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ON L^p BOUNDEDNESS OF A CLASS OF PSEUDODIFFERENTIAL OPERATORS

MICHIHIRO NAGASE (長瀬 道弘)

ABSTRACT. Let $p(x,\xi)$ be a symbol in Hörmander class $S^0_{1,\delta}$. Then it is known that the pseudodifferential operator $p(X,D_x)$ is $L^p(\mathbb{R}^n)$ bounded. In the present paper we give a class of pseudodifferential operators and study the $L^p(\mathbb{R}^n)$ boundedness of the operators. The class of operators is closely related to the Schrödinger operators with magnetic potentials.
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1. INTRODUCTION

Let $S^m_{\rho,\delta}$ be the set of Hörmander class symbols, that is,

$$S^m_{\rho,\delta} = \{p(x,\xi) : |p^{(\alpha)}_{(\beta)}(x,\xi)| \le C\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|} \text{ for any } \alpha \text{ and } \beta\}$$

Here we use that for any multiintegers $\alpha = (\alpha_1,\cdots,\alpha_n)$ and $\beta = (\beta_1,\cdots,\beta_n)$

$$p^{(\alpha)}_{(\beta)}(x,\xi) = \partial_\xi{}^\alpha D_x{}^\beta p(x,\xi)$$

and

$$\partial^\alpha_\xi = \left(\frac{\partial}{\partial\xi}\right)^\alpha = \left(\frac{\partial}{\partial\xi_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial\xi_n}\right)^{\alpha_n}$$

$$D_x{}^\beta = \left(\frac{\partial}{i\partial x}\right)^\beta = \left(\frac{\partial}{i\partial x_1}\right)^{\beta_1}\cdots\left(\frac{\partial}{i\partial x_n}\right)^{\beta_n}$$

We define the pseudodifferential operator of symbol $p(x,\xi)$ by

$$p(X,D_x)u(x) = \frac{1}{(2\pi)^n}\int e^{ix\xi}p(x,\xi)\hat{u}(\xi)d\xi$$

where the integration is taken in $\mathbb{R}^n$ and $\hat{u}(\xi)$ denotes the Fourier transform of $u(x)$, that is,

$$\hat{u}(\xi) = \int e^{-ix\xi}u(x)dx$$

We denote that the set of pseudodifferential operators with symbol of class $S^m_{\rho,\delta}$ by the same notation as the symbol class.

We say that a linear operator $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is L^p bounded if there is a constant C such that

$$\|Tu\|_{L^p(\mathbb{R}^n)} \le C\|u\|_{L^p(\mathbb{R}^n)} \text{ for any } u \in \mathcal{S}$$

We denote the set of all L^p bounded operators by $\mathcal{L}(L^p(\mathbb{R}^n))$. The following theorem is known as Calderón-Vaillancourt theorem([1]).

Theorem 1. *Let* $0 \le \delta \le \rho \le 1, \delta < 1$. *Then we have*

$$S^0_{\rho,\delta} \subset \mathcal{L}(L^2(\mathbb{R}^n))$$

For the general L^p boundedness, we can see the following theorem([2],[4]).
Theorem 2. *Let* $0 \le \delta \le \rho \le 1$, $(\delta < 1)$ *and* $1 < p < \infty$. *Then we have*

$$S^m_{\rho,\delta} \subset \mathcal{L}(L^p)$$

if and only if $m \le -n(1-\rho)\left|\frac{1}{2} - \frac{1}{p}\right|$

We want to generalize these results to a class of pseudodifferential operators which is useful to the study of Schrödinger operators with magnetic potentials.

2. Preliminary results

Let $a(x) = (a_1(x), \cdots, a_n(x))$ be an $\mathbb{R}^n$ valued function such that $\partial_x^\alpha a_j(x)$ are bounded for any multiinteger $\alpha \neq 0$. Then we define a smooth function $\lambda(x,\xi)$ by

$$\lambda(x,\xi) = \sqrt{|\xi - a(x)|^2 + 1}$$

Then it is not difficult that the function $\lambda(x,\xi)$ satisfies

(1) $\lambda(x,\xi) \ge 1$
(2) $|\partial_\xi^\alpha \partial_x^\beta \lambda(x,\xi)| \le C_{\alpha,\beta}\lambda(x,\xi)^{1-|\alpha|}$

By using the function $\lambda(x,\xi)$ we define a class $S^m_{\rho,\delta,\lambda}$ of symbols by

$$S^m_{\rho,\delta,\lambda} = \left\{p(x,\xi) : |p^{(\alpha)}_{(\beta)}(x,\xi)| \le C_{\alpha,\beta}\lambda(x,\xi)^{m-\rho|\alpha|+\delta|\beta|} \text{ for any } \alpha; and\ \beta\right\}$$

and denote

$$S^\infty_{\rho,\delta,\lambda} = \bigcup_{m\in\mathbf{R}} S^m_{\rho,\delta,\lambda}.$$

This class of symbols is useful for the study of Schrödinger operators with magnetic potentials(see for example [7]). Then it is known that if $0 \le \delta \le \rho \le 1, \delta < 1$ then the class of pseudodifferential operators with symbols $S^\infty_{\rho,\delta,\lambda}$ makes an algebra. Moreover we can show the following L^2 boundedness theorem by using the method in [5].

Theorem 3. *We assume that* $0 \le \delta < \rho \le 1$. *If a symbol* $p(x,\xi)$ *is in* $S^0_{\rho,\delta,\lambda}$, *then the pseudodifferential operator* $P = p(X, D_x)$ *is* L^2 *bounded. That is, there is a constant* C *such that*

$$\|p(X,D_x)u\| \le C\|u\|$$

where $\|\cdot\|$ *means the usual* $L^2(\mathbb{R}^n)$ *norm.*

3. L^p boundedness of pseudodifferential operators

Let $a(x) = (a_1(x), \cdots a_n(x))$ be an $\mathbb{R}^n$ valued function, and let

$$\lambda(x,\xi) = \sqrt{|\xi - a(x)|^2 + 1} \tag{1}$$

In the following we don't assume that the vector function $a(x)$ is not smooth, we need only the fact that $a(x)$ is $\mathbb{R}^n$ valued and measurable.

In the following we use always C as constant independent of variables. Hence the value of C in inequalities are not the same at each occurrence. First we give simple boundedness lemmas of the pseudodifferential operators.

Lemma 1. *If the support of symbol $p(x,\xi)$ is contained in $\{(x,\xi) : |\xi - a(x)| \leq R\}$ for some positive constant R and $p(x,\xi)$ satisfies*

$$\left|p^{(\alpha)}(x,\xi)\right| \leq C_\alpha \tag{2}$$

for any α with $|\alpha| \leq n+1$. Then the operator $p(X,D_x)$ is written as

$$p(X,D_x)u(x) = \int K(x,x-y)u(y)dy$$

where the kernel $K(x,z)$ satisfies

$$|k(x,z)| \leq \frac{C}{\langle z\rangle^{n+1}} \tag{3}$$

Proof. We can write

$$p(X,D_x)u(x) = \int K(x,x-y)u(y)dy$$

where

$$K(x,z) = \frac{1}{(2\pi)^n}\int e^{iz\xi}p(x,\xi)d\xi$$

Then for $|\alpha| \leq n+1$ we have

$$z^\alpha K(x,z) = (i)^{|\alpha|}\frac{1}{(2\pi)^n}\int e^{iz\xi}p^{(\alpha)}(x,\xi)d\xi$$

Hence we have

$$\begin{aligned} |z^\alpha K(x,z)| &\leq \frac{1}{(2\pi)^n}\int |p^{(\alpha)}(x,\xi)|d\xi \\ &\leq \frac{1}{(2\pi)^n}\int_{\xi:|\xi-a(x)|} C_\alpha d\xi \\ &\leq C \end{aligned}$$

where the last constant C is independent of the variable x. Thus we have the kernel estimate (3). □

Because of the estimate (3),we have

$$\int |K(x,z)|dz \leq M \tag{4}$$

Therefore we have

Proposition 1. *Let $p(x,\xi)$ satisfy the same assumption as in Lemma 1, then the pseudodifferential operator $p(X,D_x)$ is L^p bounded for $1 \leq p \leq \infty$ and the bound norm is estimated by M in (4).*

For $2 \leq p \leq \infty$ we have

Lemma 2. *If the support of symbol $p(x,\xi)$ contained in $\{(x,\xi) : |\xi - a(x)| \le R\}$ for some positive constant R and $p(x,\xi)$ satisfies the inequality* (2) *for $|\alpha| \le \kappa = \left[\frac{n}{2}\right]+1$, then the pseudodifferential operator $p(X,D_x)$ is L^p bounded for $2 \le p \le \infty$.*

Proof. We can write

$$p(X,D_x)u(x) = \int K(x,x-y)u(y)dy$$

where

$$K(x,z) = \frac{1}{(2\pi)^n}\int e^{iz\xi}p(x,\xi)d\xi$$

Then by the Schwarz inequality and the Plancherel formula we have

$$\begin{aligned}\int |K(x,z)|dz &\le C\left\{\int \langle z\rangle^{-2\kappa}dz\right\}^{1/2}\left\{\int \langle z\rangle^{2\kappa}|K(x,z)|^2dz\right\}^{1/2}\\ &\le C\sum_{|\alpha|\le\kappa}\left\{\int |z^\alpha K(x,z)|^2dz\right\}^{1/2}\\ &= C\sum_{|\alpha|\le\kappa}\left\{\int |p^{(\alpha)}(x,\xi)|^2d\xi\right\}^{1/2}\\ &= C\end{aligned}$$

Thus we have

$$\|p(X,D_x)u\|_{L^\infty(\mathbf{R}^n)} \le C\|u\|_{L^\infty(\mathbf{R}^n)}$$

In a similar way we have

$$\begin{aligned}\|p(X,D_x)u\|^2 &= \int\left|\int |K(x,x-y)u(y)dy\right|^2dx\\ &\le \int\left\{\int \langle x-y\rangle^{2\kappa}|K(x,x-y)|^2\right\}\left\{\int \langle x-y\rangle^{-2\kappa}|u(x)|^2dy\right\}dx\\ &= C\int\left\{\int \langle z\rangle^{2\kappa}|K(x,z)|^2\right\}\left\{\int \langle x-y\rangle^{-2\kappa}|u(x)|^2dy\right\}\\ &\le C\int\int \langle x-y\rangle^{-2\kappa}|u(x)|^2dydx\\ &= C\|u\|^2\end{aligned}$$

Hence by the Riesz-Thorin interpolation we get the Lemma. □

One of the main results in the present note is the following.

Theorem 4. *Let $a(x)$ be the same as in Lemma 1, and $\lambda(x,\xi)$ be defined by* (1). *Let $\omega(t)$ be a nonnegative and nondecreasing function on $[0,\infty)$ such that*

$$\int_0 \frac{\omega(t)}{t}dt < \infty$$

We assume that a symbol $p(x,\xi)$ satisfies

$$|p^{(\alpha)}(x,\xi)| \le C_\alpha\lambda(x,\xi)^{-|\alpha|}\omega(\lambda(x,\xi)^{-1})$$

for any α *with* $|\alpha| \leq n+1$. *Then the pseudodifferential operator* $p(X, D_x)$ *is* L^p *bounded for* $1 \leq p \leq \infty$.

Proof. By Lemma 1, we may assume that the support of the symbol $p(x,\xi)$ is contained in $\{(x,\xi) : |\xi - a(x)| \geq 2\}$. Now we take a smooth nonnegative function $f(t)$ such that the support of $f(t)$ is contained in the interval $[\frac{1}{2}, 1]$ and

$$\int_0^\infty \frac{f(t)}{t} dt = 1$$

Then since the support of the symbol $p(x,\xi)$ is contained in $\{(x,\xi) : |\xi - a(x)| \geq 2\}$, we have

$$\begin{aligned} p(X, D_x)u(x) &= \frac{1}{(2\pi)^n} \int_0^1 \frac{1}{t} dt \int\int e^{i(x-y)\xi} p(x,\xi) f(t|\xi|) u(y) dy d\xi \\ &= \frac{1}{(2\pi)^n} \int_0^1 \frac{1}{t^{n+1}} dt \int\int e^{i\frac{(x-y)}{t}\xi} e^{i(x-y)a(x)} p(x, \frac{\xi}{t} + a(x)) f(|\xi|) u(y) d\xi dy \\ &= \frac{1}{(2\pi)^n} \int_0^1 \frac{1}{t} dt \int e^{itza(x)} K_t(x,z) u(x - tz) dz \end{aligned}$$

where

$$K_t(x,z) = \frac{1}{(2\pi)^n} \int e^{iz\xi} p(x, \frac{\xi}{t} + a(x)) f(|\xi|) d\xi$$

If we put $\tilde{p}(x,\xi) = p(x, \frac{\xi}{t} + a(x))$, then it is easy to see that

$$|\tilde{p}^{(\alpha)}(x,\xi)| \leq C_\alpha \langle\xi\rangle^{-|\alpha|} \omega(\langle\xi\rangle^{-1})$$

for $|\alpha| \leq n+1$. Since the equality

$$z^\alpha K_t(x,z) = \frac{i^{|\alpha|}}{(2\pi)^n} \sum_{\alpha' \leq \alpha} \frac{1}{t^{|\alpha'|}} \binom{\alpha}{\alpha'} \int e^{iz\xi} \tilde{p}^{(\alpha')}(x, \frac{\xi}{t}) \partial_\xi^{\alpha-\alpha'} f(|\xi|) d\xi$$

holds for $|\alpha| \leq n+1$, we have

$$\begin{aligned} |z^\alpha K_t(x,z)| &\leq \frac{1}{(2\pi)^n} \sum_{\alpha' \leq \alpha} \frac{1}{t^{|\alpha'|}} \binom{\alpha}{\alpha'} \int |\tilde{p}^{(\alpha)}(x, \frac{\xi}{t}) \partial_\xi^{\alpha-\alpha'} f(|\xi|)| d\xi \\ &\leq C \sum_{\alpha' \leq \alpha} \frac{1}{t^{|\alpha'|}} \binom{\alpha}{\alpha'} \int_{1/2 \leq |\xi| \leq 1} \left|\frac{\xi}{t}\right|^{|\alpha'|} \omega(\left|\frac{\xi}{t}\right|^{-1}) d\xi \\ &\leq C\omega(t) \end{aligned}$$

for $|\alpha| \leq n+1$. Therefore we have

$$|K_t(x,z)| \leq C\langle\xi\rangle^{-n-1}\omega(t) \tag{5}$$

By the inequality (5) and the equality

$$p(X, D_x)u(x) = \frac{1}{(2\pi)^n} \int_0^1 \frac{1}{t} dt \int e^{itza(x)} K_t(x,z) u(x - tz) dz$$

we can see that the operator $p(X, D_x)$ is L^1 bounded and L^∞ bounded. That is inequalities

$$\begin{aligned} \|p(X,D_x)u\|_{L^1(\mathbf{R}^n)} &\leq C\|u\|_{L^1(\mathbf{R}^n)} \\ \|p(X,D_x)u\|_{L^\infty(\mathbf{R}^n)} &\leq C\|u\|_{L^\infty(\mathbf{R}^n)} \end{aligned}$$

holds. So by the Riesz-Thorin interpolation theorem we have the L^p boundedness for $1 \leq p \leq \infty$. □

When $2 \leq p$, we can show a little more general result than Theorem 4, by using the Plancherel Theorem.

Theorem 5. *Let $a(x)$ and $\lambda(x,\xi)$ be the same as in Theorem 4. Let $\omega(t)$ be a nonnegative and nondecreasing function on $[0,\infty)$ such that*

$$\int_0 \frac{\omega(t)}{t} dt < \infty$$

We assume that a symbol $p(x,\xi)$ satisfies

$$|p^{(\alpha)}(x,\xi)| \leq C_\alpha \lambda(x,\xi)^{-|\alpha|}\omega(\lambda(x,\xi)^{-1})$$

for any α with $|\alpha| \leq \kappa = [\frac{n}{2}] + 1$. Then the pseudodifferential operator $p(X,D_x)$ is L^p bounded for $2 \leq p \leq \infty$.

Proof. We first show the L^∞ boundedness. We write the operator $p(X, D_x)$, as in the proof of Theorem 4, by

$$p(X,D_x)u(x) = \frac{1}{(2\pi)^n}\int_0^1 \frac{dt}{t}\int e^{itza(x)}K_t(x,z)u(x-tz)dz$$

where

$$K_t(x,z) = \frac{1}{(2\pi)^n}\int e^{iz\xi}p(x,\frac{\xi}{t}+a(x))f(|\xi|)d\xi$$

Then writing $\kappa = [\frac{n}{2}] + 1$, we have

$$\begin{aligned} \int |K_t(x,z)|dz &= \int \langle z\rangle^{-\kappa}\langle z\rangle^{\kappa}|K_t(x,z)|dz \\ &\leq \left\{\int \langle z\rangle^{-2\kappa}dz\right\}^{1/2}\left\{\int \langle z\rangle^{2\kappa}|K_t(x,z)|^2dz\right\}^{1/2} \\ &\leq C\sum_{|\alpha|\leq\kappa}\left\{\int |z^\alpha K_t(x,z)|^2dz\right\}^{1/2} \end{aligned}$$

Using the Plancherel equality, we have

$$\begin{aligned} \int |z^\alpha K_t(x,z)|^2dz &= \int |\partial_\xi^\alpha\{\tilde{p}(x,\frac{\xi}{t})f(|\xi|)\}|^2d\xi \\ &\leq C_\alpha\omega(t) \end{aligned}$$

Hence we have

$$|p(X,D_x)u(x)| \leq C\|u\|_{L^\infty(\mathbf{R}^n)}$$

In order to show L^2 boundedness of the operator $p(X, D_x)$, we use the same representation

$$p(X, D_x)u(x) = \frac{1}{(2\pi)^n} \int_0^1 \frac{dt}{t} \int e^{itza(x)} K_t(x,z) u(x - tz) dz$$

where

$$K_t(x,z) = \frac{1}{(2\pi)^n} \int e^{iz\xi} p(x, \frac{\xi}{t} + a(x)) f(|\xi|) d\xi$$

From this representation we have

$$\|p(X, D_x)\|_{L^2(\mathbf{R}^n)} \leq \frac{1}{(2\pi)^n} \int_0^1 \frac{dt}{t} \left\| \int e^{itza(\cdot)} K_t(\cdot, z) u(\cdot - tz) dz \right\|_{L^2(\mathbf{R}^n)}$$

Then we have

$$\begin{aligned} \left\| \int e^{itza(\cdot)} K_t(\cdot, z) u(\cdot - tz) dz \right\|^2 &= \int \left| \int e^{itza(x)} K_t(x,z) u(x - tz) dz \right|^2 dx \\ &\leq \int \left| \int |K_t(x,z) u(x - tz)| dz \right|^2 dx \end{aligned}$$

By using the Schwarz inequality we have

$$\left| \int |K_t(x,z) u(x - tz)| dz \right|^2 \leq \int \langle z \rangle^{2\kappa} |K_t(x,z)|^2 dz \int \langle z \rangle^{-2\kappa} |u(x - tz)|^2 dz$$

As above we can see

$$\begin{aligned} \int \langle z \rangle^{2\kappa} |K_t(x,z)|^2 dz &\leq \sum_{|\alpha| \leq \kappa} \int |z^\alpha K_t(x,z)|^2 dz \\ &= \sum_{|\alpha| \leq \kappa} \int |\partial_\xi \left\{ p(x, \frac{\xi}{t} + a(x)) f(|\xi|) \right\} |^2 d\xi \\ &\leq C\omega(t)^2 \end{aligned}$$

Therefore we get

$$\begin{aligned} \left\| \int e^{itza(\cdot)} K_t(\cdot, z) u(\cdot - tz) dz \right\|^2 &\leq C \int \int \langle z \rangle^{-2\kappa} |u(x - tz)|^2 dz dx \\ &\leq C\omega(t)^2 \|u\|^2 \end{aligned}$$

Thus from the assumption of $\omega(t)$ we have the L^2 estimate

$$\|p(X, D_x)u\| \leq C\|u\|$$

Again by the Riesz-Thorin interpolation theorem we have the L^p boundedness for $2 \leq p \leq \infty$. □

Remark 1. *Theorems in this section we don't always assume that the vector function $a(x) = (a_1(x), \cdots, a_n(x))$ satisfies the estimate*

$$|\partial_x a_j(x)| \leq C$$

In several theorems we can prove the theorem under only the measurability of $a(x)$.

Remark 2. *If the vector function $a(x)$ is bounded, then the symbol class $S^m_{\rho,\delta,\lambda}$ coincide with the usual Hörmander class $S^m_{\rho,\delta}$. Hence using the similar method of usual class, the L^p boundedness in Theorem 4 can be shown(see for example, [6]). Even if $a(x)$ is not bounded, we have*

Proposition 2. *Let $a(x)$ and $\lambda(x,\xi)$ be the same as in Theorem 4. For any smooth function φ with compact support, we have*

$$\|\varphi(x)p(X,D_x)u\|_{L^p(\mathbf{R}^n)} \le C\|u\|_{L^p(\mathbf{R}^n)}$$

4. Conjecture

As we see in the previous sections we can expect that the following L^p boundedness theorem.

Conjecture 1. *If the vector function $a(x) = (a_1(x), \cdots, a_n(x))$ satisfies*

$$|\partial^\alpha a_j(x)| \le C_\alpha$$

for any $\alpha \ne 0$. Then for $1 < p < \infty$, the operator $p(X,D)$ in $S^0_{1,\delta,\lambda}$ is L^p bounded. That is,

$$S^0_{1,\delta,\lambda} \subset \mathcal{L}(L^p(\mathbb{R}^n))$$

holds.

As we stated in section 2 it is known that if the vector function $a(x)$ satisfies the estimates in the above conjecture, the operators in $S^0_{1,\delta,\lambda}$ with ($\delta < 1$) are L^2 bounded. So if we can show the weak type $(1,1)$ estimates or boundedness from $L^\infty(\mathbb{R}^n)$ to BMO, then we can get the above conjecture, that is, L^p boundedness for $1 < p < \infty$ by using the interpolation theorems(see for example [8], [3]). The fundamental conjecture is

Conjecture 2. *If the vector function $a(x) = (a_1(x), \cdots, a_n(x))$ satisfies*

$$|\partial^\alpha a_j(x)| \le C_\alpha$$

for any $\alpha \ne 0$. Then the operator $p(X,D)$ in $S^0_{1,\delta,\lambda}$ is bounded from $L^\infty(\mathbb{R}^n)$ to BMO, that is, there is a constant C such that

$$\|p(X,D_x)u\|_{BMO} \le C\|u\|_{L^\infty(\mathbf{R}^n)}$$

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Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560, Japan (大阪大学理学研究科数学教室)

E-mail address: nagase@math.wani.osaka-u.ac.jp