数理解析研究所講究録 1234 巻 2001 年 206-212

ON THE CAUCHY PROBLEM IN THE MAXWELL-CHERN-SIMONS-HIGGS SYSTEM

DONGHO CHAE†, MYEONGJU CHAE††

ABSTRACT. In this paper we shall prove the global existence of solutions of the classical Maxwell-Chern-Simons-Higgs equations in $(2+1)$-dimensional Minkowski spacetime in the temporal gauge. We also prove that the topological solution of the Maxwell-Chern-Simons-Higgs system converges to that of Maxwell-Higgs system, as κ goes to zero. Thus we reproduce the classical result by Mongrief [6] on the global existence of the Maxwell-Klein-Gordon system in $(2+1)$dimension.

1. INTRODUCTION AND MAIN RESULTS

We are concerned on the global existence problem for the Maxwell-Chern-Simons-Higgs model in $(2+1)$-spacetime which was introduced to consider a self-dual system having both Maxwell and Chern-Simons terms [1]. The Lagrangian is

$$\begin{aligned}\mathcal{L}=&-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\frac{\kappa}{4}\epsilon^{\mu\nu\rho}F_{\mu\nu}A_{\rho}-\langle D_{\mu}\phi,D^{\mu}\phi\rangle-\frac{1}{2}\partial_{\mu}N\partial^{\mu}N\\&-\frac{1}{2}(e|\phi|^{2}+\kappa N-ev^{2})^{2}-e^{2}N^{2}|\phi|^{2},\end{aligned}\tag{1.1}$$

where $g_{\mu\nu}=\mathrm{diag}(1,-1,-1)$, ϕ is a complex scalar field, N is a real scalar field, $A=(A_0,A_1,A_2)$ is a vector field, $F_{\mu\nu}=\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}$,$D_{\mu}=\partial_{\mu}-ieA_{j}$, e is the charge of the electron, and κ is a coupling constant for the Chern-Simons term.

The Euler-Lagrange equations via variation of the action taken with respect to (A,ϕ,N) are

$$\begin{aligned}&\partial_{\lambda}F^{\lambda\rho}+\frac{\kappa}{2}\epsilon^{\mu\nu\rho}F_{\mu\nu}+2eIm(\phi\overline{D^{\rho}\phi})=0,\\&D_{\mu}D^{\mu}\phi+U_{\overline{\phi}}(|\phi|^{2},N)=0,\\&\partial_{\mu}\partial^{\mu}N+U_{N}=0.\end{aligned}\tag{1.2}$$

Letting $\rho=0$ in (1.2), we obtain the Gauss-Law constraint

$$\partial_{j}F_{j0}-\kappa F_{12}-2eIm(\phi\overline{D_{O}\phi})=0.\tag{1.3}$$

† partially supported by the BK 21 Project.

The static energy functional for the system is

$$E = \int_{\mathbf{R}^2} \frac{1}{2}F_{0i}^2 + \frac{1}{2}F_{12}^2 + |D_\mu\phi|^2 + |\partial_\mu N|^2 + U(|\phi|^2, N), \tag{1.4}$$

where $U(|\phi|^2, N) = \frac{1}{2}(e|\phi|^2 + \kappa N - ev^2)^2 + e^2N^2|\phi|^2$, $(i = 1, 2, \mu = 0, 1, 2)$. We note that, if (A, ϕ, N) is a solution that makes E finite, one of the following conditions should be required;

$$\phi \to 0 \qquad \text{and } N \to \frac{ev^2}{\kappa} \qquad (\text{ non-topological }) \tag{1.5}$$

$$|\phi|^2 \to v^2 \qquad \text{and } N \to 0 \qquad (\text{ topological }). \tag{1.6}$$

The terms of non-topological solution refers to the solution satisfying (1.5) and topological solution to the solution satisfying (1.6) . [1], [2]

In the static case, above system are reduced to the system of elliptic equation. The static energy functional is

$$E = \int |(D_1 \pm iD_2)\phi|^2 + |D_0\phi \mp ie\phi N|^2$$

$$\frac{1}{2}|F_{12} \pm (e|\phi|^2 + \kappa N - e)|^2 do \pm e \int F_{12} do.$$

The solution saturating a lower bound for the energy is called self-dual solution, which studied extensively on both two conditions (1.5), (1.6) by D. Chae *et al.* ([2], [4]), and on a periodic boundary condition, by Tarantello [5]. They also studied the unifying feature of Maxwell-Chern-Simons-Higgs mathematically, which was formally discribed in [1].

For a time dependent solution to the Maxwell-Chern-Simons-Higgs, there is no result as we know, however, in [6], Mongrief proved the global existence for the classical Maxwell-Klein-Gordon equations using the Lorents gauge in (2+1) spacetime. The Lagrangian of the Maxwell-Klein-Gordon is

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \langle D_\mu\phi, D^\mu\phi\rangle.$$

He proved global existence by showing that a suitably defined higher order energy, though not strictly conserved, does not blow up in a finite time. In this article, we consider the global existence of the classical Maxwell-Chern-Simons-Higgs in the temporal gauge as well as a convergent result as the static case [3].

Before presenting main theorems, we state equations corresponding to the non-topological case in the temporal gauge.

Considering the non-topological solution of (1.2), (1.3), we put $\widetilde{N}$ to

$N - \frac{ev^2}{\kappa}$ in (1.2) to obtain the following system of semilinear wave equations with constraint($\Box = \partial_{tt} - \Delta$).

$$\Box A_1 = -\kappa\partial_0 A_2 + 2eIm(\phi\overline{D_1\phi}), \tag{1.7}$$
$$\Box A_2 = \kappa\partial_0 A_1 + 2eIm(\phi\overline{D_2\phi}),$$
$$\Box\phi = -ie\phi\partial_j A_j - 2ieA_j\partial_j\phi - e^2 {A_j}^2\phi - U_{\overline{\phi}}$$
$$\Box N = -U_{\overline{N}},$$
$$\partial_j F_{j0} - \kappa F_{12} - 2eIm(\phi\overline{\psi_0}) = 0. \tag{1.8}$$

Above equations can be rewritten as Hamiltonian formalism;

$$\begin{aligned}
\partial_0 A_j &= F_{0j}\\
\partial_0 F_{0j} &= -\epsilon^{jk}\partial_k F_{12} - \kappa\epsilon^{jk}F_{0k} - 2eIm(\phi\overline{\psi_j})\\
\partial_0 F_{12} &= \epsilon^{ij}\partial_i F_{0j}\\
\partial_0\phi &= \psi_0\\
\partial_0\psi_0 &= D_j\psi_j - U_{\overline{\psi}}\\
\partial_0\psi_j &= D_j\psi_0 - ieF_{0j}\phi\\
\partial_0 N &= \Omega_0\\
\partial_0\Omega_0 &= \partial_j\Omega_j - U_N\\
\partial_0\Omega_j &= \partial_j\Omega_0
\end{aligned} \tag{1.9}$$

supplemented by constrains,

$$\begin{aligned}
F_{jk} &= \partial_j A_k - \partial_k A_j\\
D_j\phi &= \psi_j\\
\partial_j N &= \Omega_j\\
\partial_j F_{j0} - \kappa F_{12} - 2eIm(\phi\overline{\psi_0}) &= 0.
\end{aligned} \tag{1.10}$$

For the topological solution we also have the equations corresponding to (1.7), (1.9) by introducing a new variable φ such that $\varphi + \lambda = \phi$ to give a natural boundary conditions to (1.2). Let us remark on some notations. If no confuses are arisen, u means a triple $(A, \phi, \widetilde{N})$ or (A, φ, N),

$$\|u(t,\cdot)\|_{H^s} = \|A(t,\cdot)\|_{H^s} + \|\phi(t,\cdot)\|_{H^s} + \|N(t,\cdot)\|_{H^s},$$
$$\|\partial_0 u(t,\cdot)\|_{H^s} = \|\partial A(t,\cdot)\|_{H^s} + \|\partial\phi(t,\cdot)\|_{H^s} + \|\partial N(t,\cdot)\|_{H^s},$$
$$\|u(t,\cdot)\|_{H^s\times H^{s-1}} = \|u(t,\cdot)\|_{H^s} + \|\partial_0 u(t,\cdot)\|_{H^{s-1}}.$$

Followings are our main theorems.

Theorem 1.1. *(Global smooth solutions) Consider Maxwell-Chern-Simons-Higgs. Then any finite energy H^s initial data set ($s \geq 2$) admits a unique, global solution in the temporal gauge.*

$$A, \phi, \widetilde{N} \in C([0,\infty); H^s(R^2)) \cap C^1([0,\infty); H^{s-1}(R^2))$$

in the non-topological case. Also in the same gauge, any finite energy H^s initial data set ($s \geq 2$) admits a unique, global solution

$$A, \varphi, N \in C([0,\infty); H^s(R^2)) \cap C^1([0,\infty); H^{s-1}(R^2))$$

in the topological case.

Theorem 1.2. *(Maxwell-Higgs Limit) Consider the topological case of Maxwell-Chern-Simons-Higgs. Let u_κ be the global solution with coupling constant κ of $H^s(s \geq 2)$ initial data u_0. Then $\|u_\kappa(t) - u(t)\|_{H^s} \to 0$ as $\kappa \to 0$. In the case of $\kappa = 0$, if we set N initially zero then $N(t) = 0$ for all t.*

Remark 1. In a succeeding section, we present the proof of the non-topological case only in Theorem 1.1 since the finite energy solution of the topological one can be found in the same way as non-topological case.

2. Outline of the proofs

i) local in time existence

Proposition 2.1. *Given a data set $(A, \phi, \widetilde{N}) \in H^s(s \geq 2)$ at $t = 0$, there exists T^* depending only on $\|(A, \phi, \widetilde{N})(0, \cdot)\|_{H^s}$ and a unique development $(A, \phi, \widetilde{N})$ in the temporal gauge with*

$$(A, \phi, \widetilde{N}) \in C([0,T^*); H^s(R^2)) \cap C^1([0,T^*); H^{s-1}(R^2)).$$

This solution can be continued as long as $\|(A, \phi, \widetilde{N})\|_{H^s}(t)$ remains bounded.

First we show that there exists T^* such that (1.7) has a unique solution in X_T,

$$X_T = \{(u, \partial_0 u) \in C([0,T^*); H^2 \times H^1) : \|u\|_{X_T} < \infty\},$$

where $\|u\|_{X_T} = \sup_{0 \leq t \leq T} \|u(t, \cdot)\|_{H^2 \times H^1}$. The solution is obtained by standard contraction argument using energy estimates, and has continuous dependence on the initial data. This solution can be continued as long as $\|u_n(t, \cdot)\|_{H^2 \times H^1}$ remains bounded. To complete the local existence of Maxwell-Chern-Simons-Higgs, we also show that the constraint (1.8) is preserved in time.

ii) global in time existence

The proofs follow Mongrief's method mentioned earlier and use usual a priori estimates to show $\|u(t,\cdot)\|_{H^2\times H^1}$ dose not blow up in a finite time.

Let $u=(A,\phi,N)\in C([0,T);H^s(R^2))\cap C^1([0,T);H^{s-1}(R^2))$ be a solution of Maxwell-Chern-Simons-Higgs obtained in the part *i)*. Define $E(t),E_1(t),F(t)$ as such

$$E(t)=\int_{\mathbf{R}^2}\frac{1}{2}F_{0i}^2+\frac{1}{2}F_{12}^2+|D_\mu\phi|^2+|\partial_\mu N|^2+U(|\phi|^2,N), \tag{2.1}$$

$$E_1(t)=\int_{\mathbf{R}^2}\frac{1}{2}(\partial_l F_{0i})^2+\frac{1}{2}(\partial_l F_{12})^2+|D_l\psi_\mu|^2+(\partial_l\Omega_\mu)^2,$$

$$\text{where } \psi_\mu=D_\mu\phi\ ,\ \Omega_\mu=\partial_\mu N, \tag{2.2}$$

$$F_1(t)=\|\partial_i A\|_{L^2}(t)+\|\partial_i\phi\|_{L^2}(t)+\|\partial_i N\|_{L^2}(t). \tag{2.3}$$

The global result will be established after following Lemmas.

Lemma 2.2. *Let* $u=(A,\phi,\widetilde{N})\in C([0,T);H^2(R^2))\cap C^1([0,T);H^1(R^2))$ *be a solution of* (1.7), (1.8). *Then*

(1) $E(t)=E(0)$ *for all* $t\in[0,T)$

(2) $(A,\phi,\widetilde{N})$ *are estimated in* L^2 *in terms of the initial data for all* $t\in[0,T)$*;*

$$\|u(t,\cdot)\|_{L^2}\le\|u(0,\cdot)\|_{L^2}+tE. \tag{2.4}$$

Lemma 2.3. (1) $E_1(t)$ *is differentiable for all* $t\in[0,T)$ *and satisfies*

$$\begin{aligned}\partial_0E_1(t)=\int_{\mathbf{R}^2}&-\kappa\epsilon^{ik}\partial_lF_{0i}\partial_lF_{0k}-2eIm(\psi_l\overline{\psi_i}+\phi\overline{D_l\psi_i})\\&+2Re(\overline{D_l\psi_0}\cdot ieF_{lj}\psi_j-D_lU_{\overline{\phi}}+ieF_{0l}\psi_0)\\&+2Re(\overline{D_l\psi_j}\cdot ieF_{lj}\psi_0-D_l(F_{0j}\phi)+ieF_{0l}\psi_j)-2\partial_lU_N\partial_l\Omega_0.\end{aligned} \tag{2.5}$$

(2) $E_1(t)$, $F_1(t)$ *are estimated in terms of the initial data for all* $t\in[0,T)$*,*

$$E_1(t)\le C(E,E_1(0))(1+t)^2, \tag{2.6}$$

$$F(t)\le C(E,E_1(0),F(0))(1+t)^{\frac{5}{2}}. \tag{2.7}$$

It is not clear the energy norm, $\|u(t,\cdot)\|_{H^1}$ dose not blow up in the temporal gauge, though the energy itself is preserved in Lemma 2.2. In Lemma 2.3, $E_1(t)$, the higher order energy, is shown to be initially bounded, from which $F_1(t)=\|\partial u(t,\cdot)\|_{H^1}$ can be easily estimated in terms of the initial data. Combining (2.4), (2.7) we have $\|u(t,\cdot)\|_{H^1}$ is initially bounded.

Proving two Lemmas we use the Hamiltonian formalism of this system (1.9) after taking time derivatives in $E(t), E_1(t), F(t)$. We depend on the covariant Sobolev inequalities in [7] (see Appendix of it), estimating each terms of $\partial_0 E_1(t)$ to show the right hand terms of (2.5) are at most linear with right to $E_1(t)$. Next we introduce Brezis-Gallouet inequality.

Lemma 2.4. [8] $s > 1$,

$$\|u\|_{L^\infty} \le C\|u\|_{H^1}(1+\sqrt{\log(1+\|u\|_{H^s}}).$$

Finally we carried out a priori estimate $\|u(t,\cdot)\|_{H^2\times H^1}$ to get

$$\begin{aligned}\|u\|_{H^2}(t) + \|\partial_0 u\|_{H^1}(t) \le \|u\|_{H^2}(0) + C\int_0^t \|\partial_0 u\|_{H^1} \\ + ((1+\|u\|_{L^\infty})^2 + \|u\|_{H^1}^{\frac{1}{2}}(1+\|u\|_{L^2}))\|u\|_{H^1} \\ + \|u\|_{L^\infty}\|u\|_{H^1}^2\|u\|_{H^2}^{\frac{1}{2}} + (1+\|u\|_{L^\infty})\|u\|_{H^1}^2\end{aligned} \tag{2.8}$$

by energy estimates to (1.7), then we have

$$\|u(0,\cdot)\|_{H^2} + C(t)\int_0^t \log(1+\|u(s,\cdot)\|_{H^2})\|u(s,\cdot)\|_{H^2\times H^1}ds. \tag{2.9}$$

applying above Brezis-Gallouet inequality. The desired result, thus, is given by the general Gronwall inequality.

For the case of an initial data $u \in H^s(\mathbf{R}^2)$, it is easy to obtain a local existence result as proposition 2.1. For a global result we state a next lemma omitting its simple proof.

Lemma 2.5. *Let* $(A,\phi,N) \in C([0,T);H^s(R^2)) \cap C^1([0,T);H^{s-1}(R^2))$ *be a solution of* (1.7), (1.8) *for* $s > 2$ *then* $\|u\|_{H^s}(t)$ *is estimated in terms of the initial data for all* $t \in [0,T)$.

iii) Maxwell-Higgs limit

Let u^κ be a topological solution of Maxwell-Chern-Simons-Higgs obtained in Theorem1.1 with coupling constant κ of H^2 initial data u_0. It is easy to show $\|(u^\kappa - u)(t,\cdot)\|_{H^2\times H^1}$ is estimated to be

$$\|(u^\kappa - u)(t,\cdot)\|_{H^2\times H^1} \le \int_0^t \kappa\|\partial_0 A_\kappa\|_{H_1} + C(t)\|u^\kappa - u\|_{H^2}, \tag{2.10}$$

using $\sup_{0\le\kappa\le 1}\|u^\kappa(t,\cdot)\|_{H^2\times H^1} \le C(t)$ for a smooth function in the proof of Theorem 1.1. Then applying Gronwall inequality to (2.10), we have

$$\|(u^\kappa - u)(t,\cdot)\|_{H^2} \le \kappa C(t) + \|(u^\kappa - u)(0,\cdot)\|_{H^2},$$

letting $\kappa \to 0$, we obtain the desired result.

References

[1] Lee, C., Lee, K., Min, H., *Self-dual Maxwell Chern-Simons solitons*, Phys. Lett. B **252** (1990), no.1, 99–83.

[2] Chae, D., Kim, N., *Topological multivortex solutions of the self-dual Maxwell-Chern-Simons-Higgs system*, J. Differential Equations **134** (1997), no.1, 154–182.

[3] Chae, D., Chae, M., *The global existence of the Maxwell-Chern-Simons-Higgs system*, preprint.

[4] Chae, D., Imanuvilov, O. *The existence of non-topological multivortex solutions in the relativistic self-dual Chern-Simons theory*, Comm. Math. Phys., **215** (2000), no.1, 119–142.

[5] Tarantello, G.,Ricciardi, T., *Vortices in the Maxwell-Chern-Simons Theory*, Comm. Pure Appl. Math. **53** (2000), no.7, 811–851.

[6] Moncrief, V., *Global existence of Maxwell-Klein-Gordon fields in* $(2+1)$-*dimensional spacetime*, J. Math. Phys., **21** (1980), no.9, 811–851.

[7] Ginibre, J., Velo, G., *The Cauchy problem for coupled Yang-Mills and scalar fields in the temporal gauge*, Comm. Math. Phys., **82** (1981), no.1, 1–28.

[8] Brézis, H., Gallouet, T., *Nonlinear Schrödinger evolution equations*, Nonlinear Anal. **4** (1980), no.4, 677–681.

Department of Mathematics, Seoul National University, Seoul 151-742, Korea
E-mail address: `dhchae@math.snu.ac.kr`

Department of Mathematics, Seoul National University, Seoul 151-742, Korea
E-mail address: `mjchae@math.snu.ac.kr`