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Siegel保型形式の様々な持ち上げに付随するKoecher-Maaß級数 (Koecher-Maaß Dirichlet series for various liftings of Siegel modular forms)

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1 Introduction

Let $f(Z)$ be a Siegel modular form of weight k belonging to the symplectic group $\Gamma_n = Sp_n(\mathbf{Z})$. Then $f(Z)$ has the following Fourier expansion:

$$f(Z) = \sum_A a_f(A) exp(2\pi i\ tr(AZ)),$$

where A runs over all semi-positive definite half-integral matrices over $\mathbf{Z}$ of degree n and $tr(X)$ denotes the trace of a matrix X. We then define the Koecher-Maaß Dirichlet series $L(f,s)$ by

$$L(f,s) = \sum_A \frac{a_f(A)}{e(A)(\det A)^s},$$

where A runs over a complete set of representatives of $SL_n(\mathbf{Z})$-equivalence classes of positive definite half-integral matrices of degree n, and $e(A) = \#\{A \in SL_n(\mathbf{Z});\ {}^tXAX = A\}$. We remark that in case $n=1$, $L(f,s)$ is nothing but the Hecke L-series attached to f.

Now let $F(W)$ be a certain lifting of $f(Z)$. Namely let $F(W)$ be a modular form with respect to Γ_m with some integer $m \geq n$ whose standard zeta function or spinor L-function is expressed by the standard zeta function or the spinor L-function of $f(Z)$. Then we present the following problem:

Problem 1. Express $L(F, s)$ in terms of Dirichlet series attached to f.

In this note, we consider the following two types of liftings, one the Klingen-Eisenstein lifting, and the other the Ikeda lifting. This work was partly collaborated with T. Ibukiyama.

2 Koecher-Maaß Dirichlet series for the Klingen-Eisenstein lifting

Let r, n and k be non-negative integers such that $0 \le r \le n \le k - r - 2$ and $k \equiv 0 \bmod 2$. For a cusp form f of weight k belonging to Γ_r, define $[f]_r^n(Z)$ as

$$[f]_r^n(Z) = \sum_{M \in \Delta_{n,r} \backslash \Gamma_n} f(M < Z >^*) j(M, Z)^{-k},$$

where $\Delta_{n,r} = \{ \begin{pmatrix} * & * \\ O_{n-r,n+r} & * \end{pmatrix} \in \Gamma_n \}$, and for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$ let $M < Z >^*$ denote the upper left $(r \times r)$-block of the matrix $(AZ + B)(CZ + D)^{-1}$ and $j(M, Z) = \det(CZ + D)$. We note that $[1]_0^n(Z)$ is nothing but the Siegel Eisenstein series $E_{n,k}(Z)$ of weight k. In [B], among others, Böcherer gave an explicit form of $L([f]_1^2, s)$ and $L(E_{2,k}, s)$. In [I-K1] we gave an explicit form of $L(E_{n,k}, s)$ for arbitrary n. We note that $L(E_{n,k}, s)$ is also regarded as the zeta function of prehomogeneous vector space. From this point of view, Saito gave a generalization of our result (cf. [Sa]). In relation to the above Problem 1 we should add one remark; in the explicit formula for $L([f]_1^2, s)$ by [B], a certain Dirichlet series attached to f appears. Böcherer obtained a functional equation for it from the general theory of the Koecher-Maaß Dirichlet series. This Dirichlet series is a modification of the Dirichlet series originally defined by Kohnen and Zagier [K-Z], and is of importance in its own right. Hence the following problem seems very interesting.

Problem 2. Investigate the analytic and arithmetic properties of the Dirichlet series related to f appearing in an explicit formula for $L([f]_r^n, s)$.

In this section, we give a resonable formula for $[f]_1^n$ when f is a cuspidal Hecke eigenform belonging to Γ_1 and n even. This also gives a certain generalization of Böcherer's result in [B].

Now to state our main result in this section, for the fundamental discriminant d of a quadratic field, let ψ_d denote the Kronecker character associated with d. Here we understand that $\psi_1 = 1$. For $l = \pm 1$, put

$\mathcal{F}_l = \{D_0 \in \mathbf{Z}_{>0}; lD_0$ is the fundamental discriminant of a quadratic field or 1

For an integer D such that $lD > 0$ and $D \equiv 1$ or $\equiv 0 \bmod 4$, write $D = lD_0 m^2$ with $D_0 \in \mathcal{F}_l, m > 0$, and put

$$L_D(s) = L(s, \psi_{lD_0}) \sum_{d|m} \mu(d)\psi_{lD_0}(d)d^{-s} \sum_{c|md^{-1}} c^{1-2s},$$

where $L(s, \psi_{lD_0})$ is the Dirichlet L-function attached to ψ_{lD_0}, and μ is the Möbius function. Write $L_D(s)$ as

$$L_D(s) = \sum_{e=1}^{\infty} \epsilon_D(e)e^{-s},$$

and for a cusp form $f(z) = \sum_{e=1}^{\infty} b(e)exp(2\pi iez)$ of weight k with respect to Γ_1 put

$$L(f, D, s) = \sum_{e=1}^{\infty} \epsilon_D(e)b(e)e^{-s}.$$

We note that

$$L(f, 1, s) = L(f, s).$$

Further for $l = \pm 1$

$$\mathcal{L}_l(f; \lambda, s) = \sum_D L(f, lD, \lambda)D^{-s},$$

where D runs over all positive integers such that $D \equiv l, 0 \bmod 4$. This type of Dirichlet series was originally introduced by Kohnen and Zagier [K-Z]. Assume that f is a Hecke eigenform. Then we note that

$$\mathcal{L}_l(f; \lambda, s) = \frac{\zeta^{st}(f, 2s + 2\lambda - k)\zeta(2s)}{\zeta(2s + 2\lambda - k)} \sum_{D_0 \in \mathcal{F}_l} D_0^{-s} L(f, lD_0, \lambda)$$

$$\times \prod \{(1 + \psi_{lD_0}(p)^2 p^{-2s+k-1-2\lambda})(1 + p^{-2s+k-2\lambda}) - \psi_{lD_0}(p)b(p)p^{-2s-\lambda}(1 + p^{k-2\lambda})\},$$

where $\zeta(s)$ is Riemann's zeta function and $\zeta^{st}(f,s)$ is the standard zeta function of f.

Theorem 1. *Let n be an even positive integer. Then, under the above assumption, we have*

$$L([f]_1^n, s)$$

$$= 2^{ns}\alpha_{n,k}[\frac{L(f,k-n/2)}{\zeta^{st}(f,k-1)}\zeta(2s-1)\prod_{i=1}^{n/2-1}\zeta(2s-2i-1)\zeta(2s-2k+2i+2)$$

$$\times\mathcal{L}_{(-1)^{n/2}}(f;k-1,s-k+3/2)$$

$$+(-1)^{n(n-2)/8}\frac{L(f,k-1)}{\zeta^{st}(f,k-1)}\zeta(2s-n+1)\prod_{i=1}^{n/2-1}\zeta(2s-2i)\zeta(2s-2k+2i+1)$$

$$\times\mathcal{L}_{(-1)^{n/2}}(f;k-n/2,s-k+(n+1)/2)],$$

where $\alpha_{n,k}$ is a constant depending only on n and k.

As for the proof, see [I-K2]. By the above theorem combined with the general theory of $L([f]_1^n,s)$ obtained by [M], we obtain

Corollary. *Assume that $n \equiv 2 \bmod 4$. Put*

$$\mathbf{L}_{-1}(f;\lambda,s) = \pi^{(2\lambda-2k)(s+\lambda-1/2)}\zeta(2s+4\lambda-2k)\Gamma(s+\lambda-1/2)\Gamma(s+\lambda-1)\mathcal{L}_{-1}(f;\lambda,s).$$

Then $\mathbf{L}_{-1}(f;k-n/2,s)$ can be continued analytically to a meromorphic function of s in the whole complex plane, and has the following functional equation:

$$\mathbf{L}_{-1}(f;k-n/2,n+1-s-k) = \mathbf{L}_{-1}(f;k-n/2,s).$$

Remark. If $n = 2$, the two terms in the above formula coincide with each other, and unify in one term. This is nothing but Böcherer's result [B,

3 Koecher-Maaß Dirichlet series for the Ikeda lifting

Let $f(z)$ be a normalized cuspidal Hecke eigenform of weight $2k-n$ with respect to Γ_1. Assume that n and $k-n/2$ are even positive integers. Then Duke and Imamoglu conjectured that there exists a cuspidal Hecke eigenform $I(f)^n(Z)$ of weight k with respect to Γ_n such that

$$\zeta^{st}(I(f)^n,s)=\zeta(s)\prod_{i=1}^{n}L(f,s+k-i).$$

In [I], Ikeda constructed such a Hecke eigenform explicitly. Thus we call $I(f)^n(Z)$ the Ikeda lifting of f to Γ_n. Let $\tilde{f}$ be the modular form of weight $k-n/2+1/2$ belonging to the Kohnen plus-space corresponding to f, and $E_{n/2+1/2}$ be the Cohen Eisenstein series of weight $n/2+1/2$. Let $L(\tilde{f},s)$ and $L(E_{n/2+1/2},s)$ be the Mellin transforms of $\tilde{f}$ and $E_{n/2+1/2}$, respectively, and $L(\tilde{f},s)\otimes L(E_{n/2+1/2},s)$ be the convolution product. Let

$$\tilde{f}(z)=\sum_{d_0}c(d_0)exp(2\pi i|d_0|z),$$

where d_0 runs over all integers such that $(-1)^{k-n/2}d_0\equiv 0,1 \bmod 4$. Then we note that $L(\tilde{f},s)\otimes L(E_{n/2+1/2},s)$ can be expressed as

$$L(\tilde{f},s)\otimes L(E_{n/2+1/2},s)=L(f,2s)L(f,2s-n+1)$$
$$\times\sum_{d_0}c(d_0)d_0^{-s+(n-1)/2)}\prod_{p}\{(1+p^{-2s+k-1})(1+\chi_p((-1)^{n/2}d_0)^2p^{-2s+k-2})$$
$$-\chi_p((-1)^{n/2}d_0)p^{-2s+k-3/2}\alpha_p(1+p^{1/2-n/2}\alpha_p^{-1})(1+p^{-1/2+n/2}\alpha_p^{-1})\},$$

where α_p denotes the Satake p-parameter determined by f.

Theorem 2. *Under the above notation and assumption, we have*

$$L(I(f)^n,s)$$
$$=2^{ns}\beta_{n,k}[L(\tilde{f},s)\otimes L(E_{n/2+1/2},s)\prod_{i=1}^{n/2-1}L(f,2s-2i)$$

$$+((-1)^{n/2}+1)(-1)^{n(n-2)/8}\prod_{i=1}^{n/2} L(f,2s-2i+1)],$$

where $\beta_{n,k}$ is a constant depending only on n and k.

As for the proof, see [I-K3].

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