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ISOMETRIC IMMERSIONS OF $\mathbb{E}^2$ INTO $\mathbb{E}^4$

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We seek isometric immersions of $\mathbb{E}^2$ into $\mathbb{E}^4$. We hope to find them all but at the least we would like to find some new ones. Such immersions with normal curvature zero will be the principal focus of this note.

First, let us recall what isometric immersions of $\mathbb{E}^2$ into $\mathbb{E}^4$ are known to exist. The most trivial such examples are the following: Let $X : \mathbb{E}^2 \to \mathbb{E}^3$ be an isometric immersion and consider $\mathbb{E}^3$ as a hyperplane in $\mathbb{E}^4$. Then we have the isometric immersion $X : \mathbb{E}^2 \to \mathbb{E}^4$. We want to rule out this kind of situation occurring, even infinitesimally. The *first normal space* of an immersion at a point of its domain is the span of the range of the second fundamental form at that point. If at all points of the domain of the isometric immersion

$$\dim(\text{first normal space of } X) = 2$$

we say that the immersion is *nondegenerate*. By requiring that the isometric immersion of $\mathbb{E}^2$ into $\mathbb{E}^4$ is nondegenerate we will indeed rule out that the image even infinitesimally lies in a hyperplane of $\mathbb{E}^4$.

There are two known classes of nondegenerate isometric immersions. One gets immersions of one class as follows: Let $Y : \mathbb{E}^2 \to \mathbb{E}^3$ and $Z : \mathbb{E}^3 \to \mathbb{E}^4$ be given isometric immersions, each with nowhere trivial second fundamental form. Necessarily they are to both cylindrical immersions. Then $X = Z \circ Y$ is a nondegenerate isometric immersion. Following M. do Carmo and M. Dajczer [1] we call such immersions *trivial* immersions.

A second class of immersions consists of all isometric immersions of $X : \mathbb{E}^2 \to \mathbb{S}^3$ which one can view as mappings into $\mathbb{E}^4$ by regarding $\mathbb{S}^3$ as a submanifold of $\mathbb{E}^4$. It is known [2, 4] that there is a large class of immerisons of flat tori into $\mathbb{S}^3$; by going to their covering spaces one gets a large class of immersions of $\mathbb{E}^2$ into the 3-sphere. For example, all Hopf tori give rise to isometric immersions of $\mathbb{E}^2$ into $\mathbb{E}^4$. By the same techniques that produce the immersions of flat tori, one can get additional isometric immersions of $\mathbb{E}^2$ into $\mathbb{E}^4$ which are not covering maps. It is worth mentioning now that all such immersions have trivial normal curvature (since the normal to the sphere restricted to the immersion is parallel).

To obtain more examples and hopefully all other examples we try the following strategy. Below $\mathbb{G}_{2,4}$ is the Grassmannian of oriented two-dimensional subspaces of $\mathbb{E}^4$.

1. Characterize the image of the Gauss map $G : \mathbb{E}^2 \to \mathbb{G}_{2,4}$ of a nondegenerate isometric immersion $X : \mathbb{E}^2 \to \mathbb{E}^4$.
2. Given a surface $E \subset \mathbb{G}_{2,4}$ that satisfies the characterization of 1, construct from it a nondegenerate immersion $X : E \to \mathbb{E}^4$ which induces on E a complete flat metric, so that E or its covering space is $\mathbb{E}^2$, and for which the Gauss map is the inclusion map.

Before we investigate the Gauss maps of isometric immersions of $\mathbb{E}^2$ into $\mathbb{E}^4$, let's study nondegenerate immersions $X : M \to \mathbb{E}^4$. We wish to introduce a convenient framing $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ along the immersion X. We do not want to require that it be an orthonormal framing; what we do want to require is the following:

1. $e_1(p)$ and $e_2(p)$ are tangent to $X(M)$ and $X(p)$.
2. $e_3(p)$ and $e_4(p)$ are normal to $X(M)$ and $X(p)$, i.e., $e_3(p), e_4(p) \in \nu_p(M)$, where $\nu(M)$ is the normal bundle of X.
3. $|e_i| = 1$, for $i = 1, 2, 3, 4$.

We will require more of the framing using the second fundamental form h of the immersion. Regard $h : TM \to \nu M$ as a quadratic form. The image of $\{v \in T_pM : |v| = 1\}$ under h is an ellipse in ν_pM; it is called the *curvature ellipse.* Assuming that X is nondegenerate and the Gaussian curvature $K = 0$ one may show that origin of ν_pM is in the exterior of the curvature ellipse and the tangent lines to the ellipse incident with the origin are orthogonal. Thus we may choose $\mathbf{e}_3$ and $\mathbf{e}_4$ in the direction of these tangent lines so that $\langle h, e_i\rangle$ is positive semidefinite, for $i = 3, 4$, and $\mathbf{e}_3, \mathbf{e}_4$ (in that order) induce the orientation on ν_pM. These conditions uniquely determine $\mathbf{e}_3$ and $\mathbf{e}_4$.

Let A_3 and A_4 be the Weingarten maps associated to $\mathbf{e}_3$ and $\mathbf{e}_4$, respectively. These Weingarten maps clearly are rank 1 operators; in fact, they are positive semidefinite. Choose $e_2 \in \ker A_3$ and $e_1 \in \ker A_4$ so that e_1, e_2 (in that order) induce the orientation on M. This determines $\mathbf{e}_1$ and $\mathbf{e}_2$ up to a rotation by $\frac{\pi}{2}$.

In general, $g_{12} = \langle e_1, e_2\rangle \neq 0$. Introduce $\theta : M \to (0, \pi)$ so that $\cos\theta = g_{12}$.

It is easy to show the following where N is the normal curvature of the immersion X.

Lemma 1. *$N = 0$ if and only if $g_{12} = 0$.*

Let e^1, e^2, e^3, e^4 be the dual framing, i.e. $\langle e_i, e^j\rangle = \delta_i^j$. Of course, for this framing $e_3 = e^3$ and $e_4 = e^4$. Notice that $e^1 \perp e_2$ and e_2 is a principal direction of A_3, since $A_3(e_2) = 0$. Therefore, e^1 is the other principal direction of A_3. Define k_1 by $A(e^1) = k_1e^1$; necessarily $k_1 > 0$. Similarly e^2 is a principal direction of A_4 and we define k_2 by $A(e^2) = k_2e^1$; again $k_2 > 0$.

Let ω^1, ω^2 be the framing of T^*M dual to e_1, e_1, i.e., $dX = e_1\omega^1 + e_2\omega^2$. Then by a straightforward computation one can show that

$$A_3 = \sin^2\theta\, k_1e^1\omega^1 \text{ and } A_4 = \sin^2\theta\, k_2e^2\omega^2.$$

We introduce connection forms ω_i^j defined by $de_i = e_j\omega_i^j$. Also define $\omega_{ji} = g_{jk}\omega_i^k$. The next lemma follows from our previous assertions about A_3 and A_4.

Lemma 2.

$$\begin{aligned} \omega_{31} &= \sin^2\theta\, k_1\omega^1, & \omega_{32} &= 0 \\ \omega_{41} &= 0, & \omega_{42} &= \sin^2\theta\, k_2\omega^2 \end{aligned}$$

Definition 1. *We call k_1 and k_2 the* principal curvatures *and $\omega_{31} = 0$ and $\omega_{42} = 0$ the* null directions.

In what follows i, j, k range over 1, 2 and α, β range over 3, 4. Also recall that $g_{12} \neq 0$, in general, but $g_{i\alpha} = 0$ and $g_{34} = 0$.

Remark 1. The normal curvature N can be easily computed using this framing: For any vector $v \in T_pM$, let $B_v : T_pM \to \nu_pM$ be defined by $B_v(w) = h(v, w)$. Both T_pM and ν_pM carry an element of area; hence $\det(B_v)$ makes sense. Thus define $F^\nu : TM \to \mathbb{R}$ by $F^\nu(v) = \det(B_v)$. This is a quadratic form and now let F^ν also denote its polarization. If $F_{ij}^\nu = F^\nu(e_i, e_j)$, then

$$N = \mathrm{tr}F^\nu = g^{ij}F_{ij}^\nu.$$

It is easy to show that $F^{\nu}_{ii} = \sin\theta|h^j_{\alpha i}|$, where $\omega_{\alpha i} = h^j_{\alpha i}\omega_j$, and of course, $\omega_j = g_{jk}\omega^k$. Then F^{ν}_{12} can be obtained using the standard tricks associated with the polarization.

Remark 2. As long as the immersion X is nondegenerate at $p \in M$ and the origin of $\nu_p M$ is not on the curvature ellipse we can obtain the kind of framing that we obtained above, in particular, a framing for which for which $\omega_{32} = 0$ and $\omega_{41} = 0$. In general, $\mathbf{e}_3$ is not orthogonal to $\mathbf{e}_4$. However if the origin is inside the curvature ellipse one has to complexify both the tangent space and normal space to do this.

Now let's consider the Gauss map $G : M \to \mathbb{G}_{2,4}$ associated to the immersion $X : M \to \mathbb{E}^4$. We may show the following:

Lemma 3. *Suppose the Gaussian curvature K of the immersion X is zero. Then X is nondegenerate if and only if G is an immersion.*

Identify $\mathbb{G}_{2,4}$ with the set of unit decomposable 2-vectors so that we may regard $\mathbb{G}_{2,4} \subset \mathbb{S}^5 \subset \bigwedge^2 \mathbb{E}^4$. Using the framing introduced above we may write

$$G = \frac{\mathbf{e}_1 \wedge \mathbf{e}_2}{\sin\theta}.$$

Then

$$dG = \frac{1}{\sin\theta}[\mathbf{e}_3 \wedge \mathbf{e}_2\,\omega_{31} + \mathbf{e}_1 \wedge \mathbf{e}_4\,\omega_{42}].$$

Note that the image $dG(T_pM)$ contains two one-dimensional subspaces of decomposable 2-vectors. Necessarily any of these decomposable 2-vectors is of the form (tangent vector) $\wedge$ (normal vector). Using the orientations of $G(p)$ and its orthogonal complement we obtain, easily and directly, the pair $\mathbf{e}_1, \mathbf{e}_2$ up to sign, the pair $\mathbf{e}_3, \mathbf{e}_4$ up to sign and the pair ω_{31}, ω_{42} up to sign, for each point $p \in M$. The angle θ and hence g_{12} are uniquely determined. One can not hope to obtain $\mathbf{e}_3, \mathbf{e}_4$ uniquely since $-X$ has the same Gauss map as X. The only things we cannot directly obtain from the Gauss map are k_1 and k_2. (With a knowledge of these we could obtain ω^1 and ω^2 and reconstruct the immersions $\pm X$.)

Remark 3. In the more general case where the curvature K might vary but the origin of normal space is in the exterior of the curvature ellipse, the case where $K = 0$ is characterized by the two subspaces of decomposable 2-vectors being orthogonal.

As is well-known $\mathbb{G}_{2,4} = \mathbb{S}_1 \times \mathbb{S}_2$, where $\mathbb{S}_i$ are round 2-spheres, for $i = 1, 2$. Therefore, any Gauss map $G : M \to \mathbb{G}_{2,4}$ may be represented as $G = G_1 \times G_2$, where $G_i : M \to \mathbb{S}_i$. If Ω_1 and Ω_2 are "appropriate" area elements on $\mathbb{S}_1$ and $\mathbb{S}_2$, respectively, then the following holds [3]:

$$\begin{aligned} K = 0 \ &\Leftrightarrow\ G_1^*(\Omega_1) + G_2^*(\Omega_2) \\ N = 0 \ &\Leftrightarrow\ G_1^*(\Omega_1) - G_2^*(\Omega_2) \end{aligned}$$

Thus we have the following result.

Lemma 4. *For a nondegenerate immersion $X : M \to \mathbb{E}^4$,*

$$K = N = 0 \Leftrightarrow G_1 \text{ and } G_2 \text{ are rank 1 maps.}$$

In fact, given an isometric immersion $X : \mathbb{E}^2 \to \mathbb{E}^4$ with $N = 0$, in terms of the framing introduced above

$$\begin{aligned} dG_1 &= \frac{1}{2}(\mathrm{e}_1 \wedge \mathrm{e}_4 + \mathrm{e}_2 \wedge \mathrm{e}_3)(\omega_3^1 - \omega_4^2) \\ dG_2 &= -\frac{1}{2}(\mathrm{e}_1 \wedge \mathrm{e}_4 - \mathrm{e}_2 \wedge \mathrm{e}_3)(\omega_3^1 + \omega_4^2) \end{aligned}$$

Since $N = 0$, $\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3, \mathrm{e}_4$ is an orthonormal frame, and $\omega_3^2 = \omega_4^1 = 0$. This follows immediately from Lemma 1 and thus that $\omega_B^A = \omega_{AB}$, for all $A, B = 1, 2, 3, 4$.

From now on we consider nondegenerate isometric immersions $X : \mathbb{E}^2 \to \mathbb{E}^4$ with normal curvature N identically zero. By Lemma 4, $G_i(\mathbb{E}^2) = C_i$ is a curve in $\mathbb{S}_i$, for $i = 1, 2$. All we can say at the moment is that $G(\mathbb{E}^2) \subset C_1 \times C_2$. In fact, $G : \mathbb{E}^2 \to C_1 \times C_2$ need not be surjective nor injective. We only know it is a local diffeomorphism.

By Lemma (2) with $\theta = \frac{\pi}{2}$, we know that there exist functions $u, v : \mathbb{E}^2 \to \mathbb{R}$, such that $dX = \mathrm{e}_1 u\, \omega_3^1 + \mathrm{e}_2 v\, \omega_4^2$. In fact, u and v are just the reciprocals of the principal curvatures. However, $N = 0$ implies that $d\omega_3^1 = d\omega_4^2 = 0$. Thus there exist functions $x, y : \mathbb{E}^2 \to \mathbb{R}$ such that $\omega_3^1 = dx$ and $\omega_4^2 = dy$. Note that x and y are independent since $dx \wedge dy = \omega_3^1 \wedge \omega_4^2 \neq 0$ because G is an immersion. Thus,

(1) $$dX = \mathrm{e}_1 u\, dx + \mathrm{e}_2 v\, dy.$$

Using the fact that $d(dX)) = 0 \Leftrightarrow d(d(X))^{\perp} = 0$ we obtain the following first order linear system of p.d.e's:

(2) $$\begin{cases} u_y = \omega_2^1(\frac{\partial}{\partial x})v \\ v_y = \omega_1^2(\frac{\partial}{\partial y})u \end{cases}$$

This is the normal form for a hyperbolic system of p.d.e's on $\mathbb{E}^2$. They are all that remain of the Codazzi equations.

If κ_i is the geodesic curvature of C_i (pulled back to $\mathbb{E}^2$) then we may write system (2) as follows:

(3) $$\begin{cases} u_y = \frac{1}{2\sqrt{2}}(\kappa_1 + \kappa_2)v \\ v_x = \frac{1}{2\sqrt{2}}(\kappa_1 - \kappa_2)u \end{cases}$$

Since u and v are the reciprocals of the principal curvatures, we must have

(4) $$\begin{cases} u > 0 \\ v > 0 \end{cases}$$

Definition 2. *A solution of the system* (2), *or system* (3), *is called* positive *if* (4) *holds.*

The curves $x = const.$ and $y = const.$ are the characteristic curves of the system (3). It is also clear they are the the null curves of the immersion. These curves make angles of $\frac{\pi}{4}$ with the factors C_i in the metric induced on $\mathbb{E}^2$ by G. In fact, if s^i is the arc length along

C_i (pulled back to $\mathbb{E}^2$), then at straightforward computation shows:

$$\begin{aligned} ds^1 &= \frac{1}{\sqrt{2}}(dx - dy) \\ ds^2 &= \frac{1}{\sqrt{2}}(dx + dy). \end{aligned}$$

However it is especially worth noting that all the structure we have introduced on $\mathbb{E}^2$ in fact exists on $C_1 \times C_2$ and we have pulled it back to $\mathbb{E}^2$. More precisely since $C_1 \times C_2$ is a surface in $\mathbb{G}_{2,4}$, the framing e_1, e_2, e_3, e_4, the forms ω_3^1, ω_4^2 and hence the functions x, y already exist on it. Of course, we may have to go to the simply connected covering of $C_1 \times C_2$ to achieve this. Moreover κ_1 and κ_2 exist on $C_1 \times C_2$. Thus the system (3) with unknowns u, v exists on $C_1 \times C_2$ or its simply connected covering space. What we see on $\mathbb{E}^2$ is the pull-back of this system. Setting u, v to be the reciprocals of the principal curvatures gives a multi-valued solution on $G(\mathbb{E}^2)$. It may be multi-valued since G need not be one-to-one.

We can visualize the characteristic curves as well as the image $G(\mathbb{E}^2)$ in $C_1 \times C_2$; in this fashion one "sees how the characteristic curves run" on $\mathbb{E}^2$. From this perspective and using the fact that $\mathbb{E}^2$ is complete, i.e., $u^2\, dx^2 + v^2\, dy^2$ determines a complete metric, we may show the following:

Proposition 1. *The image $G(\mathbb{E}^2)$ in $C_1 \times C_2$ (or its simply connected covering) is a convex set whose boundary in $C_1 \times C_2$ is a union of characteristic curves of the system (3). Also G is one-to-one (into the covering space).*

If, for example, C_1 and C_2 have finite length, Figure 1 shows what $G(\mathbb{E}^2)$ might look like. The rectangle represents $C_1 \times C_2$ and the shaded region represents $G(\mathbb{E}^2)$. Suppose we identify C_i with $(a_i, b_i) \subset \mathbb{R}$ by means of the arclength function s^i, for $i = 1, 2$. Let $\overline{C_1 \times C_2} = [a_1, b_1] \times [a_2, b_2]$. If the boundary of $G(\mathbb{E}^2)$ in $\overline{C_1 \times C_2}$ intersects $\{a_1\} \times C_2$ in more than a single point as it does in Figure 1, assuming C_1 is the horizontal factor, then one may show that C_1 may not be extended behond a_1 as a C^2-curve. Thus also for the situation depicted in Figure 1, the curve C_2 can not be extended as a C^2-curve in either direction.

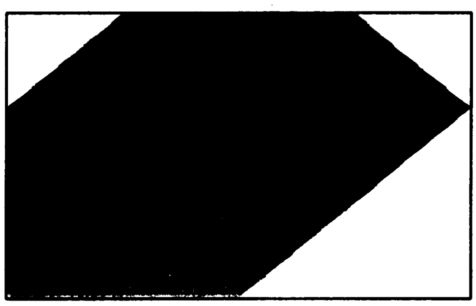

FIGURE 1. $G(\mathbb{E}^2)$ in $C_1 \times C_2$

In what follows we will identify C_i with $s^i(C_i) \subset \mathbb{R}$ and hence $C_1 \times C_2$ with a rectangular subset of $\mathbb{R}^2$. Similarly $G(\mathbb{E}^2)$ will be regarded as a subset of $\mathbb{R}^2$ and any statements regarding the topology of these sets will be relative to the topology on $\mathbb{R}^2$.

Definition 3. *Assume G is the Gauss map of immersion $X : M \to \mathbb{E}^4$ with $K = N = 0$. We say G is* normal *if the boundary of $G(\mathbb{E}^2)$ intersects each bounding edge of the rectangle $C_1 \times C_2$ in at most (and hence exactly) one point.*

If in both directions C_i is either of infinite length or can be extended as a C^2-curve, for $i = 1, 2$, then necessarily G is normal. Also if G is normal and one of the C_i has finite length then both C_1 and C_2 have the same finite length.

Conjecture 1. *If $X : \mathbb{E}^2 \to \mathbb{E}^4$ is nondegenerate isometric immersion with $N = 0$, then its Gauss map G is normal.*

What is particularly nice about normal G is that $G(\mathbb{E}^2)$ fits a characteristsic initial value problem for the system (3) so well, i.e., the domain of determinacy of such a problem on $C_1 \times C_2$ contains $G(\mathbb{E}^2)$.

Recall that s^1, s^2 and x, y are coordinate systems on $C_1 \times C_2$ whose coodinate axes make an angle of $\frac{\pi}{2}$ with each other. It is convenient to assume that $(0, 0)$ are the coordinates of some point in $G(\mathbb{E}^2)$ in both coordinate systems. Since G is one-to-one, we may also regard these as coordinate systems on $\mathbb{E}^2$. Let $(x, y)(\mathbb{E}^2) = I_x \times I_y$ where $I_x = (a_x, b_x)$ and $I_y = (a_y, b_y)$. Of course, a_x or b_x could be $-\infty$, etc. Since the metric $u^2\,dx^2 + v^2\,dy^2$ on $I_x \times I_y$ is complete,

$$\int_0^{b_x} u(x,0)dx = \int_{a_x}^0 u(x,0)dx = \int_0^{b_y} v(0,y)dy = \int_{a_y}^0 v(0,y)dy = \infty. \tag{5}$$

Can we reverse the process, i.e., starting with a surface in $\mathbb{G}_{2,4}$ that could be the image of normal Gauss map, can we find an isometric immersion of $\mathbb{E}^2$ into $\mathbb{E}^4$ with this surface as the image of its Gauss map?

We begin by choosing curves $C_i \subset \mathbb{S}_i$, for $i = 1, 2$. Either both are infinitly long in either direction, both are infinitely long in just one direction, or both have the same finite length. Let $M = C_1 \times C_2$ or its simply connected covering space and $G : M \to \mathbb{G}_{2,4}$ the inclusion map or covering map. Of course, $M = \mathbb{R}^2$. Give M the metric induced by G; this is a flat metric. Introduce coordinates $(x, y) : M \to \mathbb{R}^2$ whose level curves make an angle of $\frac{\pi}{4}$ with the factors of M such that $dx^2 + dy^2$ is the induced metric. Choose a rectangle E in x, y-coordinates which projects onto the factors of M. As above we may write $E = I_x \times I_y$ and suppose a point of E has coordinates $(0, 0)$.

The system (3) exists on M and hence E. We consider a characteristic initial value problem on E associated to system (3). We place initial data on I_x and I_y. It is well-known that given positive functions

$$\phi : I_x \to \mathbb{R} \quad \text{and} \quad \psi : I_y \to \mathbb{R},$$

there exists a unique solution of system (3) satisfying

$$\begin{cases} u(x,0) = \phi(x) & \text{for } x \in I_x \\ v(0,y) = \psi(y) & \text{for } y \in I_y. \end{cases} \tag{6}$$

If the characteristic initial value problem consisting of system (3) and initial conditions (6) has a positive solution u, v, then there exists and immersion $X : E \to \mathbb{E}^4$ whose differential is given by equation (1) and with Gauss map G. Necessarily the induced metric is flat and the normal curvature is trivial, since these conditions are determined by the Gauss map. In general, the solution u, v will not be postive and the resulting differentiable mapping X will have singularities. These are "well-behaved" singularities since X has well-defined tangent planes at all its points. If $X(E)$ has singularities then they are cuspidal edges which may intersect transversally, generically.

Assuming a positive solution of system (3) exists, one still needs to show it gives rise to a complete metric. At the very least one must require that conditions (5) hold. If $u^2\,dx^2+v^2\,dy^2$ determines a complete metric then $E = \mathbb{E}^2$ and our goal is achieved.

Some observations.

1. Replacing C_i by congruent curves, for $i = 1, 2$, but retaining the same rectangle and initial data ϕ and ψ gives rise to congruent X. So the given data for this problem is not so much C_1, C_2 but κ_1, κ_2 as well as ϕ, ψ.
2. We may show that for a positive solution to exist for system (3) the following must hold:

$$\begin{aligned} \int_0^{b_x} (\kappa_2 - \kappa_1)\phi\,dx &< \infty \\ \int_{a_x}^0 (\kappa_2 - \kappa_1)\phi\,dx &> -\infty \\ \int_0^{b_y} (\kappa_1 + \kappa_2)\psi\,dy &> -\infty \\ \int_{a_y}^0 (\kappa_1 + \kappa_2)\psi\,dx &< \infty \end{aligned}$$

   When considered along with conditions (5) these impose restrictions on the curvatures near the ends of C_1 and C_2 in order that a nondegenerate isometric immersion X of $\mathbb{E}^2$ with trivial normal curvature is to exist with these curves as the images of the factors of the Gauss map of X. For example, if $\kappa_1 = const.$ and $\kappa_2 = const.$ then there exist no such isometric immersions of $\mathbb{E}^2$ unless $\kappa_1 = \kappa_2 = 0$. More generally, no isometric immersions of $\mathbb{E}^2$ can be constructed from a given $C_1 \times C_2$ if, say, for all s^1, s^2, $\kappa_2(s^2) > \kappa_1(s^1) + c$, where $c > 0$.
3. We may show that the nondegenerate trivial immersions with $N = 0$ are characterized in terms of the Gaussian image by the condition that $\kappa_1 = \pm\kappa_2 = const.$ In light of the previous observation, we conclude that the only nondegenerate trivial immersions with $N = 0$ are the sums of two curves in orthogonal 2-planes.
4. Introduce a function $\beta : E \to \mathbb{R}$ by

$$d\beta = \frac{1}{2}(\kappa_1\,ds^1 + \kappa_2\,ds^2).$$

   Of course this only determines β up to a constant. In any case, $\beta_x = \frac{1}{2\sqrt{2}}(\kappa_1 - \kappa_2)$ and $\beta_y = -\frac{1}{2\sqrt{2}}(\kappa_1 + \kappa_2)$. Hence, system (3) can be written:

$$(7) \qquad \begin{cases} u_y = -\beta_y v \\ v_x = \beta_x u \end{cases}$$

One easily sees that

$$(8) \qquad \begin{cases} u &= \cos\beta \\ v &= \sin\beta \end{cases}$$

is a solution of system (3). If β can be chosen so that $\cos\beta > 0$ and $\sin\beta > 0$ on E, then we get examples of immersions $X : \mathbb{E}^2 \to \mathbb{S}^3$ when the metric induced on E by X is complete. In fact we get all such examples in this fashion.

Questions.

1. Is $G(\mathbb{E}^2)$ indeed normal for all nondegenerate isometric immersions with $N = 0$?
2. Assuming Gaussian images of isometric immersions of $\mathbb{E}^2$ with $N = 0$ are normal, for which κ_i do there exist positive solutions of system (3) other than solutions (8)?
3. If system (3) has positive solutions for particular κ_i, what initial conditions produce those positive solutions?
4. Which of these positive solutions induce complete metrics?

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