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ON THE ASSOCIATED CYCLES AND THE RESTRICTIONS OF QUATERNIONIC REPRESENTATIONS

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ABSTRACT. In this paper, we survey some results on the restrictions of quaternionic representations. In the process, we will compute the associated varieties and associated cycles of certain unitary quaternionic representations.

1. INTRODUCTION

This paper introduces the work of Gross and Wallach on the continuation of the quaternionic discrete series [GW1], [GW2]. It also summarizes my investigation on the restriction of these representations to quaternionic subgroups in [L3] and [L4]. In explaining [L3], I have make the following changes.

(i) I have changed the notations to the standard ones in the literatures [Vo1] [KnVo].
(ii) T. Kobayashi has studied the restriction of representations which are discretely decomposable [Ko1], [Ko2], [Ko3]. I have included some relevant results.
(iii) The connection between the associated orbits of the quaternionic representations and the associated variety is given in §4, although I am certain that these are known to the experts. Certain associated cycles are also computed. These computations give a new and shorter proof of Theorem 5.1.1.

The calculations done on the associated varieties and associated cycles are new and they are inspired from discussions with H. Yamashita, K. Nishiyama and Chengbo Zhu whom I am grateful to. The local theta correspondence of $(\mathrm{F}_{4,4}, \mathrm{G}_2)$ is also new. I have left out most of the proofs and they will appear elsewhere. Finally I would like to thank the organizers for their invitation to participate in RIMS workshop.

2. QUATERNIONIC REAL FORMS

2.1. Let $G(\mathbb{C})$ be a complex simple Lie group with Lie algebra $\mathfrak{g}$. Let G_c be a compact real form with Lie algebra $\mathfrak{g}_c$. Let τ be the complex conjugation on $\mathfrak{g}$ with respect to $\mathfrak{g}_c$. Let $\mathfrak{h}_0$ be a compact Cartan subalgebra (CSA) of $\mathfrak{g}_c$ and define

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$\mathfrak{h} = \mathfrak{h}_0 \otimes \mathbb{C}$. Choose a positive root system Φ^+ with respect to $\mathfrak{h}$ and denote its highest weight by $\tilde{\alpha}$. The roots $\pm\tilde{\alpha}$ induces an embedding

$$\mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}.$$

We can further require the embedding satisfies the following:

(i) $\mathfrak{su}_2$ is embedded into $\mathfrak{g}_c$. We will denote its image by $\mathfrak{su}_2(\tilde{\alpha})$.
(ii) Let X, Y, H denote the standard basis of $\mathfrak{sl}_2(\mathbb{C})$ and let $X_{\tilde{\alpha}}, Y_{\tilde{\alpha}}, H_{\tilde{\alpha}}$ denote their image in $\mathfrak{g}$. Then we assume that $H_{\tilde{\alpha}} \in \sqrt{-1}\mathfrak{h}_0$ and $X_{\tilde{\alpha}}$ (resp. $Y_{\tilde{\alpha}}$) spans the root space $\mathfrak{g}_{\tilde{\alpha}}$ (resp. $\mathfrak{g}_{-\tilde{\alpha}}$).

The embedding also induces an inclusion $\mathrm{SU}_2 \to G_c$. Let $\mathrm{SU}_2(\tilde{\alpha})$ denotes its image in G_c. It has Lie algebra $\mathfrak{su}_2(\tilde{\alpha})$. Let h be the nontrivial element in the center of $\mathrm{SU}_2(\tilde{\alpha})$. The *quaternionic* real form $\mathfrak{g}_0$ of $\mathfrak{g}$ is defined as

$$\mathfrak{g}_0 := \{X \in \mathfrak{g} : \tau(X) = \mathrm{ad}(h)X\}.$$

The *quaternionic* real form G_0 of $G(\mathbb{C})$ is defined as the connected component of the identity element of the group

$$\{g \in G(\mathbb{C}) : \tau g = hgh^{-1}\}.$$

2.2. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ denote the Cartan decomposition. Then $\mathfrak{k}_0 = \mathfrak{su}_2(\tilde{\alpha}) \oplus \mathfrak{m}_0$ where $\mathfrak{m}_0$ is a compact reductive Lie subalgebra. Define $\mathfrak{l}_0 := \sqrt{-1}\mathbb{R}H_{\tilde{\alpha}} \oplus \mathfrak{m}_0$.

Let $\mathfrak{g}_{(i)}$ denote the i-th eigenspace of $\mathrm{ad}H_{\tilde{\alpha}}$ on $\mathfrak{g}$. Then this defines a $\mathbb{Z}$ grading on $\mathfrak{g}$. One can show that $\mathfrak{g}_{(0)} = \mathfrak{l}$, $\mathfrak{g}_{(2)} = \mathbb{C}X_{\tilde{\alpha}}$, $\mathfrak{g}_{(-2)} = \mathbb{C}Y_{\tilde{\alpha}}$, $\mathfrak{g}_{(i)} = 0$ if $|i| > 2$.

Define

$$\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{g}_{(1)} \oplus \mathfrak{g}_{(2)}, \quad \overline{\mathfrak{q}} = \mathfrak{l} \oplus \mathfrak{g}_{(-1)} \oplus \mathfrak{g}_{(-2)}.$$

$\mathfrak{q}$ and $\overline{\mathfrak{q}}$ are opposite θ-stable parabolic subalgebras with Levi factor $\mathfrak{l}$.

Denote $V_M = \mathfrak{g}_{(-1)}$ as the representation of $\mathfrak{m}$. This is to avoid confusion with $\mathfrak{g}_{(-1)}$ later which is a representation of $\mathfrak{l}$. V_M is a self dual representation of $\mathfrak{m}$. Since $\mathfrak{g}_{(1)} \oplus \mathfrak{g}_{(2)}$ is a Heisenberg algebra, V_M has even dimension. Let $\dim V_M = 2d$.

2.3. The quaternionic real form G_0 has maximal compact subgroup of the form

$$K_0 = \mathrm{SU}_2(\tilde{\alpha}) \times_{\mu_2} M$$

where $\mathfrak{m}_0$ is the Lie algebra of M. Let G denote the (connected) two cover of G_0 with maximal compact subgroup

$$K = \mathrm{SU}_2(\tilde{\alpha}) \times M \tag{1}$$

Let G_1 denote a compact Lie group. Then a Lie group is a called a *quaternionic* Lie group if it is a cover of or covered by $G \times G_1$.

2.4. For each complex simple Lie algebra there is a unique quaternionic real form. This is evident from the above discussion. We tabulate $M(\mathbb{C})$ and V_M below where ϖ_i is the fundamental weights as given in Planches [Bou].

TABLE 1

G_0	$M(\mathbb{C})$	V_M
$\mathrm{SU}(2,d)$, $d \geq 2$	GL_d	$(\mathrm{U}_1^{d+2} \otimes \mathbb{C}^d) \oplus (\mathrm{U}_1^{-d-2} \otimes \mathbb{C}^d)^*)$
$\mathrm{Spin}(4,d)$, $d \geq 5$	$\mathrm{SL}_2 \times \mathrm{Spin}(d)$	$\mathbb{C}^2 \otimes \mathbb{C}^d$
$\mathrm{Sp}(2,2d)$	$\mathrm{Sp}(2d)$	$\mathbb{C}^{2d}$
$\mathrm{F}_{4,4}$	$\mathrm{Sp}(6)$	$\pi(\varpi_3) = \wedge^3\mathbb{C}^6 - \mathbb{C}^6$
$\mathrm{E}_{6,4} \rtimes \mathbb{Z}/2\mathbb{Z}$	$\mathrm{SL}_6 \rtimes \mathbb{Z}/2\mathbb{Z}$	$\pi(\varpi_3) = \wedge^3\mathbb{C}^6$
$\mathrm{E}_{7,4}$	$\mathrm{Spin}(12)$	$\pi(\varpi_6) = \frac{1}{2}$-spin
$\mathrm{E}_{8,4}$	simply connected E_7	$\pi(\varpi_1) = 56$ dim minuscule
$\mathrm{G}_{2,2}$	SL_2	$\mathrm{S}^3(\mathbb{C}^2)$

3. QUATERNIONIC REPRESENTATIONS

3.1. First we will introduce some notations. If V is a vector space, then $\mathrm{S}^nV = \mathrm{Sym}^nV$ and $\mathrm{S}^\bullet V = \sum_{n=0}^{\infty} \mathrm{Sym}^n V$. $\mathrm{S}^n_{\tilde{\alpha}}$ will denote $\mathrm{S}^n(\mathbb{C}^2)$ where $\mathbb{C}^2$ is the standard representation of $\mathrm{SU}_2(\tilde{\alpha})$.

3.2. Set $L = \mathrm{U}_1 \times M$ and G/L has a complex structure. Constructions of representations of G from Dolbeault cohomology $\mathrm{H}^1_{\bar{\partial}}(G/L, \mathcal{W})$ of G-equivariant holomorphic vector bundles $\mathcal{W}$ on G/L was studied by H. W. Wong [W1] [W2], who generalized the results of Schmid [S1]. These representations are globalization of certain Zuckerman modules which we will construct below.

Let H denote the Cartan subgroup of G with Lie algebra $\mathfrak{h}_0$. Let W be an finite dimensional irreducible representation of M with highest weight μ with respect to $M \cap H$. Let $W[k] = e^{k\frac{\tilde{\alpha}}{2}} \boxtimes W$ be a representation of $L = \mathrm{U}_1 \times M$. We define the following Zuckerman modules:

$$\begin{aligned} \mathcal{R}^i_{\mathfrak{q}}(W[k]) &:= \Gamma^i_{K/L}(\mathrm{Hom}_{\mathcal{U}(\mathfrak{q})}(\mathcal{U}(\mathfrak{g}), W[k+2d+2])_L) \\ \mathcal{L}^i_{\bar{\mathfrak{q}}}(W[k]) &:= \Gamma^i_{K/L}(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} W[k+2d+2]) \end{aligned}$$

In [L3], we denote $\mathcal{R}^1_{\mathfrak{q}}(W[k])$ by $\mathbf{H}(G, W[k+2d+2])$ instead.

The representations $\mathcal{L}^i_{\bar{\mathfrak{q}}}(W[k])$ and $\mathcal{R}^i_{\mathfrak{q}}(W[k])$ are the main subjects of this paper, in particular when $i = 1$. A result of Enright and Wallach states that $\mathcal{L}^i_{\bar{\mathfrak{q}}}(W[k])$ is the Hermitian dual representation of $\mathcal{R}^{2-i}_{\mathfrak{q}}(W[k])$. See Theorem 5.3 of [Vo2]. Hence we will only work with $\mathcal{L}^i_{\bar{\mathfrak{q}}}(W[k])$.

3.3. We recall some properties of $\mathcal{L}^i_{\mathfrak{q}}(W[k])$. See [GW2] and §3.3 [L2].

(i) $\mathcal{L}^i_{\mathfrak{q}}(W[k])$ has infinitesimal characters

$$\mu + \rho(M) + (d+1+k)\frac{\tilde{\alpha}}{2} = \mu + \rho(G) + k\frac{\tilde{\alpha}}{2}.$$

(ii) If $k \geq -2d$, then $\mathcal{L}^i_{\mathfrak{q}}(W[k]) \neq 0$ if and only if $i = 1$.

(iii) Suppose G is not of type A and let α' be the unique simple root that is not connected to $-\tilde{\alpha}$ in the extended Dynkin diagram. Then $\mathcal{L}^1_{\mathfrak{q}}(W[k])$ is the Harish-Chandra module of a discrete series representation if and only if $k > -\langle \mu, \alpha' \rangle$.

(iv) $\mathcal{L}^1_{\mathfrak{q}}(W[k])$ has K-types ($K = \mathrm{SU}_2(\tilde{\alpha}) \times M$)

$$\sum_{n=0}^{r} \mathrm{S}_{\tilde{\alpha}}^{k+2d+n} \boxtimes (\mathrm{S}^n(V_M) \otimes W). \tag{2}$$

Note that $\mathcal{L}^1_{\mathfrak{q}}(W[k])$ is $\mathrm{SU}_2(\tilde{\alpha})$-admissible. $\mathrm{S}_{\tilde{\alpha}}^{k+2d} \boxtimes W$ is called the *lowest* K-type.

(v) $\mathcal{L}^1_{\mathfrak{q}}(W[k]))$ have Gelfand-Kirillov dimension $\dim V_M + 1$ and Bernstein degree $\dim W \cdot \dim V_M$.

(vi) $\mathcal{L}^1_{\mathfrak{q}}(W[k])$ contains a unique quotient generated by the lowest K-type $\mathrm{S}_{\tilde{\alpha}}^{k+2d} \boxtimes W$. This follows from [W1]. An alternative proof is given after (4). We will denote this unique quotient in $\mathcal{L}^1_{\mathfrak{q}}(W[k])$ by $\overline{\mathcal{L}^1_{\mathfrak{q}}(W[k])}$ or $\overline{\mathcal{L}}$.

From now on throughout the paper, we will assume that $k \geq -2d$ so that $\mathcal{L}^1_{\mathfrak{q}}(W[k]) \neq 0$. We will call $\mathcal{L}^1_{\mathfrak{q}}(W[k])$ and $\overline{\mathcal{L}^1_{\mathfrak{q}}(W[k])}$ *quaternionic* representations or *quaternionic* Harish-Chandra modules.

4. Associated Variety

In this section if V is a complex vector space, then V^* will denote its complex dual space. Let $\mathfrak{p} = (\mathfrak{p}_0)_{\mathbb{C}}$.

4.1. We refer to [Vo3] for the definition of the associated variety $\mathcal{V}(U)$ and the associated cycle of a Harish-Chandra module U and its properties. In this subsection, we will give the definition of $\mathcal{V}(U)$ when U is either $\mathcal{L}^1_{\mathfrak{q}}(W[k])$ or $\overline{\mathcal{L}^1_{\mathfrak{q}}(W[k])}$.

Let T denote the lowest K-type of U. By §3.3(vi), T generates U. Let $\mathcal{U}_r(\mathfrak{g})$ denote the canonical filtration of $\mathcal{U}(\mathfrak{g})$. Then $\mathcal{U}_r(\mathfrak{g})T$ is an increasing filtration on U. Let $\mathrm{Gr}(U) = \sum_{n=0}^{\infty} \mathrm{Gr}_n(U)$ denote the graded module. It is a graded module over the commutative algebra $\mathrm{S}^\bullet\mathfrak{g}$, and hence over $\mathrm{S}^\bullet\mathfrak{p}$ as well. Let $\mathrm{Ann}(\mathrm{Gr}(U))$ denote the annihilator ideal of $\mathrm{Gr}U$ in $\mathrm{S}^\bullet\mathfrak{p}$. Then the *associated variety* $\mathcal{V}(U)$ of U is defined as the variety in $\mathfrak{p}^*$ cut out by this ideal.

By construction, $\mathcal{V}(U)$ is a cone, that is, $t\mathcal{V}(U) = \mathcal{V}(U)$ for any nonzero $t \in \mathbb{C}$. Suppose $\mathcal{V}(U) \supsetneq \{0\}$, we define $\mathbb{P}\mathcal{O}(U)$ as the image of $\mathcal{V}(U)\backslash\{0\}$ under the canonical projection $\mathfrak{p}^*\backslash\{0\} \to \mathbb{P}\mathfrak{p}^*$. We will call $\mathbb{P}\mathcal{O}(U)$ the *projective* associated variety.

The definition of the associated variety of $\mathcal{R}^1_{\mathfrak{q}}(W[k])$ (resp. $\overline{\mathcal{R}^1_{\mathfrak{q}}(W[k])}$) is similar and we will not give them here. It is the same as that of $\mathcal{L}^1_{\mathfrak{q}}(W[k])$ (resp. $\overline{\mathcal{L}^1_{\mathfrak{q}}}(W[k])$).

4.2. **Lie algebra action.** We will state a technical lemma which we need later. As a representation of $\mathfrak{k}$, $\mathfrak{p} \simeq \mathbb{C}^2 \boxtimes V_M$. Let $u \in \mathbb{C}^2$ and $v \in V_M$ and we identify $u \boxtimes v \in \mathfrak{p}$. Let $u_1 \in \mathrm{S}^{k+2d+n}_{\tilde{\alpha}}$ and $v_1 \otimes w_1 \in \mathrm{S}^n(V_M) \otimes W$, then $u_1 \boxtimes (v_1 \otimes w_1)$ is a K-finite vector in (2).

Lemma 4.2.1. *The Lie algebra action of $u \boxtimes v \in \mathfrak{p}$ on the vector $u_1 \boxtimes (v_1 \otimes w_1)$ in $\mathcal{L}^1_{\mathfrak{q}}(W[k])$ is*

$$(u \boxtimes v) \cdot (u_1 \boxtimes (v_1 \otimes w_1)) = w_+ \oplus w_- \tag{3}$$

where $w_\pm \in \mathrm{S}^{k+2d+n\pm 1}_{\tilde{\alpha}} \boxtimes (\mathrm{S}^{n\pm 1}(V_M) \otimes W)$ and $w_+ = uu_1 \boxtimes (vv_1 \otimes w)$.

The above lemma follows from a formula on the Lie algebra actions on Zuckerman modules in [Wa2]. The actual calculation is a little lengthy so we will omit it here. The formula for w_- is equivalent to the decomposition of tensor products of certain finite dimensional representations of M. It is rather complicated and we do not have a explicit formula for it.

4.3. Lemma 4.2.1 implies that

$$\mathcal{U}_r(\mathfrak{g})(\mathrm{S}^{k+2d}_{\tilde{\alpha}} \boxtimes W) = \sum_{n=0}^{r} \mathrm{S}^{k+2d+n}_{\tilde{\alpha}} \boxtimes (\mathrm{S}^n(V_M) \otimes W). \tag{4}$$

Hence the lowest K-type generates $\mathcal{L}^1_{\mathfrak{q}}(W[k])$ and this proves §3.3(vi).

4.4. By (4), $\mathrm{Gr}_n(\mathcal{L}^1_{\mathfrak{q}}(W[k])) = \mathrm{S}^{k+2d+n}_{\tilde{\alpha}} \boxtimes (\mathrm{S}^n(V_M) \otimes W)$. We can identify $u \boxtimes (v_1 \otimes w_1)$ in Lemma 4.2.1 with a vector in $\mathrm{Gr}_n(\mathcal{L}^1_{\mathfrak{q}}(W[k]))$. Then Lemma 4.2.1 gives the following corollary.

Corollary 4.4.1. *The Lie algebra action of $u \boxtimes v \in \mathfrak{p}$ on the vector $u_1 \boxtimes (v_1 \otimes w_1)$ in $\mathrm{Gr}_n(\mathcal{L}^1_{\mathfrak{q}}(W[k])$ is*

$$(u \boxtimes v) \cdot (u_1 \boxtimes (v_1 \otimes w_1)) = uu_1 \boxtimes (vv_1 \otimes w_1) \in \mathrm{Gr}_{n+1}(\mathcal{L}^1_{\mathfrak{q}}(W[k])). \tag{5}$$

4.5. Next we recall some elementary algebraic geometry in [Fu].

Let $V = V_M^*$ and let $p_V : V\backslash\{0\} \to \mathbb{P}V$ denote the canonical projection. Let $\mathcal{V}$ denote a projective variety in $\mathbb{P}V$ and let $\mathbb{A}^\bullet(\mathcal{V}) = \sum_{n=0}^{\infty} \mathbb{A}^\bullet(\mathcal{V})$ denote its coordinate ring. We define the cone over $\mathcal{V}$ as the variety $(p_V^{-1}\mathcal{V}) \cup \{0\}$ in V and we denote the cone by $\mathrm{Cone}(\mathcal{V})$.

Let $\mathbb{P}^1 := \mathbb{P}\mathbb{C}^2$ and let $s : \mathbb{P}^1 \times \mathcal{V} \to \mathbb{P}(\mathbb{C}^2 \times V)$ denote the Segre embedding (See Exercise 4-28 in [Fu]). The coordinate ring of the image of s can be identified with

$$\sum_{n=0}^{\infty} \mathrm{S}^n(\mathbb{C}^2) \otimes \mathbb{A}^n(\mathcal{V}).$$

From now on $\mathbb{P}^1 \times \mathbb{P}V$ is considered as a projective subvariety of $\mathbb{P}\mathfrak{p}^*$ under the Segre embedding s.

4.6. Now we review the work of Gross and Wallach [GW1] [GW2].

$\mathbb{P}V_M^*$ is a union of finitely many $M(\mathbb{C})$-orbits and there is a unique dense orbit. In other words, $\mathfrak{g}^*_{(-1)}$ is a *pre-homogenous vector space* as a representation of $L(\mathbb{C}) = \mathbb{C}^* \times M(\mathbb{C})$. Gross and Wallach considers a collection of $M(\mathbb{C})$-orbits on $\mathbb{P}V_M^*$. For every $M(\mathbb{C})$-orbits $\mathcal{O}$ in the collection, they construct a unitarizable Harish-Chandra module $\sigma_{\mathcal{O}}$ of G. This is done on a case by case basis.

Let $\mathcal{O}$ be one of these $M(\mathbb{C})$-orbits and let $I(\overline{\mathcal{O}}) = \sum_{n=m}^{\infty} I_n$ denote the homogeneous ideal of its Zariski closure $\overline{\mathcal{O}}$. Here $I_m \neq 0$ and $I_n \subset \mathrm{S}^n V_M$. Note that I_n is a representation of M and It is observed $I(\overline{\mathcal{O}}) = \mathrm{S}^*(V_M) I_m$. Then there exists k such that $\sigma_{\mathcal{O}} = \overline{\mathcal{L}^1_{\mathfrak{q}}(\mathbb{C}[k])}$ and it satisfies the following exact sequence

$$\mathcal{L}^1_{\mathfrak{q}}(I_m[k+m]) \xrightarrow{\phi} \mathcal{L}^1_{\mathfrak{q}}(\mathbb{C}[k]) \to \sigma_{\mathcal{O}} \to 0. \tag{6}$$

$\sigma_{\mathcal{O}}$ have K-types

$$\sum_{n=0}^{\infty} \mathrm{S}^{n+k+2d}_{\tilde{\alpha}}(\mathbb{C}^2) \boxtimes \mathbb{A}^n(\overline{\mathcal{O}})$$

where $\sum_n \mathbb{A}^n(\overline{\mathcal{O}})$ is the coordinate ring of $\overline{\mathcal{O}}$ in $\mathbb{P}V_M^*$. In [GW2] $\mathcal{O}$ is called the *associated orbit* of $\sigma_{\mathcal{O}}$.

4.7. The next proposition gives the associated varieties of $\mathcal{L}^1_{\mathfrak{q}}(W[k])$ and $\sigma_{\mathcal{O}}$. Its proof uses Corollary 4.4.1.

Proposition 4.7.1. (i) *Let $k \geq -2d$. Then $\mathbb{P}^1 \times \mathbb{P}V_M^*$ (in $\mathfrak{p}^*$) is the projective associated variety of $\mathcal{L}^1_{\mathfrak{q}}(W[k])$.*

(ii) *$\mathbb{P}^1 \times \overline{\mathcal{O}}$ is the projective associated variety of $\sigma_{\mathcal{O}}$. In particular $\sigma_{\mathcal{O}}$ has Gelfand-Kirillov dimension $\dim \overline{\mathcal{O}} + 2$ and Bernstein degree $\mathrm{Deg}\overline{\mathcal{O}} + 1$.*

If W is the trivial representation, then $\mathcal{R}^1_{\mathfrak{q}}(\mathbb{C}[k])$ is commonly denoted by $A^{\mathfrak{q}}(\lambda)$ where $\lambda = k\frac{\tilde{\alpha}}{2}$ (See Eq. (5.6) of [KnVo]). $A^{\mathfrak{q}}(\lambda)$ will lie in the weakly fair range if $k \geq -d-1$. By Lemma 2.7 of [Ko3], $A^{\mathfrak{q}}(\lambda)$ has associated variety $\mathrm{Ad}(K_{\mathbb{C}})\,\mathbb{P}(\mathfrak{g}^*_{(-1)}) = \mathrm{Cone}(\mathbb{P}^1 \times \mathbb{P}V_M^*)$.

Finally Lemma 1.1 in [NOT] gives the following corollary.

Corollary 4.7.2. $[\mathrm{Cone}(\mathbb{P}^1 \times \overline{\mathcal{O}})]$ *is the associated cycle of $\sigma_{\overline{\mathcal{O}}}$.*

4.8. **Groups of type F and E.** . Suppose G is a quaternionic real form of type F or E. Then $\mathbb{P}V_M^*$ is a union of four $M(\mathbb{C})$-orbits: Z, Y, X and $\mathbb{P}V_M^*\backslash\overline{X}$.

$$\mathbb{P}V_M \supsetneq \overline{X} \supsetneq \overline{Y} \supsetneq Z.$$

Here $\mathbb{P}V_M^*\backslash\overline{X}$ is Zariski dense and Z is the unique closed orbit in $\mathbb{P}V_M$. X, Y and Z are associated orbits of three unitary quaternionic representations σ_X, σ_Y, and σ_Z constructed in [GW1] and [GW2]. In particular σ_Z is annihilated by the Joseph ideal in $\mathcal{U}(\mathfrak{g})$ and it is called the *minimal* representation of G.

5. Restrictions

5.1. Next we consider restriction of quaternionic Harish-Chandra modules. Let G' denote a quaternionic subgroup of G with compact subgroup $K' = \mathrm{SU}_2(\tilde{\alpha})\times M'$. Then we get $V_{M'} \subset V_M$ and we define V_0 to be the subspace of V_M such that $V_M = V_{M'} \oplus V_0$ as representations of M'. We will abuse notation and use $\mathrm{Res}^G_{G'}U$ to denote the restriction of a quaternionic Harish-Chandra module U of G to G'. It is easy to see that the above decomposition is discrete since $\mathcal{L}^1_{\mathfrak{q}}(W[k])$ is $\mathrm{SU}_2(\tilde{\alpha})$-admissible. See [Ko3] for the definition of discrete decomposable restriction.

The inclusion $I_m(\overline{\mathcal{O}}) \subset S^m(V_M)$ and $V_M = V_{M'} \oplus V_0$ give rise to the following natural maps of M'-modules

$$\mathrm{S}^{n-m}(V_M)\otimes I_m(\overline{\mathcal{O}}) \to \mathrm{S}^{n-m}(V_M)\otimes \mathrm{S}^m(V_M) \to \mathrm{S}^n(V_M) \to \mathrm{S}^n(V_0).$$

Let r_n denote the composite of the above maps and let R_n denote its cokernel. Define $R_\bullet := \bigoplus_{n=0}^\infty R_n$. Note that R_n is a representation of M' and we write

$$R_n = \textstyle\sum_j W_{n,j}$$

where $W_{n,j}$ are the irreducible subrepresentations of M'.

Let $\mathcal{O}' = \overline{\mathcal{O}} \cap \mathbb{P}V_0^*$ and denote its coordinate ring in $\mathbb{P}V_0^*$ by $\mathbb{A}^\bullet(\mathcal{O}') = \bigoplus \mathbb{A}^n(\mathcal{O}')$. Then $\mathcal{O}'$ is cut out by $r_m(I_m(\overline{\mathcal{O}}))$ and $R_\bullet/\mathrm{Nil}(R_\bullet) = \mathbb{A}^\bullet(\mathcal{O}')$.

If $W' = \sum_i W_i'$ is a sum of irreducible M' modules, then we define the $(\mathfrak{g}', K')$-module $\overline{\mathcal{L}^1_{\mathfrak{q}'}(W'[k])} := \sum_i \overline{\mathcal{L}^1_{\mathfrak{q}'}(W_i'[k])}$. We can now state Theorem 3.3.1 and Corollary 2.8.1 of [L3].

Theorem 5.1.1. *Let* $2d_0 = \dim V_0$. *Then*

(i) $\mathrm{Res}^G_{G'}\sigma_{\mathcal{O}} = \sum\limits_{n=0}^{\infty}\overline{\mathcal{L}^1_{\mathfrak{q}'}(R_n[k+2d_0+n])} = \sum\limits_{n=0}^{\infty}\sum\limits_{j}\overline{\mathcal{L}^1_{\mathfrak{q}'}(W_{n,j}[k+2d_0+n])}$.

(ii) $\mathrm{Res}^G_{G'}\sigma_{\mathcal{O}} \supseteq \sum\limits_{n=0}^{\infty}\overline{\mathcal{L}^1_{\mathfrak{q}'}(\mathbb{A}^n(\mathcal{O}')[k+2d_0+n])}$.

Equality holds if and only if $r_m(I_m(\overline{\mathcal{O}}))$ *generates the ideal of* $\overline{\mathcal{O}}'$.

(iii) *If r_m is surjective, then r_n is surjective for $n \geq m$ and*

$$\mathrm{Res}^G_{G'}\sigma_{\mathcal{O}} = \sum_{n=0}^{m-1} \overline{\mathcal{L}^1_{\bar{\mathfrak{q}}'}(\mathrm{S}^n V_0[k+2d_0+n])}. \quad \square$$

PROOF. (ii) and (iii) follows from (i). We will sketch a proof of (i) which is a little different from that in [L2]. We write $\mathrm{Res}^G_{G'}\sigma_{\mathcal{O}} = \sum V_i$. Here the restriction is a discrete direct sum of quaternionic representations V_i since $\sigma_{\mathcal{O}}$ is unitarizable and $\mathrm{SU}_2(\tilde{\alpha})$-admissible. Let $\mathbb{A}' := \sum_n \mathrm{S}^n_{\tilde{\alpha}} \boxtimes \mathrm{S}^n V_{M'}$. Then $\mathrm{Gr}\sigma_{\mathcal{O}}$ is a direct sum of $\mathbb{A}'$-modules whose generators are the lowest K'-types of V_i. We check that $R_\bullet$ is such a minimal generating set. $\square$

5.2. Next we compute the associated variety of the restriction. Write $\mathfrak{p}^* = (\mathfrak{p}')^* \oplus (\mathbb{C}^2 \boxtimes V_M)^*$. Let $\mathrm{pr}_{\mathfrak{p}\to\mathfrak{p}'}$ denote the canonical projection from $\mathfrak{p}^*$ to $(\mathfrak{p}')^*$. Similarly we define $\mathrm{pr}_{V_M^* \to V_{M'}^*}$ using $V_M = V_{M'} \oplus V_0$.

Proposition 5.2.1. *Let J denote the annihilator ideal of* $\mathrm{Gr}\sigma_k$ *in* $\mathrm{S}^\bullet\mathfrak{p}$. *Let U' denote an irreducible Harish-Chandra module of G' on the right side of Theorem 5.1.1(i). Then the associated variety $\mathcal{V}(U')$ is defined by* $\mathrm{S}^\bullet\mathfrak{p}' \cap J$.

In particular $\mathcal{V}(U')$ contains

$$\mathrm{pr}_{\mathfrak{p}^*\to(\mathfrak{p}')^*}(\mathrm{Cone}(\mathbb{P}^1 \times \overline{\mathcal{O}})) = \mathbb{C}^2 \times (\mathrm{pr}_{V_M^*\to V_{M'}^*}(\mathrm{Cone}(\overline{\mathcal{O}})).$$

The last assertion of the above proposition is a special case of Theorem 3.1 of [Ko3].

5.3. **Restrictions of holomorphic representations.** Let G be a simple Lie group such that G/K is a bounded symmetric domain. The reducibility and unitarility of the continuation of the holomorphic discrete series representations with one dimensional lowest K-types were studied by [RV] and [Wal]. The associated cycles are documented in §7 [NOT]. Our method can also be applied to the restrictions of these representations to symmetric subgroups [L2]. On the other hand, the restriction problem for the classical groups can be easily calculated using the compact dual pairs correspondences [KV] and the Kulda's see-saw pairs argument. The restrictions of holomorphic discrete series representations are also known [Ma] [JV].

6. Realizations of orbits X, Y and Z.

6.1. Theorem 5.1.1 reduces the restriction problem to the computation of R_n. However it is still relatively difficult to determine R_n. This is the subject matter in [L4] where we treat the restrictions of $\sigma_{\mathcal{O}}$ of the exceptional quaternionic Lie groups of type F and E to to certain quaternionic Lie subgroups. For the ease of explaining we will only deal with the restriction of $\sigma_{\mathcal{O}}$ of $G = \tilde{\mathrm{E}}_{8,4}$ to $G' = \tilde{\mathrm{E}}_{7,4} \times_{\mu_2} \mathrm{SU}_2$ where the tilde above the group denotes its double cover.

6.2. **The data.** Set $G = \tilde{\mathrm{E}}_{8,4}$ (Double cover), $K = \mathrm{SU}_2 \times \mathrm{E}_7$, $M = \mathrm{E}_7$ and V_M is the 56 dimensional minuscule representation of E_7. Up to a cover, $G' = \tilde{\mathrm{E}}_{7,4} \times \mathrm{SU}_2$, $M' := \mathrm{Spin}(12) \times \mathrm{SU}_2$. $V_{M'} = \pi_{\mathrm{Spin}(12)}(\varpi_6) \boxtimes \mathbb{C}$, $V_0 = \mathbb{C}^{12} \boxtimes \mathbb{C}^2$ and $d_0 = 12$. Referring to (6), we have $(k, m, \mathcal{O}) = (-31, 4, X)$, $(-40, 3, Y)$ or $(-48, 2, Z)$.

6.3. Next we will describe the explicit realizations of orbits X, Y and Z. The basis references are [B], [GW1], [GW2], [SK]. We will first introduce the Cayley numbers $\mathbb{O}_{\mathbb{C}}$ and Jordan algebra $\mathcal{J}$.

6.3.1. *The Cayley numbers* $\mathbb{O}_{\mathbb{C}}$. $\mathbb{O}_{\mathbb{C}}$ has an anti-automorphism $z \mapsto \overline{z}$ called *conjugation.* Define $N(z) := z\overline{z} = \overline{z}z$. Then $N(z)$ is a multiplicative norm, that is, $N(zz') = N(z)N(z')$. Next define $tr(z) := z + \overline{z}$. Then $\langle z, z' \rangle := tr(z\overline{z'})$ is a bilinear symmetry form.

6.3.2. *Jordan algebra* $\mathcal{J}$. Let $\mathcal{J}$ be the Jordan algebra consisting of 3 by 3 Hermitian symmetric matrices of the form

$$J = (\gamma_1, \gamma_2, \gamma_3; c_1, c_2, c_3) := \begin{pmatrix} \gamma_1 & c_3 & \overline{c_2} \\ \overline{c_3} & \gamma_2 & c_1 \\ c_2 & \overline{c_1} & \gamma_3 \end{pmatrix}$$

where $\gamma_i \in \mathbb{C}$ and $c_i \in \mathbb{O}_{\mathbb{C}}$. The composition in $\mathcal{J}$ is given by $J_1 \circ J_2 = \frac{1}{2}(J_1 J_2 + J_2 J_1)$. We define an inner product on $\mathcal{J}$ given by $\langle X, Y \rangle = \mathrm{Tr}\,(X \circ Y)$ where Tr denotes the usual trace of matrices. There is a cubic form

$$\det(J) = \gamma_1\gamma_2\gamma_3 - \gamma_1 N(c_1) - \gamma_2 N(c_2) - \gamma_3 N(c_3) + tr(c_1(c_2 c_3))$$

on $\mathcal{J}$ which induces a trilinear form on $\mathcal{J}$ such that $(J, J, J) = \det J$. Finally we define the bilinear map $\mathcal{J} \times \mathcal{J} \longrightarrow \mathcal{J}$ such that $\langle J_1 \times J_2, J_3 \rangle = (J_1, J_2, J_3)$ for all $J_3 \in \mathcal{J}$.

6.4. Define

$$V_M := \mathbb{C} \oplus \mathcal{J} \oplus \mathcal{J} \oplus \mathbb{C}$$

and we denote a vector in V_M by (ξ, J, J', ξ'). The action of $\mathrm{E}_7(\mathbb{C})$ on V_M is given in [B].

(i) $\overline{X}$ is defined as the zeros of the equation

$$f_4(\xi, J, J', \xi') = \langle J \times J, J' \times J' \rangle - \xi \det(J) - \xi' \det(J') - \frac{1}{4}(\langle J, J' \rangle - \xi\xi')^2.$$

(ii) $\overline{Y}$ is defined by $\left\{\dfrac{\partial f_4}{\partial v} : v \in V_M\right\}$.

(iii) Z is generated by the $\mathrm{E}_7(\mathbb{C})$ action on the highest weight vector $(1,0,0,0)$.

7. Restriction to $\tilde{\mathrm{E}}_{7,4} \times \mathrm{SU}_2$

7.1. In order to apply Theorem 5.1.1, it is important to compute the coordinate rings of these intersections

$$\overline{X} \cap \mathbb{P}V_0^*, \quad \overline{Y} \cap \mathbb{P}V_0^*, \quad Z \cap \mathbb{P}V_0^*.$$

These intersections are $\mathrm{Spin}(12,\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ invariant in $\mathbb{P}V_0^*$. In general, these are rather difficult to compute. The important observation which we need is that $M'(\mathbb{C}) = \mathrm{Spin}(12,\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ has *finitely* many orbits on $\mathbb{P}V_0^*$.

7.2. **$M'(\mathbb{C})$-orbits on $\mathbb{P}V_0^*$** [GW2]. Recall $V_0^* = \mathbb{C}^{12} \boxtimes \mathbb{C}^2$. Let e_1, e_2 denote the standard basis of $\mathbb{C}^2$ and let $\langle\, , \rangle$ denote the inner product on $\mathbb{C}^{12}$. Let $v = w_1 \otimes e_2 + w_2 \otimes e_2 \in V_0^*$.

$\mathbb{P}V_0^*$ contains five $M(\mathbb{C})$-orbits: Z, Y_1, Y_2, X_1 and $\mathbb{P}V_0^* \backslash \overline{X_1}$. The orbit $\mathbb{P}V_0 \backslash \overline{X_1}$ is Zariski dense and $\overline{X_1}$ is a hypersurface defined by

$$f_4'(v) = \det \begin{pmatrix} \langle w_1, w_1 \rangle & \langle w_1, w_2 \rangle \\ \langle w_2, w_1 \rangle & \langle w_2, w_2 \rangle \end{pmatrix}.$$

$\overline{Y_1}$ is the complete intersection of the 3 quadrics

$$\langle w_1, w_1 \rangle = \langle w_1, w_2 \rangle = \langle w_2, w_2 \rangle = 0.$$

$\overline{Y_2}$ is the subvariety $\mathbb{P}^1 \times \mathbb{P}^1$. Note that $\overline{X_1} \subset Y_1 \cup Y_2$.

Let $Q \subset \mathbb{P}^{11}$ defined by $\langle w_1, w_1 \rangle = 0$, then $Z_1 = Q \times \mathbb{P}^1 = Y_1 \cap Y_2$ is the unique minimal closed orbit in $\mathbb{P}V_0^*$.

7.3. It is now possible to compute the intersections by checking whether they contain elements in the $M'(\mathbb{C})$-orbits.

Lemma 7.3.1. *$\overline{X} \cap \mathbb{P}V_0 = \overline{X_1}$, $\overline{Y} \cap \mathbb{P}V_0 = \overline{Y_1} \cup \overline{Y_2}$ and $Z \cap \mathbb{P}V_0 = Z_1$.* □

The coordinate rings of above intersections are documented in [GW2] and [L4]. We will not repeat them here. Finally we know that the homogeneous ideal of the intersection is generated by its elements of lowest degree. We can now apply Theorem 5.1.1(ii) to get the following theorem (see [L4]).

Theorem 7.3.2. *Let $V_{a,b} = \pi_{\mathrm{Spin}(12)}(a\varpi_1 + b\varpi_2)$. Then*

(i) $\mathrm{Res}^{\mathrm{E}_{8,4}}_{\mathrm{E}_{7,4} \times \mathrm{SU}_2} \sigma_Z = \sum_{n=0}^{\infty} \overline{\mathcal{L}^1_{\mathfrak{q}'}(V_{n,0}[n-24])} \boxtimes \mathrm{S}^n(\mathbb{C}^2)$.

(ii) $\mathrm{Res}^{\mathrm{E}_{8,4}}_{\mathrm{E}_{7,4}\times\mathrm{SU}_2}\sigma_Y = \sum_{a+2b+2c=n,\, bc=0} \overline{\mathcal{L}^1_{\bar{\mathfrak{q}}'}(V_{a,c}[n-16])} \boxtimes \mathrm{S}^{a+2b}(\mathbb{C}^2)$.

(iii) $\mathrm{Res}^{\tilde{\mathrm{E}}_{8,4}}_{\tilde{\mathrm{E}}_{7,4}\times\mathrm{SU}_2}\sigma_X = \sum{}^{*}\mathcal{L}^1_{\bar{\mathfrak{q}}'}(V_{a+2d,c}[n-7]) \boxtimes \mathrm{S}^{a+2b}(\mathbb{C}^2)$.

(iv) $\mathrm{Res}^{\tilde{\mathrm{E}}_{8,4}}_{\tilde{\mathrm{E}}_{7,4}\times\mathrm{SU}_2}\mathcal{L}^1_{\mathfrak{q}}(\mathbb{C}[k]) = \sum_{m=0}^{\infty}\sum{}^{*}\mathcal{L}^1_{\bar{\mathfrak{q}}'}(V_{a+2d,c}[k+n+4m]) \boxtimes \mathrm{S}^{a+2b}(\mathbb{C}^2)$ *if* $k \geq -6$.

Each summands on the right of the above equation are irreducible and unitarizable. The summation $\sum^*$ *appearing in* (iii) *and* (iv) *is taken over all nonnegative integers* a, b, c, d, n *satisfying the relations*

$$n - 2a \leq a + 2b + 2c + 4d \leq n, \quad cd = 0, \quad a \equiv n \bmod(2).$$

The right hand side of Theorem 7.3.2(i) contains the representation σ_Y of $\mathrm{E}_{7,4}$ when $n = 0$. By Theorem 3.7 in [Ko3], $\mathbb{P}^1 \times Y$ is the projective associated variety of every summand on the right hand side of (i).

7.4. **Other groups.** The restriction of $\sigma_{\overline{\mathcal{O}}}$ for other groups quaternionic groups G can be similarly computed. For example if $G = \tilde{\mathrm{F}}_{4,4} \supset G' = \widetilde{\mathrm{Spin}(4,4)} \times_{(\mathbb{Z}/2\mathbb{Z})^3} (\mathbb{Z}/2\mathbb{Z})^3$, then one can show that $Z \cap \mathbb{P}V_0 = \{\}$. Thus the restrictions of σ_Z to G' decomposes into a finite sum of irreducible representation of G' by Theorem 5.1.1(iii). We refer the reader to [L4] for details.

8. Dual pairs correspondences

8.1. **Definition.** Let G be one of the exceptional Lie group of real rank 4. A *dual pair* is a pair of subgroups (G_1, G_2) in $\mathrm{E}_{8,4}$ such that G_i is the centralizer of G_{i+1} in G for $i \in \mathbb{Z}/2\mathbb{Z}$. It is called a compact dual pair if either G_1 or G_2 is compact.

For example $G = \mathrm{E}_{8,4}$ contains the following compact dual pairs:

(i) $(\mathrm{E}_{7,4}, \mathrm{SU}_2)$, (ii) $(\mathrm{E}_{6,4}, \mathrm{SU}_3)$ (iii) $(\mathrm{Spin}(4,4), \mathrm{Spin}(8))$, (iv) $(\mathrm{F}_{4,4}, \mathrm{G}_2)$, (v) $(\mathrm{G}_{2,2}, \mathrm{F}_4)$

Partly motivated by the local theta correspondences for the Weil representation (See [Ho], [KV] and many others), it is of interest to know $\Theta(\pi)$ where $\Theta(\pi)$ is defined by the following equation

$$\mathrm{Res}^G_{G_1\times G_2}\sigma_Z = \sum \Theta(\pi) \boxtimes \pi^{\text{finite dim}}.$$

Unfortunately, in all the dual pairs above except Case (i), $M'(\mathbb{C})$ does not have a dense orbit in $\mathbb{P}V_0$. Other methods have been employed. We will briefly discuss the correspondences. The pair (iv) is given in [HPS]. The pair (i) is Theorem 7.3.2(i) and it first appeared in [GW1] and [G]. (ii) is given in [L3]. We will state (iii) and (iv) below:

8.1.1.

$$\mathrm{Res}^{\mathrm{E}_{8,4}}_{\mathrm{Spin}(4,4)\times\mathrm{Spin}(8)}\sigma_Z = \sum_{\lambda}(m_2+1)\pi_\lambda \otimes \pi_{\mathrm{Spin}(8)}(\lambda)$$

Here the sum is taken over $\lambda = m_1\varpi_1 + m_2\varpi_2 + m_3\varpi_3 + m_4\varpi_4$ where ϖ_i is the fundamental weights of D_4. π_λ is the quaternionic discrete series representation of $\widetilde{\mathrm{Spin}}(4,4)$ which has the same infinitesimal character as $\pi_{\mathrm{Spin}(8)}(\lambda)$. [L2]

8.1.2.

$$\mathrm{Res}^{\mathrm{E}_{8,4}}_{\mathrm{F}_{4,4}\times\mathrm{G}_2}\sigma_Z = \sum_{n=0}^{\infty}\Theta(a,b)\otimes\pi_{\mathrm{G}_2}(a\varpi_1+b\varpi_2).$$

where

$$\Theta(a,b) = \begin{cases} \overline{\mathcal{L}^1_{\mathfrak{q}'}(\pi_{\mathrm{Sp}_6}(a\varpi_1+b\varpi_2)[a+2b-6])} & \text{if } b\neq 0 \\ \mathcal{L}^1_{\mathfrak{q}'}(\mathrm{S}^a\mathbb{C}^6[a-6]))\oplus\mathcal{L}^1_{\mathfrak{q}'}(\mathrm{S}^{a-1}\mathbb{C}^6[a-7])) & \text{if } a\neq 0, b=0 \\ \mathcal{L}^1_{\mathfrak{q}'}(\mathbb{C}^6[-6]) & \text{if } a=b=0 \end{cases}$$

8.2. **Unitary representations.** The restriction formula and compact dual correspondences is a very efficient way of producing unitarizable quaternionic representations. For example, it helps to determine all the unitarizable quaternionic representations of $\tilde{\mathrm{F}}_{4,4}$ [L1].

8.3. **Non-compact dual pairs.** J-S Li has obtained the almost all the discrete spectrum of the restriction of σ_Z to (See [Li2])

$$\mathrm{SU}(1,1)\times_{\mu_2}\mathrm{E}_{7,3}\subset\mathrm{E}_{8,4}.$$

His method also applies to the dual pairs

$$\begin{aligned} \mathrm{SU}(1,1)\times_{\mu_2}Sp(6,\mathbb{R}) &\subset \tilde{\mathrm{F}}_{4,4} \\ \mathrm{SU}(1,1)\times_{\mu_2}\mathrm{SU}(3,3) &\subset \mathrm{E}_{6,4} \\ \mathrm{SU}(1,1)\times_{\mu_2}\mathrm{O}^*(12) &\subset \mathrm{E}_{7,4}. \end{aligned}$$

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