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On the Stokes and Navier-Stokes flows between parallel planes

Takayuki Abe
Waseda University, Tokyo
601m5016@mn.waseda.ac.jp

1. Introduction

The present paper is a shortened version of a forthcoming paper by Yoshihiro Shibata of Waseda University and the author. Let $\Omega\subset\mathbb{R}^{n}$ $(n\geq 2)$ be a domain bounded by two parallel planes, i.e.,

$$\Omega=\{x=(x',x_{n})\in\mathbb{R}^{n}\mid x'\in\mathbb{R}^{n-1},\ 0<x_{n}<1\},$$

and we consider the following initial boundary value problem of the nonstationary Stokes equation:

$$\text{(1.1)}\qquad \begin{cases} \mathbf{u}_{t}-\Delta\mathbf{u}+\nabla\mathfrak{p}=\mathbf{0}, \quad \nabla\cdot\mathbf{u}=0 & \text{in } (0,\infty)\times\Omega,\\ \mathbf{u}|_{x_{n}=0}=\mathbf{0}, \quad \mathbf{u}|_{x_{n}=1}=\mathbf{0}, & \\ \mathbf{u}(0,x)=\mathbf{a}(x) & \text{in } \Omega. \end{cases}$$

Here, $\mathbf{u}=\mathbf{u}(t,x)=(u_{1}(t,x),\cdots,u_{n}(t,x))$ and $\mathfrak{p}=\mathfrak{p}(t,x)$ denote the unknown velocity vector and the unknown pressure at point $(t,x)\in[0,\infty)\times\Omega$, respectively, while $\mathbf{a}=\mathbf{a}(x)=(a_{1}(x),\cdots,a_{n}(x))$ denotes a given initial velocity at point $x\in\Omega$. In order to prove that the nonstationary problem (1.1) generates an analytic semigroup in

$$L_{\sigma}^{p}(\Omega)=\{\mathbf{u}\in L^{p}(\Omega)^{n}\mid\nabla\cdot\mathbf{u}=0,\ \nu\cdot\mathbf{u}|_{\partial\Omega}=0\},$$

where ν is the unit outer normal to $\partial\Omega$, we investigate the corresponding resolvent problem:

$$\text{(1.2)}\qquad \begin{cases} (\lambda-\Delta)\mathbf{u}+\nabla\mathfrak{p}=\mathbf{f}, \quad \nabla\cdot\mathbf{u}=0 & \text{in } \Omega,\\ \mathbf{u}|_{x_{n}=0}=\mathbf{0}, \quad \mathbf{u}|_{x_{n}=1}=\mathbf{0}, & \end{cases}$$

where the resolvent parameter λ is contained in the union of the sector

$$\Sigma_{\varepsilon}=\{z\in\mathbb{C}\backslash\{0\}\mid|\arg z|<\pi-\varepsilon\}, \quad 0<\varepsilon<\frac{\pi}{2}$$

and the sufficiently small neighborhood of zero.

So many results of the mathematical analysis for the imcompressible viscous fluid in the whole space and in the exterior domain have been obtained. However, the case where the domain is bounded by two parallel planes has been less studied. Nazarov and Pileckas [5] and [6] treated the boundary value problem of the stationary Stokes equation between two parallel planes in the weighted L^2-framework. On the other hand, we analyze the problem (1.2) by employing the Farwig and Sohr's idea in [2]. Our main result is the following theorem.

Theorem 1.1. *Let* $1 < p < \infty$ and $0 < \varepsilon < \pi/2$. *Then there exists a sufficiently small number* $\sigma > 0$ *such that for any* $\lambda \in \Sigma_\varepsilon \cup \{z \in \mathbb{C} \mid |z| < \sigma\}$ *and any* $\mathbf{f} = (f_1, \cdots, f_n) \in L^p(\Omega)^n$ *there exists a unique* $\mathbf{u} \in W_p^2(\Omega)^n$ *which together with some* $\mathfrak{p} \in \hat{W}_p^1(\Omega)$ *solve* (1.2)*;* $\mathfrak{p}$ *is unique up to an additive constant. Moreover, there holds the following resolvent estimate:*

$$(1.3) \qquad |\lambda| \|\mathbf{u}\|_{L^p(\Omega)} + |\lambda|^{\frac{1}{2}} \|\nabla \mathbf{u}\|_{L^p(\Omega)} + \|\mathbf{u}\|_{W_p^2(\Omega)} + \|\nabla \mathfrak{p}\|_{L^p(\Omega)} \leq C_{p,n,\varepsilon} \|\mathbf{f}\|_{L^p(\Omega)}.$$

Here, $\hat{W}_p^1(\Omega) = \{\pi \in L^p_{\mathrm{loc}}(\Omega) \mid \nabla \pi \in L^p(\Omega)\}$.

Remark. Generally, $\lambda = 0$ does not belong to the resolvent set of the Stokes operator on an unbounded domain. Although Ω is also the unbounded domain, using the boundedness of Ω with respect to x_n we can prove that $\lambda = 0$ is also in the resolvent set. This is one of the outstanding features of Theorem 1.1.

Now, applying the Helmholtz projection $P_p : L^p(\Omega)^n \to L^p_\sigma(\Omega)$ to (1.2), we see that (1.2) is equivalent to

$$(\lambda + A_p)\mathbf{u} = \mathbf{f}, \qquad \mathbf{u} \in D(A_p).$$

Here, A_p is the Stokes operator defined by

$$A_p\mathbf{u} = -P_p\Delta\mathbf{u}, \quad \mathbf{u} \in D(A_p) = \{\, \mathbf{u} \in W_p^2(\Omega)^n \cap L^p_\sigma(\Omega) \mid \mathbf{u}|_{\partial\Omega} = \mathbf{0} \,\}.$$

Since by (1.3) we see that

$$\|(\lambda + A_p)^{-1}\|_{L(L^p_\sigma(\Omega))} \leq \frac{C}{|\lambda|},$$

the Stokes operator on Ω generates an analytic semigroup $\{e^{-tA_p}\}_{t\geq 0}$ and by employing the Sobolev's embedding and interpolation argument we obtain the following theorem.

Theorem 1.2. *The Stokes operator on* Ω *with Dirichlet zero boundary condition generates an analytic semigroup* $\{e^{-tA_p}\}_{t\geq 0}$ *in* $L^p_\sigma(\Omega)$ *and there holds the following* $L^p - L^q$ *estimate:*

$$(1.4) \qquad \|\nabla^k e^{-tA_p}\mathbf{a}\|_{L^q(\Omega)} \leq C_{p,q,k} e^{-\delta_{p,q}t} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{k}{2}} \|\mathbf{a}\|_{L^p(\Omega)}, \quad 1 < p \leq q < \infty$$

for any $\mathbf{a} \in L^p_\sigma(\Omega)$. *Here,* $k \geq 0$ *is an integer.*

2. Basic lemmas

Farwig and Sohr [2] analyzed the Stokes resolvent problem in the half space $\mathbb{R}^n_+$ by the Fourier multiplier method. We employ their idea in the proof of Theorem 1.1. To be more precise, by applying the Fourier transform with respect to $x' = (x_1, \cdots, x_{n-1})$ we obtain the boundary value problems of the ordinary differential equations, and we apply the Fourier multiplier theorem (cf. [3]) and the Agmon-Douglis-Nirenberg lemma (cf. [1]) to the representations of the solutions to these problems, consequently we obtain the L^p-estimates of the solutions. The Fourier multiplier theorem is the following proposition:

Proposition 2.1. *Let $1 < p < \infty$. Let $k : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be a C^n-function which satisfies the multiplier condition:*

$$\left|\partial_\xi^\alpha k(\xi)\right| \leq C_\alpha |\xi|^{-|\alpha|}, \qquad \forall\alpha,\ |\alpha| \leq n,\ \forall\xi \in \mathbb{R}^n \setminus \{0\}$$

with some constant C_α. Then there exists a constant C_p independent of C_α such that

$$\left\|\mathcal{F}_\xi^{-1}\left[k(\xi)\hat{u}(\xi)\right]\right\|_{L^p(\mathbb{R}^n)} \leq C_p \left(\max_{|\alpha|\leq n} C_\alpha\right) \|u\|_{L^p(\mathbb{R}^n)}, \qquad \forall u \in L^p(\mathbb{R}^n).$$

The basic estimates to show the above multiplier condition are as follows:

Lemma 2.1. *Let $l \in \mathbb{R}$ and let $a > 0$ be a constant. Then the following estimates are valid:*

$$\left|\lambda + |\xi|^2\right| \geq \sin\frac{\varepsilon}{2}(|\lambda| + |\xi|^2), \qquad \forall\lambda \in \Sigma_\varepsilon,\ \forall\xi \in \mathbb{R}^n, \tag{2.1}$$

$$\operatorname{Re}\sqrt{\lambda + |\xi|^2} \geq \left(\frac{1}{2}\right)^{\frac{1}{4}} \sin\frac{\varepsilon}{2}(|\lambda|^{\frac{1}{2}} + |\xi|), \qquad \forall\lambda \in \Sigma_\varepsilon,\ \forall\xi \in \mathbb{R}^n, \tag{2.2}$$

$$\left|\partial_\xi^\alpha |\xi|^l\right| \leq C_\alpha |\xi|^{l-|\alpha|}, \qquad \forall\alpha,\ |\alpha| \leq n,\ \forall\xi \in \mathbb{R}^n \setminus \{0\}, \tag{2.3}$$

$$\left|\partial_\xi^\alpha e^{-a|\xi|}\right| \leq C_\alpha |\xi|^{-|\alpha|} e^{-\frac{a}{2}|\xi|}, \qquad \forall\alpha,\ |\alpha| \leq n,\ \forall\xi \in \mathbb{R}^n \setminus \{0\}. \tag{2.4}$$

Combining the Agmon-Douglis-Nirenberg lemma with Proposition 2.1, we obtain the following lemma:

Lemma 2.2. *Let $1 < p < \infty$ and $\delta > 0$ and $u \in W_p^1(\mathbb{R}^n_+)$. Let $k : \mathbb{R}^{n-1} \setminus \{0\} \to \mathbb{R}$ be a C^{n-1}-function which satisfies the multiplier condition:*

$$\left|\partial_{\xi'}^{\alpha'} k(\xi')\right| \leq C_{\alpha'} |\xi'|^{-|\alpha'|}, \qquad \forall\alpha',\ |\alpha'| \leq n-1,\ \forall\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$$

with some constant $C_{\alpha'}$. Then there exists a constant $C_{p,n,\delta}$ independent of u such that

$$\left\|\nabla \mathcal{F}_{\xi'}^{-1}\left[k(\xi')e^{-\delta|\xi'|x_n}\hat{u}(\xi',0)\right]\right\|_{L^p(\mathbb{R}^n_+)} \le C_{p,n,\delta}\|\nabla u\|_{L^p(\mathbb{R}^n_+)}.$$

Moreover, if we assume $u \in W^2_p(\mathbb{R}^n_+)$, then there holds

$$\left\|\nabla^2 \mathcal{F}_{\xi'}^{-1}\left[k(\xi')e^{-\delta|\xi'|x_n}\hat{u}(\xi',0)\right]\right\|_{L^p(\mathbb{R}^n_+)} \le C_{p,n,\delta}\|\nabla^2 u\|_{L^p(\mathbb{R}^n_+)}.$$

The above lemma is the basic tool which often used to estimate the first and second derivatives of the solutions to (1.2).

3. Outline of the proof of Theorem 1.1

(I) The case where $\lambda \in \Sigma_\varepsilon,\ |\lambda| \ge \lambda_0 > 0$

As the first case, we consider the case where $\lambda \in \Sigma_\varepsilon$, $|\lambda| \ge \lambda_0 > 0$. Here, λ_0 is an arbitrary fixed positive number.

Step 1. We neglect the boundary condition and we shall construct $(\mathbf{U}, \Phi)$ satisfying

$$(\lambda - \Delta)\mathbf{U} + \nabla\Phi = \mathbf{F}, \quad \nabla\cdot\mathbf{U} = 0 \quad \text{in } \mathbb{R}^n. \tag{3.1}$$

Here, $\mathbf{F}$ denotes an extension of $\mathbf{f}$, which is defined as follows: First, we shall define an even and odd extension of $f : \Omega \to \mathbb{R}$. Let $\varphi \in C^\infty(\mathbb{R})$ be a cut-off function such that $\varphi(x_n) = 1$ for $x_n \le 1/3$ and $\varphi(x_n) = 0$ for $x_n \ge 2/3$. Using this cut-off function φ we put $g_0(x) = \varphi(x_n)f(x', x_n)$, $g_1(x) = (1 - \varphi(x_n))f(x', x_n)$. Then for each of g_0 and g_1 we define the even extension g_0^e, g_1^e and the odd extension g_0^o, g_1^o as follows:

$$g_0^e(x) = \begin{cases} \varphi(x_n)f(x', x_n) & x_n > 0, \\ \varphi(-x_n)f(x', -x_n) & x_n < 0, \end{cases}$$

$$g_1^e(x) = \begin{cases} (1 - \varphi(x_n))f(x', x_n) & x_n < 1, \\ (1 - \varphi(2 - x_n))f(x', 2 - x_n) & x_n > 1, \end{cases}$$

$$g_0^o(x) = \begin{cases} \varphi(x_n)f(x', x_n) & x_n > 0, \\ -\varphi(-x_n)f(x', -x_n) & x_n < 0, \end{cases}$$

$$g_1^o(x) = \begin{cases} (1 - \varphi(x_n))f(x', x_n) & x_n < 1, \\ -(1 - \varphi(2 - x_n))f(x', 2 - x_n) & x_n > 1. \end{cases}$$

Then we put $f^e = g_0^e + g_1^e$ and $f^o = g_0^o + g_1^o$. Each of f^e and f^o is an extention of f. Using this notation we define $\mathbf{F} = (f_1^e, \cdots, f_{n-1}^e, f_n^o)$.

Now, if we put

$$\mathbf{U}(x) = \mathcal{F}_\xi^{-1}\left[\frac{P(\xi)\,{}^t(\hat{f}_1^e, \cdots, \hat{f}_{n-1}^e, \hat{f}_n^o)}{\lambda + |\xi|^2}\right](x),$$

$$\Phi(x) = -\mathcal{F}_\xi^{-1}\left[\sum_{j=1}^{n-1}\frac{i\xi_j}{|\xi|^2}\hat{f}_j^e(\xi) + \frac{i\xi_n}{|\xi|^2}\hat{f}_n^o(\xi)\right](x),$$

where $P(\xi) = (P_{jk}(\xi))_{1\le j,k\le n}$, $P_{jk}(\xi) = \delta_{jk} - \xi_j\xi_k/|\xi|^2$, then $\mathbf{U}$ and Φ solve (3.1) and by the Fourier multiplier theorem we obtain the estimate

$$(3.2)\quad |\lambda|\|\mathbf{U}\|_{L^p(\mathbb{R}^n)} + |\lambda|^{\frac{1}{2}}\|\nabla\mathbf{U}\|_{L^p(\mathbb{R}^n)} + \|\nabla^2\mathbf{U}\|_{L^p(\mathbb{R}^n)} + \|\nabla\Phi\|_{L^p(\mathbb{R}^n)} \le C_{p,n,\varepsilon}\|\mathbf{f}\|_{L^p(\Omega)}.$$

Now, setting $\mathbf{u} = \mathbf{U} + \mathbf{v}$ and $\mathfrak{p} = \Phi + \pi$ the problem (1.2) is reduced to the following problem for $\mathbf{v}$ and π:

$$(3.3)\quad \begin{cases} (\lambda - \Delta)\mathbf{v} + \nabla\pi = 0, \quad \nabla\cdot\mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v}|_{x_n=0} = -\mathbf{U}|_{x_n=0}, \quad \mathbf{v}|_{x_n=1} = -\mathbf{U}|_{x_n=1}. \end{cases}$$

Remark. To adopt $\mathbf{F} = (f_1^e, \cdots, f_{n-1}^e, f_n^o)$ as an extension of $\mathbf{f}$ enable us to obtain the following estimates when $|\lambda|$ is large enough:

$$(3.4)\quad \|U_n(\cdot, a)\|_{L^p(\mathbb{R}^{n-1})} \le \frac{C}{|\lambda|}\|\mathbf{f}\|_{L^p(\Omega)}, \qquad a = 0, 1.$$

This estimate will be needed when we estimate the L^p-norm of v_n. If we use the zero extension of $\mathbf{f}$ instead of $\mathbf{F}$, we can construct $(\mathbf{U}, \Phi)$ satisfying (3.1) and the estimate (3.2), but we can only obtain

$$\|U_n(\cdot, a)\|_{L^p(\mathbb{R}^{n-1})} \le C\|\mathbf{f}\|_{L^p(\Omega)}, \qquad a = 0, 1.$$

This is the reason why we use $\mathbf{F}$ as an extension of $\mathbf{f}$ in (3.1).

Step 2. We shall construct v_n satisfying (3.3) and estimate the L^p-norms of v_n, ∇v_n and $\nabla^2 v_n$. First, let us eliminate the pressure π and $\mathbf{v}' = (v_1, \cdots, v_{n-1})$ from (3.3). Since $\mathbf{v}$ satisfies the divergence free condition, applying the divergence to the first equation of (3.3) we have

$$(3.5)\quad \Delta\pi = 0.$$

Therefore, applying the Laplacian to the n-th component of the first equation of (3.3) we have $(\lambda - \Delta)\Delta v_n = 0$. Applying the Fourier transform with respect to x' we have $(\lambda + |\xi'|^2 - \partial_n^2)(\partial_n^2 - |\xi'|^2)\hat{v}_n(\lambda, \xi', x_n) = 0$. On the other hand, applying the Fourier transform to $\nabla\cdot\mathbf{v} = 0$ with respect to x' we have

$$\frac{\partial\hat{v}_n}{\partial x_n}(\lambda, \xi', x_n) = -\sum_{j=1}^{n-1} i\xi_j\hat{v}_j(\lambda, \xi', x_n).$$

Hence, we obtain the following boundary value problem of the ordinary differential equation of fourth order:

$$(3.6)\quad \begin{cases} (\partial_n^2 - A^2)(\partial_n^2 - B^2)\hat{v}_n(\lambda, \xi', x_n) = 0, \qquad 0 < x_n < 1, \\ \hat{v}_n|_{x_n=0} = \hat{g}_1, \quad \hat{v}_n|_{x_n=1} = \hat{g}_2, \\ \left.\dfrac{\partial\hat{v}_n}{\partial x_n}\right|_{x_n=0} = \hat{h}_1, \quad \left.\dfrac{\partial\hat{v}_n}{\partial x_n}\right|_{x_n=1} = \hat{h}_2, \end{cases}$$

$$A = |\xi'|, \quad B = \sqrt{\lambda + |\xi'|^2}, \quad \hat{g}_j = \hat{U}_n|_{x_n=j-1}\ (j=1,2), \quad \hat{h}_j = \sum_{k=1}^{n-1} i\xi_k \hat{U}_k|_{x_n=j-1}\ (j=1,2).$$

The fundamental solutions of the equation are $e^{-A(1-x_n)}$, e^{-Ax_n}, $e^{-B(1-x_n)}$ and e^{-Bx_n}, we look for the solution to (3.6) in the form of $\hat{v}_n(\lambda,\xi',x_n) = a_1e^{-A(1-x_n)} + a_2e^{-Ax_n} + a_3e^{-B(1-x_n)} + a_4e^{-Bx_n}$. By the boundary condition, the constants a_1, a_2, a_3, a_4 satisfy the following simultaneous linear equations:

$$L\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} \hat{g}_1 \\ \hat{g}_2 \\ \hat{h}_1 \\ \hat{h}_2 \end{pmatrix}, \qquad \text{where } L = \begin{pmatrix} e^{-A} & 1 & e^{-B} & 1 \\ 1 & e^{-A} & 1 & e^{-B} \\ Ae^{-A} & -A & Be^{-B} & -B \\ A & -Ae^{-A} & B & -Be^{-B} \end{pmatrix}.$$

By the simple argument we can show that if $\lambda \in \mathbb{C}\setminus(-\infty,0]$ and $\xi' \neq \mathbf{0}$ then $\det L \neq 0$. Therefore $\hat{v}_n(\lambda,\xi',x_n)$ is represented as

$$\begin{aligned}\hat{v}_n(\lambda,\xi',x_n) &= \sum_{j=1}^{2}\left\{\frac{\tilde{L}_{j1}e^{-A(1-x_n)}}{\det L} + \frac{\tilde{L}_{j2}e^{-Ax_n}}{\det L} + \frac{\tilde{L}_{j3}e^{-B(1-x_n)}}{\det L} + \frac{\tilde{L}_{j4}e^{-Bx_n}}{\det L}\right\}\hat{g}_j \\ &+ \sum_{j=1}^{2}\left\{\frac{\tilde{L}_{2+j,1}e^{-A(1-x_n)}}{\det L} + \frac{\tilde{L}_{2+j,2}e^{-Ax_n}}{\det L} + \frac{\tilde{L}_{2+j,3}e^{-B(1-x_n)}}{\det L} + \frac{\tilde{L}_{2+j,4}e^{-Bx_n}}{\det L}\right\}\hat{h}_j.\end{aligned}$$

Now, we calculate the determinant of the Lopatinski matrix L and its cofactors, and by the behavior of the denominator we classify the problem into some cases. Moreover, we estimate the coefficients of $\hat{g}_j, \hat{h}_j$ by using Lemma 2.1, to estimate v_n itself we apply the Proposition 2.1 and (3.4), to estimate the first and the second derivatives of v_n we apply the Lemma 2.2 and the estimate (3.2). After those tasks we obtain the estimate

$$|\lambda|\|v_n\|_{L^p(\Omega)} + |\lambda|^{\frac{1}{2}}\|\nabla v_n\|_{L^p(\Omega)} + \|\nabla^2 v_n\|_{L^p(\Omega)} \leq C_{p,n,\varepsilon,\lambda_0}\|\mathbf{f}\|_{L^p(\Omega)}. \tag{3.7}$$

Step 3. We shall construct the pressure π satisfying (3.3) and estimate the L^p-norm of $\nabla\pi$. By (3.5) and the n-th component of the first equation of (3.3), we construct π satisfying the following problem:

$$\begin{cases} \Delta\pi = 0 & \text{in } \Omega, \\ \left.\dfrac{\partial\pi}{\partial x_n}\right|_{x_n=a} = -(\lambda-\Delta)v_n|_{x_n=a}, & a = 0,1. \end{cases}$$

Applying the Fourier transform with respect to x' we obtain the following boundary value problem of the ordinary differential equation:

$$\begin{cases} (\partial_n^2 - A^2)\hat{\pi}(\lambda,\xi',x_n) = 0 & 0 < x_n < 1, \\ \left.\dfrac{\partial\hat{\pi}}{\partial x_n}\right|_{x_n=a} = (\partial_n^2 - B^2)\hat{v}_n|_{x_n=a}, & a = 0,1. \end{cases} \tag{3.8}$$

Solving (3.8) we obtain

$$\hat{\pi}(\lambda, \xi', x_n) = -\frac{\lambda}{A}\sum_{j=1}^{2}\left\{\frac{\tilde{L}_{j1}e^{-A(1-x_n)}}{\det L} - \frac{\tilde{L}_{j2}e^{-Ax_n}}{\det L}\right\}\hat{g}_j$$
$$-\frac{\lambda}{A}\sum_{j=1}^{2}\left\{\frac{\tilde{L}_{2+j,1}e^{-A(1-x_n)}}{\det L} - \frac{\tilde{L}_{2+j,2}e^{-Ax_n}}{\det L}\right\}\hat{h}_j.$$

Therefore, by an argument similar to those in Step 2, we obtain the estimate

$$\|\nabla\pi\|_{L^p(\Omega)} \leq C_{p,n,\varepsilon,\lambda_0}\|\mathbf{f}\|_{L_p(\Omega)}. \tag{3.9}$$

Step 4. We shall construct v_k $(k = 1, \cdots, n-1)$ satisfying (3.3) and estimate the L^p-norms of v_k, ∇v_k and $\nabla^2 v_k$. By the k-th component of the first equation of (3.3), we construct v_k satisfying the following problem:

$$\begin{cases} (\lambda - \Delta)v_k + \partial_k \pi = 0 & \text{in } \Omega, \\ v_k|_{x_n=a} = -U_k|_{x_n=a}, & a = 0, 1. \end{cases}$$

Applying the Fourier transform with respect to x' we obtain the following boundary value problem of the ordinary differential equation:

$$\begin{cases} (\partial_n^2 - B^2)\hat{v}_k(\xi', x_n) = i\xi_k\hat{\pi}(\xi', x_n) & 0 < x_n < 1, \\ \hat{v}_k|_{x_n=a} = -\hat{U}_k|_{x_n=a}, & a = 0, 1. \end{cases} \tag{3.10}$$

It is easier than the case where Step 2 and Step 3 to solve the above equation and obtain the estimate

$$|\lambda|\|v_k\|_{L^p(\Omega)} + |\lambda|^{\frac{1}{2}}\|\nabla v_k\|_{L^p(\Omega)} + \|\nabla^2 v_k\|_{L^p(\Omega)} \leq C_{p,n,\varepsilon,\lambda_0}\|\mathbf{f}\|_{L^p(\Omega)}. \tag{3.11}$$

(II) The case where $\lambda \in \mathbb{C}$is close to zero

When $\lambda = 0$, because of the singularity of $|\xi'|^{-1}$ at $\xi' = 0$, the solution $\mathbf{U}$ which is constructed in Case 1 is not in $L^p(\mathbb{R}^n)$, and $\nabla\mathbf{U}$ is not in $L^p(R^n)$, either. Therefore, $\lambda = 0$ is not in the resolvent set of the Stokes operator in the whole space. However, since Ω is bounded in x_n-direction, by using the Poincaré's inequality we can prove that $\lambda = 0$ is in the resolvent set of the Stokes operator on Ω in the L^2-framework. So, from now on we shall consider the case where the resolvent parameter $\lambda \in \mathbb{C}$is close to zero. If we prove that $\lambda = 0$ is in the resolvent set, then by the perturbation method we can easily prove that the sufficiently small neighborhood of zero is also in the resolvent set. Therefore, we consider only the case where $\lambda = 0$.

Step 1. Disregarding the boundary condition, we shall construct $(\mathbf{v}, \mathbf{q})$ satisfying

$$-\Delta\mathbf{v} + \nabla\mathbf{q} = \mathbf{f}, \quad \nabla\cdot\mathbf{v} = 0 \qquad \text{in } \Omega. \tag{3.12}$$

Applying the divergence to the first equation we obtain $\Delta \mathbf{q} = \nabla \cdot \mathbf{f}$. Hence, applying the Laplacian to the n-th component of the first equation of (3.12) we have $\Delta^2 v_n = -\Delta' f_n + \nabla' \cdot \partial_n \mathbf{f}'$. Then, applying the Fourier transform with respect to x' we obtain

$$(3.13)\qquad (\partial_n^2 - |\xi'|^2)^2 \hat{v}_n(\xi', x_n) = |\xi'|^2 \hat{f}_n(\xi', x_n) + i\xi' \cdot \partial_n \hat{\mathbf{f}}'(\xi', x_n), \quad 0 < x_n < 1.$$

Now, we solve this ordinary differential equation by the variation of constants. Then, for example, $\hat{v}_n$ is represented as

$$\begin{aligned}
\hat{v}_n(\xi', x_n) = &\frac{|\xi'|}{2} \int_0^{x_n} \int_0^1 \int_0^1 \theta e^{|\xi'|(x_n - t)(1-2\theta+2\theta\eta)} (x_n - t)^2 \hat{f}_n(\xi', t) d\eta d\theta dt \\
&- \frac{|\xi'|}{2} \int_0^{x_n} \int_0^1 \int_0^1 (1-\theta) e^{|\xi'|(x_n - t)\{1-2\theta-2\eta(1-\theta)\}} (x_n - t)^2 \hat{f}_n(\xi', t) d\eta d\theta dt \\
&+ \frac{i|\xi'|}{2} \int_0^{x_n} \int_0^1 e^{-|\xi'|(x_n - t)(1-2\theta)} (x_n - t)^2 \tilde{\xi}' \cdot \hat{\mathbf{f}}'(\xi', t) d\theta dt,
\end{aligned}$$

where $\tilde{\xi}' = \xi'/|\xi'|$. Since this representaion does not have an inverse power of $|\xi'|$, we can use this representation for the analysis of the case where $|\xi'|$ is small. Similarly, the representation of $\partial_n \hat{v}_n(\xi', x_n)$ and of the $\partial_n^2 \hat{v}_n(\xi', x_n)$ does not have the reciprocal of $|\xi'|$. Therefore, applying Proposition 2.1 we obtain

$$\left\| \mathcal{F}_{\xi'}^{-1} \left[\varphi_0(\xi') \hat{v}_n(\xi', x_n) \right] \right\|_{W_p^2(\Omega)} \leq C_{p,n} \|\mathbf{f}\|_{L^p(\Omega)},$$

where $\varphi_0 \in C_0^\infty(\mathbb{R}^{n-1})$ is a cut-off function such that $\varphi_0(\xi') = 1$ for $|\xi'| \leq 1$ and $\varphi_0(\xi') = 0$ for $|\xi'| \geq 2$.

On the other hand, in the case where $|\xi'|$ is large we obtain the following estimate by applying Proposition 2.1 to the representation of $\hat{v}_n$ which is obtained by applying the Fourier transform to (3.13) with respect to x_n:

$$\left\| \mathcal{F}_{\xi'}^{-1} \left[(1 - \varphi_0(\xi')) \hat{v}_n(\xi', x_n) \right] \right\|_{W_p^2(\Omega)} \leq C_{p,n} \|\mathbf{f}\|_{L^p(\Omega)}.$$

So, we obtain

$$(3.14)\qquad \|v_n\|_{W_p^2(\Omega)} \leq C_{p,n} \|\mathbf{f}\|_{L^p(\Omega)}.$$

By the same argument we also obtain the following estimates:

$$(3.15)\qquad \|\pi\|_{W_p^1(\Omega)} \leq C_{p,n} \|\mathbf{f}\|_{L^p(\Omega)},$$

$$(3.16)\qquad \|v_k\|_{W_p^2(\Omega)} \leq C_{p,n} \|\mathbf{f}\|_{L^p(\Omega)}.$$

Step 2. Setting $\mathbf{u} = \mathbf{v} + \mathbf{w}$ and $\mathfrak{p} = \mathfrak{q} + \pi$ in (1.2) with $\lambda = 0$, it is reduced to the problem for $\mathbf{w}$ and π:

$$(3.17)\qquad \begin{cases} -\Delta \mathbf{w} + \nabla \pi = \mathbf{0}, \quad \nabla \cdot \mathbf{w} = 0 & \text{in } \Omega, \\ \mathbf{w}|_{x_n=0} = -\mathbf{v}|_{x_n=0}, \quad \mathbf{w}|_{x_n=1} = -\mathbf{v}|_{x_n=1}. \end{cases}$$

By repeating an argument similar to those in Case 2, Case 3 and Case 4 of (I) we obtain

$$\|\mathbf{w}\|_{W_p^2(\Omega)} + \|\nabla\pi\|_{L^p(\Omega)} \leq C_{p,n}\|\mathbf{f}\|_{L^p(\Omega)}. \tag{3.18}$$

4. Application

As a simple application, we shall consider the L^p-stability of the Couette flow and of the Poiseuille flow. First, we consider the following initial boundary value problem of the Navier-Stokes equation:

$$\begin{cases} \mathbf{u}_t - \Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\mathfrak{p} = \mathbf{0}, \quad \nabla\cdot\mathbf{u} = 0 & \text{in } (0,\infty)\times\Omega, \\ \mathbf{u}|_{x_n=0} = k(1,0,\dots,0), \quad \mathbf{u}|_{x_n=1} = \mathbf{0}, & \\ \mathbf{u}(0,x) = \mathbf{a}(x) & \text{in } \Omega. \end{cases} \tag{4.1}$$

The pair of functions $\mathbf{v}(x) = k(1-x_n, 0,\cdots,0)$, $\mathfrak{q}(x) = \mathfrak{q}_0$ (const.), which is called Couette flow, is a solution to the corresponding stationary problem. Now, Setting $\mathbf{u}(t,x) = \mathbf{v}(x) + \mathbf{w}(t,x)$ and $\mathfrak{p}(t,x) = \mathfrak{q}(x) + \pi(t,x)$ in (4.1), the problem on the stability for (4.1) is reduced to the following problem for $\mathbf{w}$ and π:

$$\begin{cases} \mathbf{w}_t - \Delta\mathbf{w} + k(1-x_n)\dfrac{\partial\mathbf{w}}{\partial x_1} + w_n\dfrac{\partial\mathbf{v}}{\partial x_n} + (\mathbf{w}\cdot\nabla)\mathbf{w} + \nabla\pi = \mathbf{0} & \text{in } (0,\infty)\times\Omega, \\ \nabla\cdot\mathbf{w} = 0 & \text{in } (0,\infty)\times\Omega, \\ \mathbf{w}|_{x_n=0} = \mathbf{0}, \quad \mathbf{w}|_{x_n=1} = \mathbf{0}, & \\ \mathbf{w}(0,x) = \mathbf{a}(x) - \mathbf{v}(x) \equiv \mathbf{b}(x) & \text{in } \Omega. \end{cases} \tag{4.2}$$

To solve this problem we transform (4.2) into the integral equation:

$$\mathbf{w}(t,x) = e^{-tA}\mathbf{b} - \int_0^t e^{-(t-s)A}P\left\{k(1-x_n)\frac{\partial\mathbf{w}}{\partial x_1} + w_n\frac{\partial\mathbf{v}}{\partial x_n} + (\mathbf{w}\cdot\nabla)\mathbf{w}\right\}(s)ds \tag{4.3}$$

where P is the projection from $L^p(\Omega)$ onto $L^p_\sigma(\Omega)$. Taking into consideration the boundedness of Ω with respect to x_n and the exponential decay property of the analytic semigroup $\{e^{-tA}\}_{t\geq0}$ obtained in Theorem 1.2, and employing the similar argument to [4] we can obtain the unique time global solution to (4.3) under an assumption on smallness of $|k|$ and $\|\mathbf{b}\|_{L^n(\Omega)}$. To be more precise, there holds the following theorem.

Theorem 4.1. *There is a sufficiently small number $\varepsilon > 0$ such that if $|k| + \|\mathbf{b}\|_{L^n(\Omega)} \leq \varepsilon$, then there exists a unique time-global solution*

$$\mathbf{w}(t,\cdot) \in BC([0,\infty); L^n_\sigma(\Omega))$$

to (4.2) and for any $p > n$ there holds the estimate

$$e^{\delta t}\|\mathbf{w}(t)\|_{L^n(\Omega)} + t^{\frac{1}{2}-\frac{n}{2p}}e^{\delta t}\|\mathbf{w}(t)\|_{L^p(\Omega)} + t^{\frac{1}{2}}e^{\delta t}\|\nabla\mathbf{w}(t)\|_{L^n(\Omega)} \leq C, \qquad \forall t > 0.$$

The stability of the Poiseuille flow $\mathbf{v}(x) = k(x_n(1-x_n)/2, 0, \cdots, 0)$, $\mathfrak{q}(x) = kx_1$ for

$$(4.4)\qquad \begin{cases} \mathbf{u}_t - \Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\mathfrak{p} = \mathbf{0}, \quad \nabla\cdot\mathbf{u} = 0 & \text{in } (0,\infty)\times\Omega, \\ \mathbf{u}|_{x_n=0} = \mathbf{0}, \quad \mathbf{u}|_{x_n=1} = \mathbf{0}, & \\ \mathbf{u}(0,x) = \mathbf{a}(x) & \text{in } \Omega \end{cases}$$

is also proved similarly. Setting $\mathbf{u}(t,x) = \mathbf{v}(x) + \mathbf{w}(t,x)$ and $\mathfrak{p}(t,x) = \mathfrak{q}(x) + \pi(t,x)$ in (4.4), the problem on the stability of (4.4) is reduced to the following problem for $\mathbf{w}$ and π:

(4.5)

$$\begin{cases} \mathbf{w}_t - \Delta\mathbf{w} + \dfrac{k}{2}x_n(x_n-1)\dfrac{\partial\mathbf{w}}{\partial x_1} + w_n\dfrac{\partial\mathbf{v}}{\partial x_n} + (\mathbf{w}\cdot\nabla)\mathbf{w} + \nabla\pi = \mathbf{0} & \text{in } (0,\infty)\times\Omega, \\ \nabla\cdot\mathbf{w} = 0 & \text{in } (0,\infty)\times\Omega, \\ \mathbf{w}|_{x_n=0} = \mathbf{0}, \quad \mathbf{w}|_{x_n=1} = \mathbf{0}, & \\ \mathbf{w}(0,x) = \mathbf{a}(x) - \mathbf{v}(x) \equiv \mathbf{b}(x) & \text{in } \Omega. \end{cases}$$

Solving the corresponding integral equation we obtain the following theorem.

Theorem 4.2. *There is a sufficiently small number $\varepsilon > 0$ such that if $|k| + \|\mathbf{b}\|_{L^n(\Omega)} \le \varepsilon$, then there exists a unique time-global solution*

$$\mathbf{w}(t,\cdot) \in BC([0,\infty); L^n_\sigma(\Omega))$$

to (4.5) *and for any $p > n$ there holds the estimate*

$$e^{\delta t}\|\mathbf{w}(t)\|_{L^n(\Omega)} + t^{\frac{1}{2}-\frac{n}{2p}}e^{\delta t}\|\mathbf{w}(t)\|_{L^p(\Omega)} + t^{\frac{1}{2}}e^{\delta t}\|\nabla\mathbf{w}(t)\|_{L^n(\Omega)} \le C, \qquad \forall t > 0.$$

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