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ON THE STEADY FLOW OF COMPRESSIBLE VISCOUS FLUID AND ITS STABILITY WITH RESPECT TO INITIAL DISTURBANCE

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1 Introduction

This note is based on a joint work with Prof. Y. Shibata, Waseda University [7].

The motion of a compressible viscous isotropic Newtonian fluid is formulated by the following initial value problem of the Navier-Stokes equation for viscous compressible fluid:

$$\begin{cases} \rho_t + \nabla\cdot(\rho v) = G(x), \\ v_t + (v\cdot\nabla)v = \dfrac{\mu}{\rho}\Delta v + \dfrac{\mu+\mu'}{\rho}\nabla(\nabla\cdot v) - \dfrac{\nabla(P(\rho))}{\rho} + F(x), \\ (\rho, v)(0,x) = (\rho_0, v_0)(x), \end{cases} \tag{1.1}$$

where $t \geq 0$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$; $\rho = \rho(t,x)\,(>0)$ and $v = (v_1(t,x), v_2(t,x), v_3(t,x))$ denote the density and velocity respectively, which are unknown; $P(\cdot)$ $(P' > 0)$ denotes the pressure; μ and μ' are the viscosity coefficients which satisfy the condition: $\mu > 0$ and $\mu' + 2\mu/3 \geq 0$; $F(x) = (F_1(x), F_2(x), F_3(x))$ is a given external force and $G(x)$ is a given mass source. The stationary problem corresponding to the initial value problem (1.1) is

$$\begin{cases} \nabla\cdot(\rho v) = G(x), \\ (v\cdot\nabla)v = \dfrac{\mu}{\rho}\Delta v + \dfrac{\mu+\mu'}{\rho}\nabla(\nabla\cdot v) - \dfrac{\nabla(P(\rho))}{\rho} + F(x), \end{cases} \tag{1.2}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$; $\rho = \rho(x)\,(>0)$ and $v = (v_1(x), v_2(x), v_3(x))$ are unknown functions; $F(x)$, $G(x)$ and the other symbols are the same as in (1.1). In this note, we consider the case where the external force F is given by following form

$$F = \nabla\cdot F_1 + F_2. \tag{1.3}$$

Before stating our results, we introduce some function spaces. Let L_p denote the usual L_p space, $\mathscr{S}'$ the set of all tempered distributions both on $\mathbb{R}^3$. We put

$$H^k = \{\, u \in L_{1,loc} \mid \|u\|_k < \infty \,\} = \{\, u \in \mathscr{S}' \mid \|\mathscr{F}^{-1}[(1+|\xi|^2)^{k/2}\hat{u}]\| < \infty \,\},$$
$$\hat{H}^k = \{\, u \in L_{1,loc} \mid \nabla u \in H^{k-1} \,\}, \quad \|u\| = \|u\|_{L_2}, \quad \|u\|_k = \textstyle\sum_{\nu=0}^k \|\nabla^\nu u\|_{L_2}$$

and furthermore for short we use the notation:

$$\mathscr{H}^{k,\ell} = \{\, (\sigma, v) \mid \sigma \in H^k,\ v \in H^\ell \,\}, \quad \hat{\mathscr{H}}^{k,\ell} = \{\, (\sigma, v) \mid \sigma \in \hat{H}^k,\ v \in \hat{H}^\ell \,\}$$
$$\mathscr{H}^{j,k,\ell} = \{\, (\sigma, v, w) \mid \sigma \in H^j,\ v \in H^k,\ w \in H^\ell \,\},$$
$$\|(\sigma, v)\|_{k,\ell} = \|\sigma\|_k + \|v\|_\ell, \quad \|(\sigma, v, w)\|_{j,k,\ell} = \|\sigma\|_j + \|v\|_k + \|w\|_\ell.$$

Definition 1

$$I_\epsilon^k = \{\, \sigma \in H^k \mid \|\sigma\|_{I^k} < \epsilon \,\}, \quad J_\epsilon^k = \{\, u \in \hat{H}^k \mid \|v\|_{J^k} < \epsilon \,\},$$

where

$$\|\sigma\|_{I^k} = \|\sigma\|_{L_6} + \left\|\frac{\sigma}{|x|}\right\| + \textstyle\sum_{\nu=1}^k \|(1+|x|)^\nu \nabla^\nu \sigma\| + \|(1+|x|)^2 \sigma\|_{L_\infty},$$

$$\|v\|_{J^k} = \|v\|_{L_6} + \left\|\frac{v}{|x|}\right\| + \textstyle\sum_{\nu=1}^k \|(1+|x|)^{\nu-1} \nabla^\nu v\| + \sum_{\nu=0}^1 \|(1+|x|)^{\nu+1} \nabla^\nu v\|_{L_\infty}.$$

Moreover we put

$$\mathscr{I}_\epsilon^{k,\ell} = \{\, (\sigma, v) \mid \sigma \in I_\epsilon^k,\ v \in J_\epsilon^\ell \,\},$$

$$\dot{\mathscr{I}}_\epsilon^{k,\ell} = \{\, (\sigma, v) \in \mathscr{I}_\epsilon^{k,\ell} \mid \nabla\cdot v = \nabla\cdot V_1 + V_2 \quad \textit{for some} \ \ V_1,\ V_2$$
$$\textit{such that} \ \ \|(1+|x|)^3 V_1\|_{L_\infty} + \|(1+|x|)^{-1} V_2\|_{L_1} \le \epsilon \,\},$$

$$\|(\sigma, v)\|_{\mathscr{I}^{k,\ell}} = \|\sigma\|_{I^k} + \|v\|_{J^\ell}.$$

The first theorem is about the existence of stationary solution for (1.2) and its weighted-L_2, L_∞ estimates.

Theorem 1 *Let $\bar\rho$ be any positive constant. Then, there exist small constants $c_0 > 0$ and $\epsilon > 0$ depending on $\bar\rho$ such that if F and G satisfy the estimate:*

$$\textstyle\sum_{\nu=0}^3 \|(1+|x|)^{\nu+1}\nabla^\nu F\| + \|(1+|x|)^3 F\|_{L_\infty} + \|(1+|x|)^2 F_1\|_{L_\infty} + \|F_2\|_{L_1}$$
$$+\|(1+|x|)G\| + \textstyle\sum_{\nu=1}^4 \|(1+|x|)^\nu \nabla^\nu G\|$$
$$+\textstyle\sum_{\nu=0}^1 \|(1+|x|)^{\nu+2}\nabla^\nu G\|_{L_\infty} + \|(1+|x|)^{-1} G\|_{L_1} \le c_0 \epsilon,$$

then (1.2) admits a solution of the form: $(\rho, v) = (\bar\rho + \sigma, v)$ where $(\sigma, v) \in \dot{\mathscr{I}}_\epsilon^{4,5}$. Furthermore the solution is unique in the following sense:

> *There exists an ϵ_1 with $0 < \epsilon_1 \le \epsilon$ such that if $(\bar\rho + \sigma_1, v_1)$ and $(\bar\rho + \sigma_2, v_2)$ satisfy (1.2) with the same F and G, and $\|(\sigma_1, v_1)\|_{\mathscr{I}^{3,4}}, \|(\sigma_1, v_1)\|_{\mathscr{I}^{3,4}} \le \epsilon_1$, then $(\sigma_1, v_1) = (\sigma_2, v_2)$.*

Next we consider the stability of the stationary solution of (1.2) with respect to initial disturbance. Let (ρ^*, v^*) be a solution of (1.2). The stability of (ρ^*, v^*) means the solvability of the non-stationary problem (1.1). Let us introduce the class of functions which solutions of (1.1) belong to.

Definition 2

$$\mathscr{C}(0,T\,;\mathscr{H}^{k,\ell}) = \{\, (\sigma, v) \mid \sigma(t,x) \in C^0(0,T\,;H^k) \cap C^1(0,T\,;H^{k-1}),$$
$$w(t,x) \in C^0(0,T\,;H^\ell) \cap C^1(0,T\,;H^{\ell-2}) \,\}.$$

Then, we have the following theorem.

Theorem 2 *There exist $C > 0$ and $\delta > 0$ such that if $\|(\rho_0 - \rho^*, v_0 - v^*)\|_{3,3} \le \delta$ then (1.1) admits a unique solution: $(\rho, v) = (\rho^* + \sigma, v^* + w)$ globally in time, where $(\sigma, w) \in \mathscr{C}(0,\infty\,;\mathscr{H}^{3,3})$, $\nabla\sigma$, $w_t \in L_2(0,\infty\,;H^2)$, $\nabla w \in L_2(0,\infty\,;H^3)$. Moreover the (σ, w) satisfies the estimate:*

$$\|(\sigma, w)(t)\|_{3,3}^2 + \int_0^t \|(\nabla\sigma, \nabla w, w_t)(s)\|_{2,3,2}^2\, ds \le C\|(\rho_0 - \rho^*, v_0 - v^*)\|_{3,3}^2 \tag{1.4}$$

for any $t \ge 0$.

Remark 1 When Theorem 1.2 holds, we shall say that the stationary solution (ρ^*, v^*) of (1.2) is stable in the H^3-framework with respect to small initial disturbance.

Matsumura and Nishida [4] first proved the stability of constant state $(\bar{\rho}, 0)$ in H^3-framework with respect to initial disturbance, namely they proved Theorem 1.2 in the case where $(\rho^*, v^*) = (\bar{\rho}, 0)$. When the external force is given by the potential: $F = \nabla\Phi$, $F_2 = G = 0$ in (1.2) and (1.3) where Φ is a scalar function, the stationary solution $(\rho^*, v^*)(x)$ of (1.2) in a neighborhood of $(\bar{\rho}, 0)$ in $\mathscr{H}^{2,2}$ has the form:

$$\int_{\bar{\rho}}^{\rho^*(x)} \frac{P'(\eta)}{\eta}\, d\eta + \Phi(x) = 0, \quad v^*(x) = 0.$$

In this case, Matsumura and Nishida [5] proved the stability of $(\rho^*(x), 0)$ in the H^3-framework with respect to initial disturbance in an exterior domain.

The purpose of this note is to consider the case where the external force is given by the general formula (1.3) and also mass source G appears. In this case, the stationary solution $(\rho^*, v^*)(x)$ are both non-trivial in general. We are interested only in strong solutions. Then, when F is small enough in a certain norm and $G = 0$, Novotny and Padula [6] proved a unique existence theorem of solutions to (1.2) in an exterior domain. In their proof, they decomposed the equations into the Stokes equation, transport equation and Laplace equation. Since we consider the problem in $\mathbb{R}^3$, that is, the boundary condition is not imposed, we can solve (1.2) without any such decomposition technique. In fact, in §2, we establish the corresponding linear theory to (1.2) in the L_2-framework by the usual Banach closed range theorem, after obtaining some weighted-L_2 estimates for solutions.

The stability of the stationary solutions $(\rho^*, v^*)(x)$ of (1.2) in H^3-framework has not been studied yet. As we stated in Remark 1, Theorem 2 tells us the stability of stationary solutions $(\rho^*, v^*)(x)$ in H^3-framework. The main step of our proof of Theorem 2 is to obtain a priori estimate for solutions of (1.1) as usual. In §3, we shall obtain a priori estimates by choosing several multipliers and using the integration by parts. Compared with the case where $v^* = 0$, we have to give more consideration to choice of multipliers.

Recently, Kawashita [3] and Danchin [1,2] consider the optimal class of initial data regarding the regularity. We think that our result will be improved in this direction.

2 Sketch of proof of Theorem 1

Now, we shall give a rough idea of proof of Theorem 1. Take any constant $\bar{\rho} > 0$. Substituting $\rho = \bar{\rho} + \sigma$ into (1.2) and putting $\gamma = P'(\bar{\rho})$, (1.2) is reduced to the equation:

$$\begin{cases} \nabla\cdot v + \Big(\dfrac{v}{\bar{\rho}+\sigma}\cdot\nabla\Big)\sigma = \dfrac{G}{\bar{\rho}+\sigma}, \\ -\mu\Delta v - (\mu+\mu')\nabla(\nabla\cdot v) + \gamma\nabla\sigma = -(\bar{\rho}+\sigma)(v\cdot\nabla)v \\ \qquad\qquad - \big[P'(\bar{\rho}+\sigma) - P'(\bar{\rho})\big]\nabla\sigma + (\bar{\rho}+\sigma)F. \end{cases} \tag{2.1}$$

We consider the following linearized equation:

$$\begin{cases} \nabla\cdot v + (a\cdot\nabla)\sigma = g, & (2.2) \\ -\mu\Delta v - (\mu+\mu')\nabla(\nabla\cdot v) + \gamma\nabla\sigma = f, & (2.3) \end{cases}$$

where $(\tilde{\sigma}, \tilde{v})(x) \in \dot{\mathscr{I}}_{\epsilon}^{4,5}$ is given and a, f, g is defined by

$$a = \frac{\tilde{v}}{\bar{\rho}+\tilde{\sigma}}, \quad f = -\bar{\rho}(\tilde{v}\cdot\nabla)\tilde{v} + f_*, \quad g = \frac{G}{\bar{\rho}+\tilde{\sigma}},$$
$$f_* = -\tilde{\sigma}(\tilde{v}\cdot\nabla)\tilde{v} - \big[P'(\bar{\rho}+\tilde{\sigma}) - P'(\bar{\rho})\big]\nabla\tilde{\sigma} + (\bar{\rho}+\tilde{\sigma})F.$$

By a successive approximation method based on the L_2 estimate, weighted-L_2 estimate and L_∞ estimate, we construct the stationary solution to (2.1).

L_2 estimate: First, we estimate L_2 norm of the solution by using the energy method. Multiplying (2.2) and (2.3) by σ and v respectively, and using integration by parts, we have

$$(f,v) = \mu\|\nabla v\|^2 + (\mu+\mu')\|\nabla\cdot v\|^2 + \gamma(\nabla\sigma, v),$$
$$(g,\sigma) = -(v,\nabla\sigma) + (a\cdot\nabla\sigma,\sigma).$$

Canceling the term of $(\nabla\sigma, v)$ in the above two relations, we obtain

$$\mu\|\nabla v\|^2 \le \gamma|(a\cdot\nabla\sigma,\sigma)| + |(f,v)| + \gamma|(g,\sigma)|.$$

Differentiating (2.2)–(2.3), and employing the same argument, we have

$$\mu\|\nabla^2 v\|^2 \le \gamma|(\nabla(a\cdot\nabla\sigma),\nabla\sigma)| + |(\nabla f,\nabla v)| + \gamma|(\nabla g,\nabla\sigma)|.$$

Adding the above two inequalities, we have

$$\mu\|\nabla v\|_1^2 \le \sum_{\nu=0}^{1}\Big[\gamma|(\nabla^\nu(a\cdot\nabla\sigma),\nabla^\nu\sigma)| + |(\nabla^\nu f,\nabla^\nu v)| + \gamma|(\nabla^\nu g,\nabla^\nu\sigma)|\Big]. \tag{2.4}$$

Since

$$\|\nabla\sigma\|^2 \le C_{\gamma,\mu,\mu'}\{\|\nabla^2 v\|^2 + \|f\|^2\}$$

as follows from (2.3), it follows from (2.4) that

$$\begin{aligned}\|(\nabla\sigma,\nabla v)\|_{0,1}^2 &\le C\sum_{\nu=0}^{1}|(\nabla^\nu(a\cdot\nabla\sigma),\nabla^\nu\sigma)| \\ &+ C\Big[\|f\|^2 + \sum_{\nu=0}^{1}\{|(\nabla^\nu f,\nabla^\nu v)| + |(\nabla^\nu g,\nabla^\nu\sigma)|\}\Big] \equiv I_1 + I_2,\end{aligned} \tag{2.5}$$

where the constant $C > 0$ depends only on μ, μ' and γ. Here, integration by parts and the Hardy inequality imply that

$$\begin{aligned} I_1 &\le C\Big[\Big|\Big(|x|a\cdot\nabla\sigma, \frac{\sigma}{|x|}\Big)\Big| + \sum_{i=1}^{3}\Big\{\Big|\Big(\frac{\partial a}{\partial x_i}\cdot\nabla\sigma, \frac{\partial\sigma}{\partial x_i}\Big)\Big| + \frac{1}{2}\Big|\Big((\nabla\cdot a)\frac{\partial\sigma}{\partial x_i}, \frac{\partial\sigma}{\partial x_i}\Big)\Big|\Big\}\Big] \\ &\le C\{\|(1+|x|)a\|_{L_\infty} + \|\nabla a\|_{L_\infty}\}\|\nabla\sigma\|^2 \le C\epsilon\|\nabla\sigma\|^2, \\ I_2 &\le \frac{1}{2}\|(\nabla\sigma,\nabla v)\|_{0,1}^2 + C\{\|(1+|x|)(f,g)\|^2 + \|\nabla g\|^2\}.\end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6), we have

$$\|(\nabla\sigma,\nabla v)\|_{0,1} \le C\{\|(1+|x|)(f,g)\| + \|\nabla g\|\}.$$

Differentiating (2.2)–(2.3) and by repeated use of the same argumet, we can show that

$$\|(\nabla\sigma,\nabla v)\|_{3,4}\le C\{\|(1+|x|)(f,g)\|+\|(\nabla f,\nabla g)\|_{2,3}\}. \tag{2.7}$$

Weighted-L_2 estimate: The second step is to have the weighted-L_2 estimate. We apply ∂_x^α ($1\le|\alpha|\le 4$) to (2.2) and (2.3); multiply the resultant equation by $(1+|x|)^{2|\alpha|}\partial_x^\alpha\sigma$ and $(1+|x|)^{2|\alpha|}\partial_x^\alpha v$ respectively. Then using the same techniques as above, we obtain

$$\sum_{\nu=1}^{4}\|(1+|x|)^{\nu}(\nabla^\nu\sigma,\nabla^{\nu+1}v)\|\le C\Big[\|\tilde v\|^2_{\mathcal{J}^5}+\|\nabla v\|+\sum_{\nu=1}^{4}\|(1+|x|)^{\nu}(\nabla^{\nu-1}f_*,\nabla^\nu g)\|\Big], \tag{2.8}$$

where $C>0$ is a constant depending only on μ,μ' and γ.

L_∞ estimate: At last, in order to get L_∞ estimate, we employ the Helmholtz decomposition: $v=w+\nabla p$ $(\nabla\cdot w=0)$. Putting this formula into (2.2)–(2.3), we have the following system of three equations:

$$\begin{cases}\Delta p+(a\cdot\nabla)\sigma=g,\\ -\mu\Delta w+\nabla\Phi=f,\\ \qquad\qquad \Phi=\gamma\sigma-(2\mu+\mu')\Delta p.\end{cases}$$

Using the Fourier transform, we have the representations for Φ, w_j ($j=1,2,3$) and p:

$$\Phi=\sum_{k=1}^{3}\frac{\partial E_0}{\partial x_k}*f_k,\quad w_j(x)=\sum_{k=1}^{3}E_{jk}*f_k(x),\quad p=E_0*\{-(a\cdot\nabla)\sigma+g\},$$

where E_0 and E_{jk} denote the fundamental solution of the Laplace equation and Stokes equation respectively. Therefore, integration by parts and the Sobolev inequality imply that

$$\begin{aligned}\|(1+|x|)^2\nabla^2\sigma\|_{L_\infty}+\sum_{\nu=0}^{1}\|(1+|x|)^{\nu+1}\nabla^\nu v\|_{L_\infty}&\le C\Big[\epsilon\sum_{\nu=1}^{4}\|(1+|x|)^{\nu}\nabla^\nu\sigma\|\\ +\|(1+|x|)^3 f\|_{L_\infty}+\|(1+|x|)^2 f_1\|_{L_\infty}+\|f_2\|_{L_1}&+\sum_{\nu=0}^{1}\|(1+|x|)^{\nu+2}\nabla^\nu g\|_{L_\infty}\Big],\end{aligned} \tag{2.9}$$

where f_1, f_2 are defined by the appropriate decomposition of f into the form: $f=\nabla\cdot f_1+f_2$.

Combining (2.7)–(2.9) and returning to definition of f, g, we get

$$\|(\sigma,v)\|_{\mathcal{J}^{4,5}}\le C\{\epsilon^2+K\},$$

if we take $\epsilon>0$ small enough, where K is the same as in Theorem 1 and $C>0$ is a constant depending only on μ,μ' and γ. This is the way to close our process of estimation.

3 Sketch of proof of Theorem 2

Finally, we shall give a sketch of proof of Theorem 2. The proof consists of the following two steps: One is local existence and the other is a priori estimate. Concerning the local existence, we can apply the Matsumura-Nishida [4] method directly. So, we will discuss how to get the a priori estimate. Let $\bar{\rho}$ be a positive constant and we denote the corresponding stationary solution obtained in Theorem 1 by (ρ^*, v^*). We put

$$\rho(t,x) = \rho^*(x) + \sigma(t,x), \quad v(t,x) = v^*(x) + w(t,x)$$

into (1.1), then we have the system of equation for (σ, w):

$$\begin{cases} \sigma_t(t) + \nabla\cdot\{(\rho^* + \sigma(t))w(t)\} = -\nabla\cdot(v^*\sigma(t)), & (3.1) \\ w_t(t) - \dfrac{1}{\rho^*}\left[\mu\Delta w(t) + (\mu+\mu')\nabla(\nabla\cdot w(t))\right] + A(t)\nabla\sigma(t) = f(t), & (3.2) \\ (\sigma, w)(0,x) = (\rho_0 - \rho^*, v_0 - v^*)(x), & (3.3) \end{cases}$$

where

$$\begin{aligned} f(t) =& -(v^*\cdot\nabla)w(t) - (w(t)\cdot\nabla)(v^* + w(t)) \\ & - \frac{1}{\rho^*}\{P'(\rho^*+\sigma(t)) - P'(\rho^*)\}\nabla\rho^* - \frac{\sigma(t)}{\rho^*(\rho^*+\sigma(t))}\Big[\mu\Delta(v^* + w(t)) \\ & \qquad + (\mu+\mu')\nabla\{\nabla\cdot(v^*+w(t))\} - P'(\rho^*+\sigma(t))\nabla\rho^*\Big], \\ A(t) =& \frac{P'(\rho^*+\sigma(t))}{\rho^*+\sigma(t)}. \end{aligned}$$

Let $(\sigma, w)(t) \in \mathscr{C}(0, t_1; \mathscr{H}^{3,3})$ be a solution to (3.1)–(3.2) satisfying $\|(\sigma, w)(t)\|_{3,3} \leq \epsilon$. We also suppose that $\|(\rho^* - \rho_0, v^*)\|_{\mathscr{I}^{4,5}} \leq \epsilon$.

Estimates for $\nabla w(t)$ and its derivatives up to $\nabla^4 w(t)$: Applying ∂_x^α $(0 \leq |\alpha| \leq 3)$ to (3.1) and (3.2); multiplying resultant equation by $\partial_x^\alpha\sigma(t)$ and $(\rho + \sigma(t))A(t)^{-1}\partial_x^\alpha w(t)$ respectively, we have

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha\sigma(t)\|^2 - \big((\rho^*+\sigma(t))\partial_x^\alpha w(t), \nabla\partial_x^\alpha\sigma(t)\big) = \big(-\partial_x^\alpha(v^*\sigma(t)) + I_\alpha(t), \nabla\partial_x^\alpha\sigma(t)\big),$$

$$\begin{aligned} &\big(B(t)\partial_x^\alpha w_t(t), \partial_x^\alpha w(t)\big) - \Big(\frac{B(t)}{\rho^*}\partial_x^\alpha\{\mu\Delta w(t) + (\mu+\mu')\nabla(\nabla\cdot w(t))\}, \partial_x^\alpha w(t)\Big) \\ &\qquad + \big((\rho^*+\sigma(t))\nabla\partial_x^\alpha\sigma(t), \partial_x^\alpha w(t)\big) = \big(\partial_x^\alpha f(t) + J_\alpha(t), B(t)\partial_x^\alpha w(t)\big), \end{aligned}$$

where $I_\alpha(t)$ and $J_\alpha(t)$ are defined by

$$I_\alpha(t) = \sum_{\beta<\alpha}\binom{\alpha}{\beta}\big(\partial_x^{\alpha-\beta}(\rho^*+\sigma(t))\big)\,\partial_x^\beta w(t),$$

$$J_\alpha(t) = \sum_{\beta<\alpha}\binom{\alpha}{\beta}\Big[\Big(\partial_x^{\alpha-\beta}\frac{1}{\rho^*}\Big)\partial_x^\beta\{\mu\Delta w(t) + (\mu+\mu')\nabla(\nabla\cdot w(t))\} + \big(\partial_x^{\alpha-\beta}A(t)\big)\nabla\partial_x^\beta w(t)\Big].$$

Canceling the term of $\big((\rho+\sigma(t))\partial_x^\alpha w(t), \nabla\partial_x^\alpha\sigma(t)\big)$ by the above two formulas and writing the first term of second formula as follows:

$$\big(B(t)\partial_x^\alpha w(t), \partial_x^\alpha w(t)\big) = \frac{1}{2}\frac{d}{dt}\big(B(t)\partial_x^\alpha(t), \partial_x^\alpha w(t)\big) - \frac{1}{2}\big(B_t(t)\partial_x^\alpha(t), \partial_x^\alpha w(t)\big),$$

and using integration by parts for the second term of second formula, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\{\|\partial_x^\alpha \sigma(t)\|^2 + \big(B(t)\partial_x^\alpha w(t), \partial_x^\alpha w(t)\big)\} + B_0\mu\|\nabla\partial_x^\alpha w(t)\|^2\\
&\le \big|\big(\partial_x^\alpha(v^*\sigma(t)), \nabla\partial_x^\alpha\sigma(t)\big)\big| + \big|\big(\partial_x^\alpha f(t), B(t)\partial_x^\alpha w(t)\big)\big|\\
&\quad + \Big[\big|\big(I_\alpha(t), \nabla\partial_x^\alpha\sigma(t)\big)\big| + \big|\big(J_\alpha(t), B(t)\partial_x^\alpha w(t)\big)\big|\Big] + \frac{1}{2}\big|\big(B_t(t)\partial_x^\alpha w(t), \partial_x^\alpha w(t)\big)\big| \qquad (3.4)\\
&\quad + \left[\mu\left|\left(\left(\nabla\frac{B(t)}{\rho^*}\right)\nabla\partial_x^\alpha w(t), \partial_x^\alpha w(t)\right)\right| + (\mu+\mu')\left|\left(\left(\nabla\frac{B(t)}{\rho^*}\right)\nabla\partial_x^\alpha w(t), \partial_x^\alpha w(t)\right)\right|\right]\\
&\equiv K_1 + K_2 + K_3 + K_4 + K_5,
\end{aligned}
$$

where $B_0 = \min_{\rho_0/2\le s\le 2\rho_0} s^2/P'(s)$. Now, we estimate the right hand side of (3.4) using the Sobolev inequality and the Gagliard-Nirenberg inequality. In order to estimate K_4, we use (3.1), and then we have

$$
\begin{aligned}
K_4 &= \big|\big(\tilde{B}(t)\sigma_t(t)\,\partial_x^\alpha w(t), \partial_x^\alpha w(t)\big)\big|\\
&= \big|\big(\nabla\cdot\{(\rho^* + \sigma(t))w(t) + v^*\sigma(t)\}, \tilde{B}(t)\,\partial_x^\alpha w(t)\cdot\partial_x^\alpha w(t)\big)\big|\\
&\le C\big|\big(w(t) + v^*\sigma(t), \nabla\{\partial_x^\alpha w(t)\cdot\partial_x^\alpha w(t)\} + \{\nabla\tilde{B}(t)\}\,\partial_x^\alpha w(t)\cdot\partial_x^\alpha w(t)\big)\big| \qquad (3.5)\\
&\le C\{\big(\|w(t)\|_{L_3} + \|v^*\|_{L_6}\|\sigma(t)\|_{L_6}\big)\|\nabla\partial_x^\alpha w(t)\|\|\partial_x^\alpha w(t)\|_{L_6}\\
&\qquad + \|(w,\sigma)(t)\|_{L_6}\|(\nabla\rho^*, \nabla\sigma(t))\|\|\partial_x^\alpha w(t)\|_{L_6}^2\} \le C\epsilon\|\nabla\partial_x^\alpha w(t)\|^2,
\end{aligned}
$$

where $\tilde{B}(t)$ is defined by

$$
\tilde{B}(t) = \frac{\rho^* + \sigma(t)}{P'(\rho^* + \sigma(t))}\left[2 - \frac{P''(\rho^* + \sigma(t))}{P'(\rho^* + \sigma(t))}(\rho^* + \sigma(t))\right].
$$

The other terms are estimated as follows:

$$
\begin{aligned}
K_1 &\le \begin{cases} C\|(1+|x|)v^*\|_{L_\infty}\left\|\dfrac{\sigma(t)}{|x|}\right\|\|\nabla\sigma(t)\| \le C\epsilon\|\nabla\sigma(t)\|^2 & \text{if } \alpha = 0,\\ C\epsilon\|\nabla\sigma(t)\|_{|\alpha|-1}^2 & \text{if } 1\le|\alpha|\le 3,\end{cases}\\
K_2 &\le \begin{cases} C\epsilon\|(\nabla\sigma, \nabla w)(t)\|^2 & \text{if } \alpha = 0,\\ C(\epsilon+\lambda)\|(\nabla\sigma(t), \nabla w(t))\|_{|\alpha|-1,|\alpha|}^2 + C\lambda^{-1}\|\nabla^{|\alpha|}w(t)\|^2 & \text{if } 1\le|\alpha|\le 3,\end{cases} \qquad (3.6)\\
K_3 &\le C\epsilon\|(\nabla\sigma(t), \nabla w(t))\|_{|\alpha|-1,|\alpha|}^2,\\
K_5 &\le C\|(\nabla\rho^*, \nabla\sigma(t))\|_{L_3}\|\nabla\partial_x^\alpha w(t)\|\|\partial_x^\alpha w(t)\|_{L_6} \le C\epsilon\|\nabla\partial_x^\alpha w(t)\|^2.
\end{aligned}
$$

Combining (3.4)-(3.6), we obtain the following estimate:

$$
\begin{aligned}
&\frac{d}{dt}\big[\|\sigma(t)\|^2 + \big(B(t)w(t), w(t)\big)\big] + \alpha_0\|\nabla w(t)\|^2 \le C\epsilon\|\nabla\sigma(t)\|^2,\\
&\frac{d}{dt}\big[\|\nabla^k\sigma(t)\|^2 + \big(B(t)\nabla^k w(t), \nabla^k w(t)\big)\big] + \alpha_k\|\nabla^{k+1}w(t)\|^2 \qquad (3.7)\\
&\qquad \le C(\epsilon+\lambda)\|(\nabla\sigma, w_t)(t)\|_{k-1,k-1}^2 + C\lambda^{-1}\|\nabla w(t)\|_{k-1}^2
\end{aligned}
$$

for $1\le k\le 3$ and any λ with $0<\lambda<\lambda_0$, if we take ϵ, $\lambda_0>0$ small enough. Here, $C>0$ is a constant depending only on μ and μ'.

Estimates for $w_t(t)$ and its derivatives up to $\nabla^2 w_t(t)$: Applying ∂_x^α $(0 \le |\alpha| \le 2)$ to (3.2), multiplying the resultant equation by $\partial_x^\alpha w_t(t)$ and using (3.1), we have

$$\begin{aligned}\frac{d}{dt}\big(w(t), \nabla\sigma(t)\big) + \beta_1\|w_t(t)\|^2 &\le C\epsilon\|\nabla\sigma(t)\|^2 + C\|\nabla w(t)\|_1^2,\\ \frac{d}{dt}\big(\nabla^{k-1}w(t), \nabla^k\sigma(t)\big) + \beta_k\|\nabla^{k-1}w_t(t)\|^2 &\le C\|(\nabla\sigma, \nabla w, \nabla^{k-2}w_t)(t)\|_{k-2,k,0}^2\end{aligned} \tag{3.8}$$

for $2 \le k \le 3$. Here, $C > 0$ is a constant depending only on μ and μ'.

Estimates for $\nabla\sigma(t)$ and its derivatives up to $\nabla^3\sigma(t)$: Similarly, applying ∂_x^α $(0 \le |\alpha| \le 2)$ to (3.2) and multiplying the resultant equation by $\nabla\partial_x^\alpha\sigma(t)$, we have

$$\|\nabla\sigma(t)\|^2 \le \|(\nabla w, w_t)(t)\|_{1,0}^2, \quad \|\nabla^k\sigma(t)\|^2 \le C\|(\nabla\sigma, \nabla w, \nabla^{k-1}w_t)(t)\|_{k-2,k,0}^2 \tag{3.9}$$

for $2 \le k \le 3$, where $C > 0$ is a constant depending only on μ and μ'.

Combining (3.7)–(3.9), we obtain

$$\frac{d}{dt}\Big\{\sum_{\nu=0}^{3}\alpha_\nu[\nabla^\nu\sigma, \nabla^\nu w]_B + \sum_{\nu=1}^{3}\beta_\nu\big(\nabla^{\nu-1}w, \nabla^\nu\sigma\big)\Big\} + \|(\nabla\sigma, \nabla w, w_t)\|_{2,3,2}^2 \le 0,$$

where

$$[\sigma, w]_B(t) \equiv \|\sigma(t)\|^2 + \big(B(t)w(t), w(t)\big), \quad B(t) = \frac{(\rho^* + \sigma(t))^2}{P'(\rho^* + \sigma(t))}.$$

Integration of this formula on $[0, t]$ implies that our a priori estimate.

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