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A generalization of the Lieb-Thirring inequality and its applications

東北大・理・数学　立澤 一哉 (Kazuya Tachizawa)
Mathematical Institute, Tohoku University

1 Introduction

In 1976 Lieb and Thirring proved the following theorem([9]).

Theorem 1.1 *Let $n \in \boldsymbol{N}$ and γ be a non-negative number such that*

$$\begin{aligned} \gamma > \frac{1}{2} \quad & \text{if} \quad n = 1, \\ \gamma > 0 \quad & \text{if} \quad n = 2, \\ \gamma \geq 0 \quad & \text{if} \quad n \geq 3. \end{aligned}$$

Suppose that $V \in L^{n/2+\gamma}(\boldsymbol{R}^n)$ and $V \geq 0$. Let $\lambda_1 \leq \lambda_2 \leq \cdots$ be the negative eigenvalues of the Schrödinger operator $-\Delta - V$. Then we have

$$\sum_i |\lambda_i|^\gamma \leq c_{n,\gamma} \int_{\boldsymbol{R}^n} V^{n/2+\gamma} dx.$$

Remark

(i) The Lieb-Thirring inequality holds for $n = 1$ and $\gamma = 1/2$ (Weidl[17]).

(ii) The Lieb-Thirring inequality does not hold for $n = 1, \gamma < 1/2$ or $n = 2, \gamma = 0$ ([9]).

The Lieb-Thirring inequality has important applications in the study of the stability of matter or the estimate of the dimension of attractors of nonlinear equations.

In 1995 Egorov-Kondrat'ev provided a generalization of the Lieb-Thirring inequality([3]).

Theorem 1.2 *Let $n \in \boldsymbol{N}$, $q \geq \dfrac{n}{2}$ and γ be a non-negative number such that*

$$\begin{aligned} \gamma > q \quad & \text{if} \quad n = 1, \\ \gamma > 0 \quad & \text{if} \quad n = 2 \\ \gamma \geq 0 \quad & \text{if} \quad n \geq 3. \end{aligned}$$

Suppose $V \geq 0$ *and* $\int_{\boldsymbol{R}^n} V^{q+\gamma}|x|^{2q-n}\,dx < \infty$. *Let* $\lambda_1 \leq \lambda_2 \leq \cdots$ *be the negative eigenvalues of the Schrödinger operator* $-\Delta - V$. *Then we have*

$$\sum_i |\lambda_i|^\gamma \leq c_{n,\gamma,q} \int_{\boldsymbol{R}^n} V^{q+\gamma}|x|^{2q-n}\,dx.$$

Theorem 1.2 is a special case of Egorov-Kondrat'ev's result in [3]. In fact Egorov and Kondrat'ev proved a generalization of Theorem 1.2 for an elliptic operator of order $2m$. In this paper we give a generalization of Egorov-Kondrat'ev's result for certain degenerate elliptic partial differential operator, for which the rate of degeneracy is regulated by the weight $w \in A_2$.

First we recall the definition of A_p-weights. By a cube in $\boldsymbol{R}^n$ we mean a cube which sides are parallel to coordinate axes. A locally integrable function w on $\boldsymbol{R}^n$ and $w > 0$ *a.e.* is an A_p-weight for some $p \in (1,\infty)$ if there exists a positive constant C such that

$$\frac{1}{|Q|}\int_Q w(x)\,dx \left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} \leq C$$

for all cubes $Q \subset \boldsymbol{R}^n$. We say that w is an A_1-weight if there exists a positive constant C such that

$$\frac{1}{|Q|}\int_Q w(y)\,dy \leq Cw(x) \qquad a.e.\, x \in Q$$

for all cubes $Q \subset \boldsymbol{R}^n$. We write A_p for the class of A_p-weights.

Next we consider an elliptic partial differential operator of order $2m$. For $m \in \boldsymbol{N}$ and $f \in C_0^\infty(\boldsymbol{R}^n)$ let

$$L_0 f(x) = \sum_{|\alpha|=|\beta|=m} (-1)^m D^\alpha\left(a_{\alpha\beta}(x) D^\beta f(x)\right),$$

where

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} \quad \text{for } \alpha = (\alpha_1,\ldots,\alpha_n) \in (\boldsymbol{N}\cup\{0\})^n,$$

$$a_{\alpha\beta} \in H^m_{loc}(\boldsymbol{R}^n), \quad \text{and} \quad a_{\alpha\beta} = \overline{a_{\beta\alpha}}.$$

In the above definition the space $H^m_{loc}(\boldsymbol{R}^n)$ denotes the set of all $f \in L^2_{loc}(\boldsymbol{R}^n)$ such that $D^\alpha f \in L^2_{loc}(\boldsymbol{R}^n)$ for all $|\alpha| \leq m$.

$$a(f,g) = \int_{\boldsymbol{R}^n} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) D^\beta f(x) \overline{D^\alpha g(x)}\, dx$$

for $f, g \in C_0^\infty(\boldsymbol{R}^n)$ and $\|\cdot\|$ be the norm of $L^2(\boldsymbol{R}^n)$.

We have the following theorem.

Theorem 1.3 *Let $n > 2m, q \geq n/(2m)$ and $\gamma \geq 0$. We assume that there exists a $w \in A_2$ such that*

$$(1) \qquad (L_0 f, f) \geq \int_{\boldsymbol{R}^n} w(x) \sum_{|\alpha|=m} |D^\alpha f(x)|^2\, dx$$

for all $f \in C_0^\infty(\boldsymbol{R}^n)$. Suppose that u is a non-negative locally integrable function on $\boldsymbol{R}^n$ which satisfies $uw^{-q} \in A_q$ and

$$(2) \qquad |Q|^{2m/n+1} \leq c_1 \int_Q w\, dx \left(\int_Q \frac{u}{w^q}\, dx \right)^{1/q}$$

for all cubes $Q \subset \boldsymbol{R}^n$, where c_1 is a positive constant not depending on Q. For a non-negative measurable function V on $\boldsymbol{R}^n$ we assume that

$$(3) \qquad \int_{\boldsymbol{R}^n} V^{q+\gamma} \frac{u}{w^q}\, dx < \infty.$$

Let $\mathcal{H}$ be the completion of $C_0^\infty(\boldsymbol{R}^n)$ with respect to the norm

$$\|f\|_{\mathcal{H}} = \{a(f,f) + \|f\|^2\}^{1/2}.$$

Then we have the following.

(i) There exists a unique self-adjoint operator L in $L^2(\boldsymbol{R}^n)$ with domain $\mathcal{D} \subset \mathcal{H}$ such that

$$(Lf, g) = a(f,g) - \int_{\boldsymbol{R}^n} V f \bar{g}\, dx$$

for all $f \in \mathcal{D}$ and $g \in \mathcal{H}$.

(ii) The negative spectrum of L is discrete.

(iii) There exists a positive constant c such that

$$(4) \qquad \sum_i |\lambda_i|^\gamma \leq c \int_{\boldsymbol{R}^n} V^{q+\gamma} \frac{u}{w^q}\, dx,$$

where $\{\lambda_i\}$ is the set of all negative eigenvalues of L and c does not depend on V.

Remark

(i) Let $L_0 = -\Delta, m = 1, w \equiv 1$, and $u = |x|^{2q-n}$. Then we have the Egorov-Kondrat'ev theorem for $n \geq 3$.

(ii) If $u \equiv 1$ and $q = n/(2m)$, then (2) is trivial by the Hölder inequality.

Next we consider the lower dimensional cases. First we recall the definition of dyadic cubes. For $j \in \boldsymbol{Z}$ and $k \in \boldsymbol{Z}^n$ the cube

$$Q = \{(x_1, \dots, x_n) : k_i \leq 2^j x_i < k_i + 1, \ i = 1, \dots, n\}$$

is called a dyadic cube. Let $\mathcal{Q}$ be the set of all dyadic cubes in $\boldsymbol{R}^n$. For each $Q \in \mathcal{Q}$ there is a unique $Q' \in \mathcal{Q}$ such that $Q \subset Q'$ and the side-length of Q' is the double of that of Q. We call Q' the parent of Q in this paper.
We have the following theorem.

Theorem 1.4 *Let $n \leq 2m, q \geq n/(2m), \gamma > 0$ and $q + \gamma > 1$. We assume that there exists a $w \in A_2$ such that*

$$(L_0 f, f) \geq \int_{\boldsymbol{R}^n} w(x) \sum_{|\alpha|=m} |D^\alpha f(x)|^2 \, dx \tag{5}$$

for all $f \in C_0^\infty(\boldsymbol{R}^n)$. We assume that

$$\int_{Q'} w \, dx \leq 2^{2m} \int_Q w \, dx \tag{6}$$

for all dyadic cubes Q and its parent Q'. Suppose that u is a non-negative locally integrable function on $\boldsymbol{R}^n$ which satisfies $uw^{-q} \in A_{q+\gamma}$ and

$$|Q|^{2m/n+1} \leq c_1 \int_Q w \, dx \left(\int_Q \frac{u}{w^q} \, dx \right)^{1/q} \tag{7}$$

for all cubes $Q \subset \boldsymbol{R}^n$, where c_1 is a positive constant not depending on Q. For a non-negative measurable function V on $\boldsymbol{R}^n$ we assume that

$$\int_{\boldsymbol{R}^n} V^{q+\gamma} \frac{u}{w^q} \, dx < \infty. \tag{8}$$

Let $\mathcal{H}$ be the completion of $C_0^\infty(\boldsymbol{R}^n)$ with respect to the norm

$$\|f\|_{\mathcal{H}} = \{a(f, f) + \|f\|^2\}^{1/2}.$$

Then we have the following.

(i) There exists a unique self-adjoint operator L in $L^2(\boldsymbol{R}^n)$ with domain $\mathcal{D} \subset \mathcal{H}$ such that

$$(Lf, g) = a(f, g) - \int_{\boldsymbol{R}^n} V f \bar{g}\, dx$$

for all $f \in \mathcal{D}$ and $g \in \mathcal{H}$.

(ii) The negative spectrum of L is discrete.

(iii) There exists a positive constant c such that

$$\sum_i |\lambda_i|^\gamma \leq c \int_{\boldsymbol{R}^n} V^{q+\gamma} \frac{u}{w^q}\, dx, \tag{9}$$

where $\{\lambda_i\}$ is the set of all negative eigenvalues of L and c does not depend on V.

Remark

(i) Let $L_0 = -\Delta, m = 1, w \equiv 1$, and $u = |x|^{2q-n}$. Then we have the Egorov-Kondrat'ev theorem for $n = 1$ or 2.

(ii) Since $w \in A_2$, there exists a positive constant c such that

$$\int_{Q'} w\, dx \leq c \int_Q w\, dx$$

for all dyadic cubes Q and its parent Q' (c.f. Prop.3.1 (iv) in Section 3). Hence the condition (6) is satisfied if m is sufficiently large.

In the proofs of Theorems 1.3 and 1.4 we use Meyer's wavelet basis.

2 Wavelets

First we recall the definition of Meyer's wavelet basis. Let θ be a function which satisfies the following condition.

- θ is an even function in $C_0^\infty(\boldsymbol{R})$.
- $0 \leq \theta(\xi) \leq 1$ and $\operatorname{supp} \theta \subset [-4\pi/3, 4\pi/3]$.
- $\theta(\xi) = 1$ for all $\xi \in [-2\pi/3, 2\pi/3]$.

- $\theta(\xi)^2 + \theta(2\pi - \xi)^2 = 1$ for all $\xi \in [0, 2\pi]$.

We define a function $\psi \in L^2(\boldsymbol{R})$ by

$$\hat{\psi}(\xi) = \{\theta(\xi/2)^2 - \theta(\xi)^2\}^{1/2} e^{-i\xi/2}.$$

For integers j, k we set $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$. Then it turns out that $\{\psi_{j,k}\}_{j,k\in\boldsymbol{Z}}$ is an orthonormal basis of $L^2(\boldsymbol{R})$([10]) which we call Meyer's wavelet basis.

We define n-dimensional Meyer's wavelet basis as follows. Let φ be a function in $L^2(\boldsymbol{R})$ such that $\hat{\varphi}(x) = \theta(\xi)$. Set $E = \{0,1\}^n \setminus \{\mathbf{0}\}$ and

$$\psi^0(x) = \varphi(x), \quad \psi^1(x) = \psi(x).$$

For $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in E$ and $x = (x_1, \dots, x_n) \in \boldsymbol{R}^n$ we define

$$\psi^\varepsilon(x) = \psi^{\varepsilon_1}(x_1) \cdots \psi^{\varepsilon_n}(x_n).$$

Let $\Lambda = \{ (\varepsilon, j, k) \; : \; \varepsilon \in E, \; j \in \boldsymbol{Z}, \; k \in \boldsymbol{Z}^n \}$. For $\lambda = (\varepsilon, j, k) \in \Lambda, x \in \boldsymbol{R}^n$, set

$$\psi_\lambda(x) = 2^{nj/2}\psi^\varepsilon(2^j x - k).$$

Then $\{\psi_\lambda\}_{\lambda\in\Lambda}$ is Meyer's wavelet basis of $L^2(\boldsymbol{R}^n)$.

3 Weighted inequalities

First we recall some properties of A_p-weights which will be used in the following sections. Let M be the Hardy-Littlewood maximal operator, that is,

$$M(f)(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(y)| \, dy,$$

where the supremum is taken over all cubes Q which contain x.

Proposition 3.1

(i) *Let $1 < p < \infty$ and w be a non-negative locally integrable function on $\boldsymbol{R}^n$. Then M is bounded on $L^p(w)$ if and only if $w \in A_p$.*

(ii) *Let $1 < p < \infty$ and $w \in A_p$. Then there exists a $q \in (1, p)$ such that $w \in A_q$.*

(iii) *Let* $0 < \tau < 1$ *and* f *be a locally integrable function on* $\mathbf{R}^n$ *such that* $M(f)(x) < \infty$ *a.e.. Then* $(M(f))^\tau \in A_1$.

(iv) *Let* $1 \leq p < \infty$ *and* $w \in A_p$. *Then there exists a positive constant* c *such that*

$$\int_{2Q} w\,dx \leq c \int_Q w\,dx$$

for all cubes $Q \in \mathbf{R}^n$, *where* $2Q$ *denotes the double of* Q.

The proofs of these facts are in [6, Chapter IV] or [15, Chapter V]. Property (iv) is called the doubling property of A_p-weights.

Next we state some weighted inequalities. For $\alpha \geq 0$ and $f \in C_0^\infty(\mathbf{R}^n)$ we define via inverse Fourier transform

$$(-\Delta)^{\alpha/2} f(x) = \mathcal{F}^{-1}(|\xi|^\alpha \hat{f})(x).$$

For $\lambda = (\varepsilon, j, k) \in \Lambda$, set

$$Q(\lambda) = \{(x_1, \ldots, x_n) : k_i \leq 2^j x_i < k_i + 1,\ i = 1, \ldots, n\}.$$

Proposition 3.2 *Let* $\alpha \geq 0$ *and* $w \in A_2$. *Then there exist positive constants* c_1 *and* c_2 *such that*

$$\begin{aligned} c_1 \int_{\mathbf{R}^n} |(-\Delta)^{\alpha/2} f|^2 w\,dx &\leq \sum_{\lambda \in \Lambda} |Q(\lambda)|^{-2\alpha/n} |(f, \psi_\lambda)|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} w\,dx \\ &\leq c_2 \int_{\mathbf{R}^n} |(-\Delta)^{\alpha/2} f|^2 w\,dx \end{aligned}$$

for all $f \in C_0^\infty(\mathbf{R}^n)$.

This proposition is proved in [16, Prop. 2.2] for the φ-transform of Frazier-Jawerth. We can prove Proposition 3.2 by Proposition 2.2 in [16] by similar arguments in [5, p.72]. In our case we need the boundedness property of an almost orthogonal matrix on weighted spaces. This property is proved by the vector valued weighted inequality for maximal operators in [1] and similar arguments in [4, p.54].

4 Outline of the proof of Theorem 1.3

We shall prove Theorem 1.3 for the case $\gamma = 0$. The general case is proved by this special case. The detail of the proof is in [16]. By (ii) of Proposition 3.1 there exists a constant s such that $1 < s < q$ and $uw^{-q} \in A_{q/s}$. Let $v(x) = (M(V^s)(x))^{1/s}$. By the properties of the maximal operator we have $V(x) \leq v(x)$ *a.e.*. By (i) of Proposition 3.1 we get

$$\int_{\boldsymbol{R}^n} \left(\frac{v}{w}\right)^q u\,dx = \int_{\boldsymbol{R}^n} \frac{M(V^s)^{q/s}}{w^q} u\,dx \leq c_1 \int_{\boldsymbol{R}^n} \left(\frac{V}{w}\right)^q u\,dx < \infty.$$

Furthermore v is an A_1-weight by (iii) of Proposition 3.1.

Now we fix a $\delta > 0$ and set

$$\mathcal{I} = \{\lambda \in \Lambda : \int_{Q(\lambda)} v(x)\,dx \geq \delta |Q(\lambda)|^{-2m/n} \int_{Q(\lambda)} w(x)\,dx\}.$$

Lemma 4.1 *$\mathcal{I}$ is a finite set.*

For $f \in C_0^\infty(\boldsymbol{R}^n)$ we have

$$\int |f|^2 V\,dx \leq \int |f|^2 v\,dx \leq c_2 \sum_{\lambda\in\Lambda} |(f,\psi_\lambda)|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v\,dx,$$

where we used Proposition 3.2 and the fact $v \in A_1 \subset A_2$. The last quantity is bounded by

$$\begin{aligned} c_2 \sum_{\lambda\in\mathcal{I}} |(f,\psi_\lambda)|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v\,dx + c_2 \sum_{\lambda\notin\mathcal{I}} |(f,\psi_\lambda)|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v\,dx \\ \leq c_2 K \sum_{\lambda\in\mathcal{I}} |(f,\psi_\lambda)|^2 + c_2\delta \sum_{\lambda\notin\mathcal{I}} |(f,\psi_\lambda)|^2 |Q(\lambda)|^{-2m/n} \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} w\,dx \\ \leq c_2 K \|f\|_2^2 + c_3 \delta \int |(-\Delta)^{m/2} f(x)|^2 w(x)\,dx, \end{aligned}$$

where

$$K = \max_{\lambda\in\mathcal{I}} \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v\,dx$$

and we used Proposition 3.2.

Now we use the following lemma([16, Lemma 3.2]).

Lemma 4.2 *Let $m \in \boldsymbol{N}$ and $w \in A_2$. Then there exists a positive constant $c > 0$ such that*

$$\int_{\boldsymbol{R}^n} |(-\Delta)^{m/2} f(x)|^2 w(x)\,dx \leq c \int_{\boldsymbol{R}^n} \{\sum_{|\alpha|=m} |D^\alpha f(x)|^2\} w(x)\,dx$$

for all $f \in C_0^\infty(\boldsymbol{R}^n)$.

By Lemma 4.2 and the condition (1) we have

$$\begin{aligned}\int_{\boldsymbol{R}^n} |f|^2 V\,dx &\le c_2 K\|f\|_2^2 + c_4\delta \int_{\boldsymbol{R}^n} \{\sum_{|\alpha|=m} |D^\alpha f(x)|^2\} w(x)\,dx \\ &\le c_2 K\|f\|_2^2 + c_4\delta (L_0 f, f).\end{aligned}$$

We choose δ such that $c_4\delta < 1$. Then we have

$$a(f,f) - \int_{\boldsymbol{R}^n} V|f|^2\,dx \ge -c_2 K\|f\|_2^2$$

for all $f \in C_0^\infty(\boldsymbol{R}^n)$. Hence

$$b(f,g) = a(f,g) - \int_{\boldsymbol{R}^n} V f\bar{g}\,dx$$

is a lower semi-bounded quadratic form on $\mathcal{H}$.

We can show that $b(f,g)$ is a closed form on $\mathcal{H}$. Since $b(f,g)$ is a closed and lower semi-bounded quadratic form on $\mathcal{H}$, there exists a unique self-adjoint operator L in $L^2(\boldsymbol{R}^n)$ with domain $\mathcal{D} \subset \mathcal{H}$ such that

$$(Lf,g) = a(f,g) - \int_{\boldsymbol{R}^n} V f\bar{g}\,dx$$

for all $f \in \mathcal{D}$ and $g \in \mathcal{H}$([11, Theorem VIII.15]).

We shall estimate the number of negative eigenvalues of L. Let

$$F = \{f \in \mathcal{D} : (f, \psi_\lambda) = 0 \text{ for all } \lambda \in \mathcal{I}\}.$$

Then the similar arguments as before lead to the estimate

$$\int |f|^2 V\,dx \le c_4\delta(L_0 f, f) \quad (f \in F).$$

Hence we get

$$(Lf,f) \ge 0 \quad (f \in F).$$

Therefore by Theorem 12 in [8, Chap.1] the negative spectrum of L is discrete. Furthermore we have

$$N \le \operatorname{codim} F = \sharp\mathcal{I},$$

where N is the number of negative eigenvalues of L.

We shall estimate $\sharp\mathcal{I}$. The following arguments are similar to those in [13, p.201]. Let

$$\mathcal{B} = \{Q \in \mathcal{Q} : \int_Q v(x)\,dx \geq \delta|Q|^{-2m/n} \int_Q w(x)\,dx\}.$$

Let $\tilde{\mathcal{B}}$ be the set of all $Q \in \mathcal{B}$ which satisfy the following condition: there exists a half size dyadic sub-cube $\tilde{Q} \subset Q$ such that $\tilde{Q}$ does not contain any dyadic cubes in $\mathcal{B}$.

Then we have the following lemma.

Lemma 4.3 $\sharp\mathcal{B} \leq 2\sharp\tilde{\mathcal{B}}$.

Lemma 4.3 is proved in Rochberg and Taibleson's paper([14, Lemma 1]). Let $Q \in \tilde{\mathcal{B}}$ and $\tilde{Q}$ be a dyadic cube which satisfies the condition in the definition of $\tilde{\mathcal{B}}$. Then we get

$$1 \leq c_5 \int_{\tilde{Q}} \left(\frac{v}{w}\right)^q u\,dx.$$

For each $Q \in \tilde{\mathcal{B}}$ we choose a $\tilde{Q}$ as above. Then these $\{\tilde{Q}\}$ are disjoint. Therefore we get

$$\sharp\tilde{\mathcal{B}} = \sharp\{\tilde{Q}\} \leq \sum_{\tilde{Q}} c_5 \int_{\tilde{Q}} \left(\frac{v}{w}\right)^q u\,dx$$
$$\leq c_5 \int_{\mathbf{R}^n} \left(\frac{v}{w}\right)^q u\,dx \leq c_6 \int_{\mathbf{R}^n} \left(\frac{V}{w}\right)^q u\,dx.$$

Hence we conclude

$$N \leq \sharp\mathcal{I} = (2^n - 1)\sharp\mathcal{B} \leq c_7 \int_{\mathbf{R}^n} \left(\frac{V}{w}\right)^q u\,dx.$$

Therefore we proved Theorem 1.3 for the case $\gamma = 0$.

5 Outline of the proof of Theorem 1.4

By (ii) of Proposition 3.1 there exists a constant s such that $1 < s < q + \gamma$ and $uw^{-q} \in A_{(q+\gamma)/s}$. Let $v(x) = (M(V^s)(x))^{1/s}$. Then we have $V(x) \leq v(x)$ *a.e.*. By (i) of Proposition 3.1 we get

$$\int_{\mathbf{R}^n} v^{q+\gamma} \frac{u}{w^q}\,dx = \int_{\mathbf{R}^n} M(V^s)^{(q+\gamma)/s} \frac{u}{w^q}\,dx \leq c_1 \int_{\mathbf{R}^n} V^{q+\gamma} \frac{u}{w^q}\,dx < \infty.$$

Furthermore v is an A_1-weight by (iii) of Proposition 3.1. By Proposition 3.2 and Lemma 4.2 we have the following lemmata.

Lemma 5.1 *There exists a positive constant* α *such that*

$$\alpha \sum_{\lambda \in \Lambda} |Q(\lambda)|^{-2m/n} |(f, \psi_\lambda)|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} w \, dx \le \int_{\boldsymbol{R}^n} \{ \sum_{|\alpha|=m} |D^\alpha f|^2 \} w \, dx$$

for all $f \in C_0^\infty(\boldsymbol{R}^n)$.

Lemma 5.2 *There exists a positive constant* β *such that*

$$\int_{\boldsymbol{R}^n} |f|^2 v \, dx \le \beta \sum_{\lambda \in \Lambda} |(f, \psi_\lambda)|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v \, dx$$

for all $f \in C_0^\infty(\boldsymbol{R}^n)$.

Now we set

$$\mathcal{I} = \{ \lambda \in \Lambda \, : \, \beta \int_{Q(\lambda)} v(x) \, dx > \alpha |Q(\lambda)|^{-2m/n} \int_{Q(\lambda)} w(x) \, dx \}.$$

Then the following lemma holds.

Lemma 5.3 *There exists a* $c > 0$ *such that*

$$\sum_{\lambda \in \mathcal{I}} \left(\frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v \, dx \right)^\gamma \le c \int_{\boldsymbol{R}^n} v^{q+\gamma} \frac{u}{w^q} \, dx$$

For $f \in C_0^\infty(\boldsymbol{R}^n)$ we have

$$\int |f|^2 V \, dx \le \int |f|^2 v \, dx \le \beta \sum_{\lambda \in \Lambda} |(f, \psi_\lambda)|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v \, dx,$$

where we used Lemma 5.2. The last quantity is bounded by

$$\begin{aligned} &\beta \sum_{\lambda \in \mathcal{I}} |(f, \psi_\lambda)|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v \, dx + \beta \sum_{\lambda \notin \mathcal{I}} |(f, \psi_\lambda)|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v \, dx \\ &\le \beta K \sum_{\lambda \in \mathcal{I}} |(f, \psi_\lambda)|^2 + \alpha \sum_{\lambda \notin \mathcal{I}} |(f, \psi_\lambda)|^2 |Q(\lambda)|^{-2m/n} \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} w \, dx \\ &\le \beta K \|f\|_2^2 + \int_{\boldsymbol{R}^n} \{ \sum_{|\alpha|=m} |D^\alpha f|^2 \} w \, dx \end{aligned}$$

where

$$K = \max_{\lambda \in \mathcal{I}} \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v \, dx$$

and we used Lemma 5.1.

By the condition (5) we have

$$\int_{\mathbf{R}^n} |f|^2 V\,dx \le \beta K \|f\|_2^2 + (L_0 f, f).$$

Hence we have

$$a(f,f) - \int_{\mathbf{R}^n} V|f|^2\,dx \ge -\beta K \|f\|_2^2$$

for all $f \in C_0^\infty(\mathbf{R}^n)$. Therefore

$$b(f,g) = a(f,g) - \int_{\mathbf{R}^n} V f \bar{g}\,dx$$

is a lower semi-bounded quadratic form on $\mathcal{H}$. We can show that $b(f,g)$ is a closed form on $\mathcal{H}$. Since $b(f,g)$ is a closed and lower semi-bounded quadratic form on $\mathcal{H}$, there exists a unique self-adjoint operator L in $L^2(\mathbf{R}^n)$ with domain $\mathcal{D} \subset \mathcal{H}$ such that

$$(Lf,g) = a(f,g) - \int_{\mathbf{R}^n} V f \bar{g}\,dx$$

for all $f \in \mathcal{D}$ and $g \in \mathcal{H}$([11, Theorem VIII.15]).

We set

$$\lambda_1 = \inf_{f \in \mathcal{D}, \|f\|=1} (Lf, f)$$

and

$$\lambda_k = \sup_{\phi_1,\cdots,\phi_{k-1} \in L^2} \inf_{f \in \mathcal{D}, \|f\|=1, f \perp \phi_1,\cdots,\phi_{k-1}} (Lf, f)$$

for $k \in \mathbf{N}, k \ge 2$. There are two cases.

(i) $\lambda_1 \le \lambda_2 \le \cdots$ are eigenvalues of L.

(ii) $\lambda_1 \le \cdots \le \lambda_{k_0}$ are eigenvalues of L. Furthermore we have $\lambda_{k_0+1} = \lambda_{k_0+2} = \cdots$ which value is the infimum of the essential spectrum of L.

The following lemma holds.

Lemma 5.4 *For $A > 0$ we set*

$$\mathcal{I}_A = \{\lambda \in \Lambda : \alpha |Q(\lambda)|^{-1-2m/n} \int_{Q(\lambda)} w\,dx - \beta |Q(\lambda)|^{-1} \int_{Q(\lambda)} v\,dx \le -A\}.$$

Then $\mathcal{I}_A$ is a finite set.

Let $\{\mu_k\}_{k=1}^{\infty}$ be the non-decreasing rearrangement of

$$\left\{\alpha|Q(\lambda)|^{-1-2m/n}\int_{Q(\lambda)} w\,dx-\beta|Q(\lambda)|^{-1}\int_{Q(\lambda)} v\,dx\right\}_{\lambda\in\mathcal{I}}.$$

Then

$$\mu_1\le\mu_2\le\cdots$$

and

$$\lim_{k\to\infty}\mu_k=0.$$

When

$$\mu_k=\alpha|Q(\lambda)|^{-1-2m/n}\int_{Q(\lambda)} w\,dx-\beta|Q(\lambda)|^{-1}\int_{Q(\lambda)} v\,dx,$$

we set $\psi_k=\psi_\lambda$. Then we have

$$\begin{aligned}\lambda_k &\ge \inf_{f\in\mathcal{D},\|f\|=1,f\perp\psi_1,\dots,\psi_{k-1}}(Lf,f)\\ &\ge \inf_{f\in\mathcal{D},\|f\|=1,f\perp\psi_1,\dots,\psi_{k-1}}\sum_{j=1}^{\infty}|(f,\psi_j)|^2\mu_j\\ &\ge \mu_k\sup_{f\in\mathcal{D},\|f\|=1,f\perp\psi_1,\dots,\psi_{k-1}}\sum_{j=k}^{\infty}|(f,\psi_j)|^2\ge\mu_k,\end{aligned}$$

where we used the fact $\mu_k<0$.

Since

$$\lim_{k\to\infty}\mu_k=0,$$

the negative spectrum of L is discrete. By these inequalities we have

$$\begin{aligned}&\sum_{k,\lambda_k<0}|\lambda_k|^\gamma\le\sum_{k=1}^{\infty}|\mu_k|^\gamma\\ &=\sum_{\lambda\in\mathcal{I}}\left(\beta|Q(\lambda)|^{-1}\int_{Q(\lambda)} v\,dx-\alpha|Q(\lambda)|^{-1-2m/n}\int_{Q(\lambda)} w\,dx\right)^\gamma\\ &\le\sum_{\lambda\in\mathcal{I}}\left(\beta|Q(\lambda)|^{-1}\int_{Q(\lambda)} v\,dx\right)^\gamma\\ &\le c\int_{\mathbf{R}^n} v^{q+\gamma}\frac{u}{w^q}\,dx\le c\int_{\mathbf{R}^n} V^{q+\gamma}\frac{u}{w^q}\,dx,\end{aligned}$$

where we used Lemma 5.3.

6 The Sobolev-Lieb-Thirring inequality

As an application of Theorem 1.1 Lieb and Thirring proved the following inequality.

Theorem 6.1 *Suppose $n \in \boldsymbol{N}$, $\phi_i \in H^1(\boldsymbol{R}^n)$ $(i = 1, \dots, N)$, and that $\{\phi_i\}_{i=1}^N$ is an orthonormal family in $L^2(\boldsymbol{R}^n)$. Then we have*

$$\int_{\boldsymbol{R}^n} \rho^{1+2/n} dx \le c_n \sum_{i=1}^N \int_{\boldsymbol{R}^n} |\nabla \phi_i|^2 dx,$$

where

$$\rho(x) = \sum_{i=1}^N |\phi_i(x)|^2.$$

This inequality has important applications such as the stability of matter or the estimates of the dimension of attractors of nonlinear equations.

A generalization of the Sobolev-Lieb-Thirring inequality is known([7]).

Theorem 6.2 *Let $n, m \in \boldsymbol{N}$ and $\phi_i \in H^m(\boldsymbol{R}^n)$ $(i = 1, \dots, N)$. Suppose that $\{\phi_i\}_{i=1}^N$ is an orthonormal family in $L^2(\boldsymbol{R}^n)$. Then we have*

$$\int_{\boldsymbol{R}^n} \rho^{1+2m/n} dx \le c \sum_{i=1}^N \int_{\boldsymbol{R}^n} \sum_{|\alpha|=m} |D^\alpha \phi_i|^2 dx,$$

where

$$\rho(x) = \sum_{i=1}^N |\phi_i(x)|^2.$$

By Theorem 1.3 we have the following generalization of Theorem 6.2.

Theorem 6.3 *Let $m, n \in \boldsymbol{N}$, and $n > 2m$. Let w be a weight in $A_2 \cap H^m_{loc}(\boldsymbol{R}^n)$ such that $w^{-n/(2m)} \in A_{n/(2m)}$. Suppose that $\{\phi_i\}_{i=1}^N$ is an orthonormal family in $L^2(\boldsymbol{R}^n)$ such that*

$$\sum_{i=1}^N \int_{\boldsymbol{R}^n} \left\{ \sum_{|\alpha|=m} |D^\alpha \phi_i(x)|^2 \right\} w(x)\, dx < \infty.$$

Then we have

$$\int_{\boldsymbol{R}^n} \rho(x)^{1+2m/n} w(x)\, dx \le c \sum_{i=1}^N \int_{\boldsymbol{R}^n} \left\{ \sum_{|\alpha|=m} |D^\alpha \phi_i(x)|^2 \right\} w(x)\, dx,$$

$$\rho(x) = \sum_{i=1}^{N} |\phi_i(x)|^2$$

and c is a positive constant which does not depend on $\{\phi_i\}_{i=1}^N$.

Example of weights Let a be a number satisfying $m - n/2 < a < 2m$. Then

$$w(x) = |x|^a$$

is an example of weights which satisfy the conditions of Theorem 6.3.

We have a similar theorem in low dimensional cases.

Theorem 6.4 *Let* $m, n \in \boldsymbol{N}$, *and* $n \leq 2m$. *Let* w *be a weight in* $A_2 \cap H^m_{loc}(\boldsymbol{R}^n)$ *such that* $w^{-n/(2m)} \in A_{1+n/(2m)}$ *and*

$$\int_{Q'} w\, dx \leq 2^{2m} \int_Q w\, dx$$

for all dyadic cubes Q, Q' *such that* Q' *is the parent of* Q. *Suppose that* $\{\phi_i\}_{i=1}^N$ *is an orthonormal family in* $L^2(\boldsymbol{R}^n)$ *such that*

$$\sum_{i=1}^{N} \int_{\boldsymbol{R}^n} \left\{ \sum_{|\alpha|=m} |D^\alpha \phi_i(x)|^2 \right\} w(x)\, dx < \infty.$$

Then we have

$$\int_{\boldsymbol{R}^n} \rho(x)^{1+2m/n} w(x)\, dx \leq c \sum_{i=1}^{N} \int_{\boldsymbol{R}^n} \left\{ \sum_{|\alpha|=m} |D^\alpha \phi_i(x)|^2 \right\} w(x)\, dx,$$

where

$$\rho(x) = \sum_{i=1}^{N} |\phi_i(x)|^2$$

and c is a positive constant which does not depend on $\{\phi_i\}_{i=1}^N$.

The proofs of these theorems will appear elsewhere.

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Mathematical Institute, Tohoku University, Sendai, Japan
e-mail : tachizaw@math.tohoku.ac.jp
current address :
Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo 060-0810, Japan
e-mail : tachizaw@math.sci.hokudai.ac.jp