pi ith)^{n/2}}e^{iS^h(t,x,y)/h}a^h(t, x, y) \tag{2.27}$$

for $|t| \le \delta$ and δ and the constants appeared in the estimates (1.11) and (1.13) for S^h and a^h can be chosen independently of $\lambda \ge 1$.

Lemma 2.2 *Let $\psi \in C_0^\infty([0,\infty))$ and $\psi \in C_0^\infty(\mathbb{R})$ be as in* Lemma 2.1. *Then, for any $N, \ell = 0, 1, \ldots$, there exists C_1 and $\varepsilon > 0$ such that*

$$\sup_{|t|\le\varepsilon} \|H^\ell(e^{-itH^h/h} - e^{-it\tilde{H}^h/h})\Phi_\lambda(x,D)\| \le C_{N\ell}\lambda^{-N} \tag{2.28}$$

for a positive constant $C_{N\ell}$ independent of $\ell \ge 1$.

Lemma 2.2 can be proved as follows. We may write via the Duhamel formula:

$$\begin{aligned} &H^\ell(e^{-itH^h/h} - e^{-it\tilde{H}^h/h})\Phi_\lambda(x,D)u \\ &\quad = ih^{-1}\int_0^t H^\ell e^{-i(t-s)H}(V_h - \tilde{V}_h)e^{-it\tilde{H}^h/h}\Phi_\lambda(x,D)udt. \end{aligned}$$

By using (2.27) and the stationary phase method, we estimate $H^\ell(V_h - \tilde{V}_h)e^{-it\tilde{H}^h/h}\Phi_\lambda(x,D)$. We find it is $O(h^N)$ for any N as there are no stationary phase point for x in the support of $(V_h - \tilde{V}_h)$. This follows because classical particles of energy λ cannot enter the support of $V_h - \tilde{V}_h$.

2.2 Proof of Strichartz inequlity

We take $\phi \in C_0^\infty((2^{-3}, 2^3))$ such that $\psi(x) = 1$ for $2^{-2} \le x \le 2^2$ and set $\Phi_j(x,\xi) = \phi(a(x,\xi)/2^j)$. Define $h_j = 2^{-j(\frac{1}{2}-\frac{1}{m})}$ and

$$U_j(t) = e^{-i(t/h_j)H_j/h_j}, \quad H_j = \tilde{H}^{h_j}$$

By virtue of (2.27), the integral kernel $E_j(t,x,y)$ of $U_j(t)$ satisfies

$$|E_j(t,x,y)| \le C|t|^{-n/2}, \quad |t| \le \varepsilon h_j$$

with C independent of j and the argument of the proof of (1.2) implies

$$\left(\int_{|t|\le\varepsilon h_j} \|U_j(t)\Phi_j(x,D)u\|_p^\theta dt\right)^{1/\theta} \le C\|u\|_2. \tag{2.29}$$

By virtue of Lemma 2.2 and obvious Sobolev embedding, we have

$$\sup_{|t|\le\varepsilon h_j} \|(e^{-itH} - U_j(t))\Phi_j(x,D)u\|_p \le C_{Np}2^{-Nj}\|u\|_2. \tag{2.30}$$

Combining (2.29) and (2.30), we obtain for p and θ of Theorem 1.3

$$\left(\int_{|t|\le\varepsilon h_j} \|e^{-itH}u\|_p^\theta dt\right)^{1/\theta} \le C\|u\|_2. \tag{2.31}$$

with the contants $\varepsilon > 0$ and $C > 0$ independent of $j = 0, 1, \ldots$.

We let u_{0j} be as in the begining of this section. Minkowski's inequlity then implies that for any small $\delta > 0$

$$\left(\int_0^T \|e^{-itH}u_0\|_p^\theta dt\right)^{1/\theta} \le \sum_{j=0}^{\infty}\left(\int_0^T \|e^{-itH}u_{0j}\|_p^\theta dt\right)^{1/\theta}. \tag{2.32}$$

We then break up the interval $[0, T]$ as

$$0 = t_0 < t_1 < \ldots < t_{L_j} = T, \quad \tau_k = t_k - t_{k-1} < \varepsilon h_j, \quad L_j \sim T/\varepsilon h_j \tag{2.33}$$

and write the integral on the right of (2.32) in the following form, where $v_{jk} = e^{-it_{k-1}H}u_{0j}$:

$$\int_0^T \|e^{-itH}u_{0j}\|_p^\theta dt = \sum_{k=1}^{L_j}\int_{t_{k-1}}^{t_k} \|e^{-itH}u_{0j}\|_p^\theta dt = \sum_{k=1}^{L_j}\int_0^{\tau_k} \|e^{-itH}v_{jk}\|_p^\theta dt.$$

Then, (2.31) and the unitarity of e^{-itH} imply that the right hand side is bounded by

$$\sum_{k=1}^{L_j} C\|u_{0j}\|_2^\theta \le C(T/\varepsilon h_j)\|u_{0j}\|_2^\theta \le C_{T\varepsilon}\|\langle H\rangle^{\frac{1}{\theta}(\frac{1}{2}-\frac{1}{m})}u_{0j}\|_p^\theta.$$

Summing up the right hand side with respect to j and combining the result with (2.32), we obtain Theorem 1.3.

2.3 Proof of local smoothing property

Using Lemma 2.1 and the pseudo-differential calculus, we estimate

$$\begin{aligned}\int_0^T \|\Psi(x)e^{-itH}\langle H\rangle^{\frac{1}{2m}}u_0\|^2 dt &= \int_0^T \|\sum_{j=0}^{\infty}\Psi(x)e^{-itH}\langle H\rangle^{\frac{1}{2m}}u_{0j}\|^2 dt \\ &\le \sum_{j=0}^{\infty}\int_0^T \|\Psi(x)\Phi_j(x,D)e^{-itH}\langle H\rangle^{\frac{1}{2m}}u_{0j}\|^2 dt + C_T\|\langle H\rangle^{-\frac{1}{2}}u_0\|^2.\end{aligned} \tag{2.34}$$

Breaking up $[0,T]$ as in (2.33), we write the integral on the right as

$$\sum_{k=1}^{L_j}\int_0^{\tau_k}\|\Psi(x)\Phi_j(x,D)e^{-itH}u_{0j}^{(k)}\|^2dt,\quad u_{0j}^{(k)}=e^{-it_{k-1}H}\langle H\rangle^{\frac{1}{2m}}u_{0j}. \tag{2.35}$$

We approximate e^{-itH} by $U_j(t)$ using the dual statement of Lemma 2.2. Changing the variable $t\longrightarrow th_j$, we estimate, with negligible error, the integral on the right of (2.35) by

$$h_j\int_0^{\tau_k/h_j}\|\Psi(x)\Phi_j(x,D)e^{-itH_j/h_j}u_{0j}^{(k)}\|^2dt. \tag{2.36}$$

Define $K_j(x,\xi)=\Psi(x)^2\phi(a(x,h_j^{-1}\xi)/2^j)^2$. Then, the integral (2.36) is equal to

$$h_j\int_0^{\tau_k/h_j}(e^{itH_j/h_j}K_j(x,h_jD)e^{-itH_j/h_j}u_{0j}^{(k)},u_{0j}^{(k)}) \tag{2.37}$$

modulo errors whose sum over j,k is bounded by $C\|\langle H\rangle^{-\frac{1}{4}}u_0\|^2$. We then construct the parametrix $K_j(t,x,h_jD)$ of $e^{itH_j/h_j}K_j(x,h_jD)e^{-itH_j/h_j}$ by a procedure standard for proving Egorov's theorem, requiring

$$\begin{aligned}&(d/dt)e^{-itH_j/h_j}K_j(t,x,h_jD)e^{itH_j/h_j}\\&\qquad=e^{-itH_j/h_j}(\partial K_j/\partial t-i[H_j,K_j])e^{itH_j/h_j}\sim 0.\end{aligned}$$

This produces pseudo-differential operators $K_j^N(t,x,h_jD)$ such that

$$\|e^{itH_j/h_j}K_j(x,h_jD)e^{-itH_j/h_j}-K_j^N(t,x,h_jD)\|\le C_Nh_j^{N+1} \tag{2.38}$$

with constant C_N independent of $j=1,2,\ldots$. The symbols of $K_j^N(t,x,h_jD)$ are computable by using trajectories of (1.9). In particular, they are supported by $\Gamma(-t)(\text{supp }K_j)$, where $\Gamma(t):(y,k)\rightarrow(x(t,y,k),p(t,y,k))$ and the remark after Theorem 1.2 about the sojourn time of the particle of large velocity implies

$$\left|\int_0^{\tau_k/h_j}K_j^N(t,x,\xi)dt\right|\le C_{\alpha\beta N}2^{-\frac{j}{m}}. \tag{2.39}$$

Here again the constant $C_{\alpha\beta N}$ independent of $j=1,2,\ldots$.

Completion of the proof. The integral (2.37) is equal to

$$h_j \int_0^{\tau_k/h_j} (K_j^N(t,x,h_jD)u_{0j}^{(k)}, u_{0j}^{(k)})dt + O(2^{-Nj})$$

by virtue of (2.38), and (2.39) implies that the integral is bounded by

$$Ch_j 2^{-\frac{j}{m}} \|u_{0j}^{(k)}\|^2 = Ch_j 2^{-\frac{j}{m}} \|\langle H\rangle^{\frac{1}{2m}} u_{0j}\|^2 \leq Ch_j \|u_{0j}\|^2$$

Summing up over $L_j \sim \varepsilon h_j^{-1}$ number of k's, we obtain

$$\int_0^T \|\Psi(x)\Phi_j(x,D)e^{-itH}\langle H\rangle^{\frac{1}{2m}} u_{0j}\|^2 dt \leq C\|u_{0j}\|^2$$

and, therefore,

$$\sum_{j=0}^{\infty} \int_0^T \|\Psi(x)\Phi_j(x,D)e^{-itH}\langle H\rangle^{\frac{1}{2m}} u_{0j}\|^2 dt \leq C\|u_0\|^2,$$

which implies Theorem 1.2. ∎

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